Modification of the "Coulomb" interaction at small distances in finite quantum electrodynamics

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We investigate the "Coulomb" interaction in finite QED at small distances. By finite QED it is meant that we sum all photon self-energy subgraphs in renormalized QED and fix α, the renormalized fine-structure constant, as the (infinite order) zero of the Callan-Symanzik function: $\beta(\alpha)=0^{\infty}$. We show that for $mc\,|\vec{x}-\vec{x}\,'|/\hbar \ll 1$, the Coulomb interaction between two heavy point test bodies, with renormalized charges e_1 and e_2 at $\vec{x}$ and $\vec{x}\,'$, respectively, is given by $V(|\vec{x}-\vec{x}\,'|)\simeq (e_1e_2/4\pi|\vec{x}-\vec{x}\,'|)[q_1(\alpha)-q_2(\alpha)\times mc\,|\vec{x}-\vec{x}\,'|/\hbar+O(m^2c^2|\vec{x}-\vec{x}\,'|^2/\hbar^2)]$, where $1<q_1(\alpha)<\infty$, $0<q_2(\alpha)<\infty$.

I. INTRODUCTION

The purpose of this paper is to formally investigate the classic problem of the modification of the "Coulomb" interaction at small distances in finite quantum electrodynamics (QED). To this end the knowledge of the rate of "decrease" of the photon spectral function at high energies is essential. The rate of decrease of the photon spectral function at high energies in finite QED was investigated by the author some time ago.[1] By finite QED it is meant here that we sum all photon self-energy subgraphs in renormalized QED and fix α, the renormalized fine-structure constant, as the (infinite-order) zero of the Callan-Symanzik function: $\beta(\alpha)=0^{\infty}$ in the sense of Ref. 2.

II. DERIVATION

The potential energy between two heavy point test bodies with renormalized charges e_1 and e_2, situated at $\vec{x}$ and $\vec{x}'$, respectively, is given by the well-known expression[3-5]

$$V(|\vec{x}-\vec{x}'|)=\frac{e_1e_2}{4\pi|\vec{x}-\vec{x}'|}\left[1+\int_0^{\infty}\frac{d\mu^2}{\mu^2}\rho(\mu^2)e^{-(\mu^2)^{1/2}|\vec{x}-\vec{x}'|}\right], \tag{1}$$

where $\rho(\mu^2)$ is the photon spectral function defined through

$$D(Q^2)=\frac{1}{Q^2-i\epsilon}+\int_0^{\infty}\frac{d\mu^2}{\mu^2}\frac{\rho(\mu^2)}{(\mu^2+Q^2-i\epsilon)},\quad \epsilon\to+0 \tag{2}$$

and

$$D_{\mu\nu}(Q)=\left[g_{\mu\nu}-\frac{Q_\mu Q_\nu}{Q^2}\right]D(Q^2)+\text{gauge-fixing terms} \tag{3}$$

is the renormalized photon propagator.

Let $(Q^2>0)$

$$D(Q^2)=d_c(Q^2)/Q^2\,. \tag{4}$$

Then the Callan-Symanzik equation for $[\alpha d_c(Q^2)]^{-1}$ is given by[2]

$$\left[m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right]\alpha d_c(Q^2]^{-1}=[1+\delta(\alpha)]\tilde{\Gamma}_{\gamma\gamma S}(Q^2)\,, \tag{5}$$

where

$$\tilde{\Gamma}_{\gamma\gamma S}Q^2=m_0\frac{\partial}{\partial m_0}\pi(Q^2)\,, \tag{6}$$

$$d_c^{-1}(Q^2)=1+\alpha[\pi(Q^2)-\pi(0)]\,, \tag{7}$$

$$\frac{m}{m_0}\frac{d}{dm}m_0=[1+\delta(\alpha)]\,,\qquad Z_3^{-1}m\frac{d}{dm}Z_3=\beta(\alpha)\,, \tag{8}$$

and $\beta(\alpha)$, $\delta(\alpha)$, and $\tilde{\Gamma}_{\gamma\gamma S}(Q^2)$, in particular, are all cutoff independent. Here m_0 denotes the unrenormalized mass and m and denotes the renormalized one.

In turn the Callan-Symanzik equation for $\Gamma_{\gamma\gamma S}$ is given by[1]

$$\left[m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}-[1+\delta(\alpha)]\right]\tilde{\Gamma}_{\gamma\gamma S}(Q^2)=[1+\delta(\alpha)]\tilde{\Gamma}_{\gamma\gamma SS}(Q^2)\,, \quad (9)$$

where

$$\tilde{\Gamma}_{\gamma\gamma SS}(Q^2)=m_0{}^2\frac{\partial^2}{\partial m_0{}^2}\pi(Q^2)\,, \quad (10)$$

and $\tilde{\Gamma}_{\gamma\gamma SS}$ is also cutoff independent.[1] In perturbation theory $\tilde{\Gamma}_{\gamma\gamma SS}$ vanishes like (m^2/Q^2) up to powers of logarithms. (Note that $\tilde{\Gamma}_{\gamma\gamma S}$, $\tilde{\Gamma}_{\gamma\gamma SS}$ are all infrared cutoff independent as well, since they do not contain any proper subdiagrams in them with all their lines being photon lines.[6])At the eigenvalue $\beta(\alpha)=0$, we then obtain from (5)–(10) (Ref. 1)

$$\alpha d_c(Q^2)\underset{Q^2\to+\infty}{\sim} q(\alpha)+\alpha q_0(\alpha)\left[\frac{m^2}{Q^2}\right]^{1+\delta(\alpha)}\,, \quad (11)$$

where, at least to lowest order in α, $\delta(\alpha)>0$.[2] [The finiteness of the self-mass $\delta m=m-m_0$, or more precisely the dynamical origin of the mass, implies that $\delta(\alpha)>0$. We note that (cf. Ref. 5) $q(\alpha)=\alpha+O(\alpha^2)$, and from the work of Källén and Sabry,[7] as the coefficient of (m^2/Q^2) for $Q^2\to\infty$ in $d_c(Q^2)$, the following expression may be extracted[1]: $q_o(\alpha)=2\alpha/\pi+O(\alpha^2)$.]

We may write

$$d_c\left[\frac{Q^2}{m^2}\right]=1+Q^2\int_0^\infty\frac{d\mu^2}{\mu^2}\frac{\rho(\mu^2/m^2)}{(\mu^2+Q^2)} \quad (12)$$

and

$$q_1(\alpha)=1+\int_0^\infty\frac{d\mu^2}{\mu^2}\rho(\mu^2/m^2)\,, \quad (13)$$

where $q(\alpha)=\alpha q_1(\alpha)$. The finiteness of $q_1(\alpha)$ (or equivalently of $1/Z_3$) implies that $\rho(0)=0$ and $\rho(\infty)=0$. In (12) we note that taking the limit $Q^2\to\infty$ is equivalent to taking the limit $m^2\to 0$. We cannot *a priori* take the limit $m^2\to 0$ inside the integral in (12) without a knowledge of the asymptotic behavior of $\rho(\mu^2)$. Instead consider the following expression:

$$m^2\frac{\partial}{\partial m^2}\left[m^2\frac{\partial}{\partial m^2}-1\right]d_c(Q^2)=2\int_0^\infty\frac{x\,dx}{(1+x)^3}\rho\left[x\frac{Q^2}{m^2}\right]\,. \quad (14)$$

At the eigenvalue, we then obtain, in particular from (11) and (14), by an elementary dimensional reasoning that[1]

$$\rho\left[\frac{\mu^2}{m^2}\right]\underset{\mu^2\to\infty}{\sim} C_0(\alpha)\left[\frac{m^2}{\mu^2}\right]^{1+\delta(\alpha)} \quad (15)$$

and

$$\pi\csc[-\delta(\alpha)\pi]C_0(\alpha)=-q_0(\alpha)\,, \quad (16)$$

where positivity requires that $C_0(\alpha)>0$. [We note that since $q_0(\alpha)=2\alpha/\pi+O(\alpha^2)$, (16) implies that $C_0(\alpha)=3\alpha^2/\pi^2+O(\alpha^3)$, and this is consistent with positivity and coincides with the coefficient of (m^2/μ^2) coming from the so-called photon self-energy proper diagrams in Ref. 7.] The rapid vanishing property of $\rho(\mu^2/m^2)$ in (15) is to be noted.

We rewrite the potential in (1) in the form

$$V(|\vec{x}-\vec{x}'|)=\frac{e_1e_2}{4\pi|\vec{x}-\vec{x}'|}[q_1(\alpha)-q_2(\alpha)m\,|\vec{x}-\vec{x}'|+R(|\vec{x}-\vec{x}'|)]\,, \quad (17)$$

where

$$R(|\vec{x}-\vec{x}'|)=\int_0^\infty\frac{d\mu^2}{\mu^2}\rho\left[\frac{\mu^2}{m^2}\right][e^{-(\mu^2)^{1/2}|\vec{x}-\vec{x}'|}-1+(\mu^2)^{1/2}|\vec{x}-\vec{x}'|], \quad (18)$$

$$q_2(\alpha)=\int_0^\infty\frac{d\mu^2}{\mu^2}\left[\frac{\mu^2}{m^2}\right]^{1/2}\rho\left[\frac{\mu^2}{m^2}\right]>0\,. \quad (19)$$

The factor $(\mu^2)^{1/2}$ in (19) improves the low-energy behavior of the integrand in it for $\mu^2 \to 0$. This, together with the finiteness of $q_1(\alpha)(1/Z_3)$ in (13) and the power-law behavior obtained for $\rho(\mu^2/m^2)$ in (15) for $\mu^2 \to \infty$, implies that $q_2(\alpha)$ is finite.

Finally we use the following inequalities:

$$|e^{-a}-1| = a\int_0^1 e^{-ax}dx \le a \; , \qquad (20)$$

$$|e^{-a}-1+a| \le a\int_0^1 |e^{-ax}-1|\,dx \le a^2\int_0^1 x\,dx = \frac{a^2}{2} \; , \qquad (21)$$

for $a>0$, to bound $R(|\vec{x}-\vec{x}'|)$ in (18) as

$$|R(|\vec{x}-\vec{x}'|)| \le \frac{m^2|\vec{x}-\vec{x}'|^2}{2} \times \int_0^\infty d\left[\frac{\mu^2}{m^2}\right]\rho\left[\frac{\mu^2}{m^2}\right] , \qquad (22)$$

and with the asymptotic behavior of $\rho(\mu^2/m^2)$ obtained in (15), there is no question of the existence of the integral in (22).

Upon readjusting the units and hence introducing the fundamental constants c and $\hbar$, we obtain from (17) and (22)

$$V(|\vec{x}-\vec{x}'|) \simeq \frac{e_1 e_2}{4\pi|\vec{x}-\vec{x}'|}\left[q_1(\alpha)-q_2(\alpha)\frac{mc|\vec{x}-\vec{x}'|}{\hbar}+O\left[\frac{m^2c^2|\vec{x}-\vec{x}'|^2}{\hbar^2}\right]\right] \qquad (23)$$

for $mc|\vec{x}-\vec{x}'|/\hbar \ll 1$.[8]

As very little is known about the full Callan-Symanzik function beyond its infinite-order-zero nature (if indeed it does have a zero) and beyond its low-order structure, we will not go into the philosophical implications of (23).

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[1] E. B. Manoukian, Phys. Rev. D 11, 3616 (1975).

[2] S. L. Adler, Phys. Rev. D 5, 3021 (1972); 7, 1984 (E) (1973) and references therein.

[3] J. Schwinger, Phys. Rev. 151, 1048, (1966); 151, 1055 (1966); *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1970), Vol. I.

[4] R. Serber, Phys. Rev. 48, 49 (1935); A. E. Uehling, *ibid.* 48, 55 (1935); P. A. M. Dirac, Proc. Cambridge Philos. Soc. 30, 150 (1934); W. Heisenberg, Z. Phys. 90, 209 (1934).

[5] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[6] E. B. Manoukian, J. Math. Phys. 21, 1218 (1980); 21, 1662 (1980).

[7] G. Källén and A. Sabry, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 29, No. 17, 1 (1955).

[8] Technically an upper bound may be readily obtained for $mc|\vec{x}-\vec{x}'|/\hbar$ for the validity of (23). Note that

$$q_1(\alpha) > \int_0^\infty \frac{dx}{x}\rho(x)$$

and

$$q_2(\alpha) = \int_0^\infty \frac{dx}{x} x^{1/2}\rho(x) \le \left[\int_0^\infty \frac{dx}{x}\rho(x)\right]^{1/2}\left[\int_0^\infty dx\,\rho(x)\right]^{1/2} .$$

Accordingly

$$q_1(\alpha)-q_2(\alpha)\frac{mc|\vec{x}-\vec{x}'|}{\hbar} \ge \frac{m^2c^2|\vec{x}-\vec{x}'|^2}{2\hbar^2}\int_0^\infty dx\,\rho(x)\;[\ge |R(|\vec{x}-\vec{x}'|)|] ,$$

and, hence, (23) rigorously holds for

$$\frac{mc|\vec{x}-\vec{x}'|}{\hbar} \ll \frac{1}{\sqrt{3}}\left[\int_0^\infty \frac{dx}{x}\rho(x)\right]^{1/2} \times \left[\int_0^\infty dx\,\rho(x)\right]^{-1/2} .$$