Minimum-uncertainty coherent states for certain time-dependent systems

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Nieto and Simmons have defined and studied coherent states for arbitrary potentials $V(q)$. We show how to extend their results to certain time-dependent potentials $V(q,t)$. For each $V(q)$ there is a $V(q,t)$, for which we can construct time-dependent coherent states.

I. INTRODUCTION

In a series of papers Nieto and Simmons[1] have constructed generalized coherent states for systems with Hamiltonians

$$H=\tfrac{1}{2}p^2+V(q) \tag{1}$$

for various potentials $V(q)$. These coherent states have been further studied in later papers.[2-4] The hope is that these generalized coherent states can be used to describe the interactions between molecules and lasers.

In this paper we point out that generalized Nieto-Simmons coherent states can be constructed for certain time-dependent systems

$$H=\tfrac{1}{2}p^2+V(q,t) \ . \tag{2}$$

We use the results of our recent papers[5,6] on systems described by Hamiltonians

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2+\frac{1}{x^2}f(q/x) \ , \tag{3}$$

where $\omega(t)$, $f(q/x)$ are arbitrary functions and x is a c-number solution to the auxiliary equation

$$\ddot{x}+\omega^2(t)x=1/x^3 \ . \tag{4}$$

Systems such as (3) are called Ermakov systems and possess an invariant I given by

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2+\tfrac{1}{2}(q/x)^2+f(q/x) \ . \tag{5}$$

II. TIME-DEPENDENT COHERENT STATES

The solution to the Schrödinger equation for (3) involves solving the equation

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\tfrac{1}{2}\sigma^2+f(\sigma)\right]\phi_n(\sigma)=\lambda_n\phi_n(\sigma) \ . \tag{6}$$

Since the function f in (6) is arbitrary, this equation is the time-independent Schrödinger equation for an arbitrary potential. Thus, we can perform the Nieto-Simmons construction for the time-independent equation (6) and then map the states so obtained back to the original system via the inverse transformations. We shall illustrate this procedure using the time-dependent harmonic oscillator for which we have recently constructed coherent states.[7]

We consider the Hamiltonian (3) with $f=0$,

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2 \ , \tag{7}$$

and the Lewis invariant

$$I=\tfrac{1}{2}(xp-\dot{x}q)^2+\tfrac{1}{2}(q/x)^2 \ . \tag{8}$$

The eigenvalue problem for the invariant I is

$$I\psi_n(q,t)=\lambda_n\psi_n(q,t) \ . \tag{9}$$

Under the unitary transformation $U=e^{-i\dot{x}q^2/(2\hbar x)}$ this becomes

$$I'\phi_n=\lambda_n\phi_n \ , \tag{10}$$

with

$$I'=UIU^\dagger=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\tfrac{1}{2}\sigma^2 \tag{11}$$

and

$$\phi_n=x^{1/2}U\psi_n, \quad \sigma=q/x \ . \tag{12}$$

Equation (10) has the form

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\tfrac{1}{2}\sigma^2\right]\phi_n(\sigma)=\lambda_n\phi_n(\sigma) \ , \tag{13}$$

whose solution is well known,

$$\lambda_n=\hbar(n+\tfrac{1}{2}) \tag{14}$$

and

$$\phi_n(\sigma)=\frac{1}{(\pi^{1/2}\hbar^{1/2}2^n n!)^{1/2}}e^{-\sigma^2/2\hbar}H_n(\sigma/\hbar^{1/2}) \ . \tag{15}$$

For the "Hamiltonian" I' we define the lowering and raising operators

$$a'=\frac{1}{(2\hbar)^{1/2}}\left[\sigma+i\frac{\hbar}{i}\frac{\partial}{\partial\sigma}\right], \tag{16}$$

$$a'^{\dagger}=\frac{1}{(2\hbar)^{1/2}}\left[\sigma-i\frac{\hbar}{i}\frac{\partial}{\partial\sigma}\right], \tag{17}$$

which factors I',

$$I'=\hbar(a'^{\dagger}a'+\tfrac{1}{2}) . \tag{18}$$

Coherent states for I' have the form

$$\phi_\alpha(\sigma,t)=e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{(n!)^{1/2}}e^{i\alpha_n(t)}\phi_n(\sigma) , \tag{19}$$

where from Ref. 5

$$\alpha_n(t)=-(n+\tfrac{1}{2})\int_0^t \frac{dt'}{x^2(t')} .$$

The factor $e^{i\alpha_n(t)}$ in (19) is dictated by the requirement that when $\omega(t)\to\omega_0=\text{const}$ and

$$x(t)\to x_0=\text{const}=1/\omega_0^{1/2} ,$$

the coherent states $\phi_\alpha(q,t)$ become the correct coherent states for the time-independent harmonic oscillator. The states $\phi_\alpha(\sigma,t)$ are eigenstates of a' with eigenvalue $\alpha\exp[-i\int_0^t dt'/x^2(t')]$. Note that as $\omega\to\omega_0$ this becomes $\alpha e^{-i\omega_0 t}$, the usual result. Coherent states for the time-dependent harmonic oscillator are now obtained by the inverse transformation on $\phi_\alpha(\sigma,t)$ and have the form

$$\psi_\alpha(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_\alpha(\sigma,t) . \tag{20}$$

The coherent states (20) are just the coherent states for the time-dependent harmonic oscillator derived in Ref. 7. There are several ways to prove this latter statement. For example, consider the eigenvalue problem

$$a'\phi_\alpha(\sigma,t)=\alpha\exp\left[-i\int dt'/x^2\right]\phi_\alpha(\sigma,t) . \tag{21}$$

Transforming this via the inverse transformation yields

$$a\psi_\alpha(q,t)=\alpha\exp\left[-i\int dt'/x^2\right]\psi_\alpha(q,t) , \tag{22}$$

where

$$a=e^{i\dot{x}q^2/(2\hbar x)}a'e^{-i\dot{x}q^2/(2\hbar x)}$$
$$=\frac{1}{(2\hbar)^{1/2}}[q/x+i(xp-q\dot{x})] , \tag{23}$$

which is exactly the lowering operator associated with the Lewis invariant (8). The eigenvalue of a in (22) is also correct.[7] The operator (23) was originally introduced bv Lewis[8] to factor the Lewis invariant (8) as

$$I=\hbar(a^{\dagger}a+\tfrac{1}{2}) \tag{24}$$

in the first exact quantum treatment of the time-dependent harmonic oscillator. Thus, coherent states for the time-dependent harmonic oscillator can be derived by (i) finding coherent states for the time-independent problem (13), $\phi_\alpha(\sigma,t)$, and (ii) transforming to $\psi_\alpha(q,t)$ via (20). These states $\psi_\alpha(q,t)$ are then the generalized Nieto-Simmons coherent states for the time-dependent harmonic oscillator. Note that these time-dependent coherent states have the Schrödinger property of giving the *exact* classical motion for $\langle\psi_\alpha|q|\psi_\alpha\rangle$ as is proven in Ref. 7.

III. NIETO-SIMMONS COHERENT STATES FOR GENERAL POTENTIALS

For other Ermakov systems we can construct Nieto-Simmons coherent states in the same manner as just described for the time-dependent harmonic oscillator. First we consider the time-independent Schrödinger equation (6),

$$I'\phi_n(\sigma)=\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\sigma^2}+\tfrac{1}{2}\sigma^2+f(\sigma)\right]\phi_n(\sigma)$$
$$=\lambda_n\phi_n(\sigma) . \tag{25}$$

We use I' to construct Nieto-Simmons coherent states $\phi_\alpha(\sigma,t)$ following Ref. 1. We first define classical variables $X_c(\sigma),P_c(\sigma)$ associated with the classical motion of a particle with Hamiltonian I',

$$X_c'=\frac{dX_c(\sigma)}{d\sigma}=\frac{\omega_c}{2^{1/2}}\frac{(A_c^2-X_c^2)^{1/2}}{[I'-V(\sigma)]^{1/2}} , \tag{26}$$

$$P_c=\dot{X}_c=p_\sigma X_c' , \tag{27}$$

where ω_c, A_c are constants and $p_\sigma=\dot{\sigma}$. The function $X_c(\sigma)$ is found by solving (26) and then P_c is found from (27). Next we construct the Nieto-Simmons operators X, P,

$$X(\sigma)=X_c(\sigma) , \tag{28}$$

$$P=\frac{\hbar}{2i}\left[\frac{\partial}{\partial\sigma}X'(\sigma)+X'(\sigma)\frac{\partial}{\partial\sigma}\right] . \tag{29}$$

We then use these operators to construct minimum-uncertainty coherent states $\phi_\alpha(\sigma)$ following Ref. 1,

$$\frac{1}{2}\left[\frac{X}{\Delta X}+i\frac{P}{\Delta P}\right]\phi_\alpha(\sigma)=\alpha\phi_\alpha(\sigma)\ , \quad (30)$$

where

$$\Delta A=(\langle A^2\rangle-\langle A\rangle^2)^{1/2}\ , \quad (31)$$

for any quantity A. The states $\phi_\alpha(\sigma)$ have the form

$$\phi_\alpha(\sigma)=\sum c_n\phi_n(\sigma)\ , \quad (32)$$

where c_n are constants and $\phi_n(\sigma)$ are the eigenstates of I'. Next we map these states and operators using the unitary operator

$$V=\exp\left[-i\frac{I'}{\hbar}\int dt'/x^2\right]$$

which gives the coherent states

$$\phi_\alpha(\sigma,t)=\exp\left[-i\frac{I'}{\hbar}\int dt'/x^2\right]\phi_\alpha(\sigma)$$

$$=\sum c_n e^{i\alpha_n(t)}\phi_n(\sigma)\ , \quad (33)$$

and new operators $\bar{X}$, $\bar{P}$ defined by

$$\bar{X}=VXV^\dagger,\ \ \bar{P}=VPV^\dagger\ . \quad (34)$$

The operator V introduces the time-dependent phase factor $e^{i\alpha_n(t)}$ into (33). This is necessary in order to obtain the correct coherent states in the time-independent limit $\omega\rightarrow\omega_0$, $x\rightarrow 1/\omega_0{}^{1/2}$. In this limit

$$H_0=\omega_0 I_0\ , \quad (35)$$

$$E_n=\omega_0\lambda_n\ , \quad (36)$$

$$U=e^{i\dot{x}q^2/(2\hbar x)}=1\ , \quad (37)$$

$$\alpha_n=-E_n/\hbar\ , \quad (38)$$

where

$$H_0=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega_0{}^2q^2+\omega_0 f(\omega_0{}^{1/2}q)\ , \quad (39)$$

and the E_n's are the eigenvalues of H_0. Note that in this time-independent limit there is no need to introduce I_0, since it differs by only a constant from H_0. Thus, in the time-independent limit we are back exactly to the Nieto-Simmons case of time-independent Hamiltonians. The final step in obtaining the coherent states for the Ermakov Hamiltonian (3) is to apply the unitary operator $U^\dagger=e^{i\dot{x}q^2/(2\hbar x)}$, which gives the states

$$\psi_\alpha(q,t)=\frac{1}{x^{1/2}}e^{i\dot{x}q^2/(2\hbar x)}\phi_\alpha(\sigma,t)\ , \quad (40)$$

and the operators $\bar{X}'$, $\bar{P}'$ defined by

$$\bar{X}'=U^\dagger\bar{X}U,\ \ \bar{P}'=U^\dagger\bar{P}U\ . \quad (41)$$

These states $\psi_\alpha(q,t)$ are coherent states for the time-dependent system described by the Hamiltonian (3).

IV. CONCLUSIONS

It is important to notice that by construction the coherent states (40) satisfy the time-dependent Schrödinger equation (7). Thus one can, following the work of Gutschick and Nieto,[2] time-evolve the coherent states $\psi_\alpha(q,t)$ and study their evolution. Here, this will be a somewhat more involved problem since one must calculate $x(t)$ and $\alpha_n(t)$. In the constant-frequency case the functions x and α_n have trivial time dependence.

The physical interpretation of time-dependent coherent states is discussed in Ref. 7. For the time-dependent harmonic oscillator they are states that are associated with the exact classical motion. They should be useful in describing the radiation field of a single-mode laser as the laser is tuned. If the Nieto-Simmons coherent states have practical applications for molecule-laser interactions, then the time-dependent coherent states derived here should have similar applications involving such time-dependent systems.

As a final result, we mention that most of the results of this paper can be generalized to N-dimensional Ermakov systems (Ref. 9).

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