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Simplified solution of the Dirac equation with a Coulomb potential

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It is shown that the Dirac equation with a Coulomb potential has a simplified solution where each component contains one term of a confluent hypergeometric function only instead of two terms. This solution reduces to the usual free-field solution when the Coulomb potential is turned off. Thus the Dirac Coulomb equation has a solution which is not very different from the corresponding Schrödinger or Klein-Gordon equations.

I. INTRODUCTION

The traditional Dirac equation with a Coulomb potential is written as

$$(E-\vec{\alpha}\cdot\vec{p}-\beta m-V)\Psi=0\ , \tag{1.1}$$

where α_1, α_2, α_3, and β are the usual matrices, obeying

$$\alpha_1{}^2=\alpha_2{}^2=\alpha_3{}^2=\beta^2=1\ , \tag{1.2}$$

$$\alpha_1\alpha_2+\alpha_2\alpha_1=\alpha_1\alpha_3+\alpha_3\alpha_1=\alpha_2\alpha_3+\alpha_3\alpha_2=0\ , \tag{1.3}$$

$$\alpha_1\beta+\beta\alpha_1=\alpha_2\beta+\beta\alpha_2=\alpha_3\beta+\beta\alpha_3=0\ , \tag{1.4}$$

$$V=-Ze^2/r\ . \tag{1.5}$$

One of the representations for α and β is

$$\alpha_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\quad \alpha_2=\begin{bmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{bmatrix},$$
$$\alpha_3=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix},\quad \beta=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}. \tag{1.6}$$

The solution to Eq. (1.1) in spherical coordinates was obtained by Darwin[1] and Gordon,[2] and a detailed derivation can be found in Bethe and Salpeter.[3] The radial solution can be expressed in terms of two functions $g(r)$ and $f(r)$, where each function is a sum of two confluent hypergeometric functions.

Martin and Glauber[4] introduced an operator suggested by Lippmann and Johnson[5]

$$\mathscr{L}=-K\beta-iZe^2\alpha_r\ , \tag{1.7}$$

where

$$K=\beta(\vec{\sigma}\cdot\vec{L}+1)\ , \tag{1.8}$$

$$\alpha_r=\vec{\alpha}\cdot\vec{r}/r\ . \tag{1.9}$$

The eigenvalue of $\mathscr{L}$ is

$$\gamma=\pm|\gamma|=\pm[\kappa^2-(Ze^2)^2]^{1/2}\ , \tag{1.10}$$

where

$$\kappa^2=(j+\tfrac{1}{2})^2\ . \tag{1.11}$$

However, they treated the solutions to the first-order Dirac equation as linear combinations of eigenfunctions of $\mathscr{L}$. Thus their solution to the first-order Dirac equation still contained sums of confluent hypergeometric functions.

Biedenharn[6] found a transformation S which diagonalizes $\mathscr{L}$. Explicitly,

$$S=\exp[-\tfrac{1}{2}\rho_2\vec{\sigma}\cdot\hat{r}\tanh^{-1}(Ze^2/K)] \tag{1.12}$$

such that

$$S\mathscr{L}S^{-1}=-\rho_3K\,|[1-(Ze^2/K)^2]^{1/2}|\ . \tag{1.13}$$

Thus in this representation $\mathscr{L}$ can be considered to have sharp eigenvalues. Choosing the convention

$$\mathrm{sign}(\gamma)=\mathrm{sign}(\kappa) \tag{1.14}$$

Biedenharn obtained a solution to the Dirac

Coulomb equation for the continuum. His solution is simpler than the previous ones in that each component contains only one term of a confluent hypergeometric function, and agrees with the usual plane-wave solution when the Coulomb potential is turned off.

We wish to make two modifications on Biedenharn's work. First one notices that Biedenharn's Dirac equation differs from the traditional Dirac equation (1.1) in the sign of E. We shall keep the sign of E the same as in Eq. (1.1), but we shall change the sign of m. That is, we shall start with the equation

$$(E-\vec{\alpha}\cdot\vec{\mathrm{p}}+\beta m-V)\Psi=0 \ . \tag{1.15}$$

Second, Biedenharn did not discuss the bound-state case. We shall show that the bound-state case can be treated in the same way as the continuum. Thus we shall show in this paper the following results.

1. The Dirac Coulomb equation has a simple solution where each component contains one term of a confluent hypergeometric function only.

2. The solution is valid for both the bound state and the continuum. Moreover, they reduce to the traditional plane-wave solution when the potential is turned off.

3. Our solution is obtained if one writes the Dirac Coulomb equation in the form of Eq. (1.15).

One obvious conclusion from our result is that the Dirac Coulomb equation has basically the same type of solution as the corresponding Schrödinger and Klein-Gordon equations. Therefore the Dirac equation is really not too difficult to deal with, and should be used more often, because it is, after all, more accurate.

In a subsequent paper we shall apply our solution to the Coulomb scattering of fast electrons.

II. TRADITIONAL DIRAC EQUATION WITH A COULOMB POTENTIAL

The traditional Dirac equation for a free field is[7]

$$(E-\vec{\alpha}\cdot\vec{\mathrm{p}}-\beta m)\Psi=0 \ . \tag{2.1}$$

We use units such that $\hbar=1$, $c=1$.

If one includes a Coulomb potential $V=-Ze^2/r$, then one obtains the traditional Dirac Coulomb equation

$$(E-\vec{\alpha}\cdot\vec{\mathrm{p}}-\beta m-V)\Psi=0 \ . \tag{2.2}$$

Using the matrices for α and β given in Eq. (1.6), one obtains from (2.2) four partial differential equations in agreement with Bethe and Salpeter[3] (Ref. 3, Eq. 14.2). The solution to these equations, originally obtained by Darwin[1] and Gordon,[2] can be found in Ref. 3. One notes that the solution is complicated, and that each component contains the sum of two confluent hypergeometric functions.

It is the purpose of this paper to show that the solution to the Dirac Coulomb equation can be written in a simplified form, where each component contains one term of a confluent hypergeometric function only. Thus the solution is very similar to the corresponding solution of the Schrödinger or Klein-Gordon equation.

An improvement over the traditional solution was made by Biedenharn[6] for the continuum case. Biedenharn starts with the equation

$$[\rho_2\vec{\sigma}\cdot\vec{\nabla}+\rho_3(E/\hbar c-\alpha Z/r)+mc/\hbar]\Psi=0 \ . \tag{2.3}$$

It can be seen that (2.3) differs from (2.2) in the sign of E.

The main contributions of Biedenharn's work are the following. (1) He uses the "Lippmann-Johnson" or "Martin-Glauber" operator $\mathscr{L}$ [Eq. (1.7)] and finds a transformation S [Eq. (1.12)] that diagonalizes $\mathscr{L}$. (2) He obtains a recurrence relation for the confluent hypergeometric functions, which are solutions to the second-order Dirac Coulomb equation. Thus he was able to obtain a solution to the Dirac Coulomb equation (2.3) where each component contains one term of a confluent hypergeometric function only. Moreover, he shows that this solution easily goes over to the plane-wave case when the Coulomb potential is turned off.

In the next section we shall make some modifications on Biedenharn's work, and obtain some simple and general results for the Dirac Coulomb equation.

The new features of our work are as follows.

1. We shall start with a Dirac Coulomb equation which differs from the traditional equation (1.1) in the sign of the mass term m. Our equation differs from Biedenharn's equation in the sign of both E and m. The equation we start with is therefore

$$(E-\vec{\alpha}\cdot\vec{\mathrm{p}}+\beta m-V)\Psi=0 \ . \tag{2.4}$$

2. Biedenharn did not discuss the case of the bound state. We shall show that the solution for the bound state is just as simple as the solution for the continuum. Moreover, they both agree with the traditional solution of the free field when the

potential is turned off. Finally, we find that the solution of the Dirac Coulomb equation is very similar to the solution of the corresponding Schrödinger and Klein-Gordon equations.

In a subsequent paper we shall apply our results to the Coulomb scattering of fast electrons, obtaining results in agreement with McKinley and Feshbach.[8]

III. SIMPLIFIED SOLUTION OF THE DIRAC COULOMB EQUATION

First one notes that the sign of $\vec{\alpha}\cdot\vec{p}$ and βm in Eq. (1.1) is somewhat arbitrary. In fact, Biedenharn's Eq. (2.3) is equivalent to changing the sign of E or both $\vec{\alpha}\cdot\vec{p}$ and βm. Thus we shall start with the Dirac Coulomb equation

$$(E-\vec{\alpha}\cdot\vec{p}+\beta m-V)\Psi=0\ , \tag{3.1}$$

where α and β will still have the same properties as in (1.2), (1.3), and (1.4), and we shall use the same representation for α and β as Eq. (1.6).

Equation (3.1) can also be written as

$$[\rho_2\vec{\sigma}\cdot\vec{\nabla}-\rho_3(E+Ze^2/r)-m]\Psi=0\ , \tag{3.2}$$

where

$$\rho_i\rho_j=-\rho_j\rho_i=i\rho_k\ . \tag{3.3}$$

The subscripts i, j, k, are cyclic permutations of 1, 2, 3.

Now consider the second-order Dirac equation

$$O_-O_+\Phi=[\rho_2\vec{\sigma}\cdot\vec{\nabla}-\rho_3(E+Ze^2/r)-m][\rho_2\vec{\sigma}\cdot\vec{\nabla}-\rho_3(E+Ze^2/r)+m]\Phi=0\ . \tag{3.4}$$

Equation (3.4) can be written as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{K^2-Z^2e^4}{r^2}+\frac{2Ze^2E}{r}+\frac{\rho_3K+iZe^2\rho_1\vec{\sigma}\cdot\hat{r}}{r^2}+E^2-m^2\right]\Phi=0\ , \tag{3.5}$$

where e^2 is sometimes written as

$$\alpha=\text{the fine-structure constant}\ , \tag{3.6}$$

$$K=\rho_3(\vec{\sigma}\cdot\vec{L}+1)\ . \tag{3.7}$$

Equation (3.7) is the same as (1.8), since $\rho_3=\beta$.

Next define the Martin-Glauber operator $\mathscr{L}$,

$$\mathscr{L}=-(\rho_3K+iZe^2\rho_1\vec{\sigma}\cdot\hat{r})\ . \tag{3.8}$$

Then (3.5) becomes

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{\mathscr{L}(\mathscr{L}+1)}{r^2}+\frac{2Ze^2E}{r}+E^2-m^2\right]\Phi=0\ . \tag{3.9}$$

Let us now discuss the solution for the bound state, when $m^2-E^2>0$. Introduce the transformation

$$\rho=2\mu r=2(m^2-E^2)^{1/2}r\ , \tag{3.10}$$

$$\mu=(m^2-E^2)^{1/2}\ , \tag{3.11}$$

$$w/4=Ze^2E/\mu\ . \tag{3.12}$$

Equation (3.9) becomes

$$\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}-\frac{\mathscr{L}(\mathscr{L}+1)}{\rho^2}+\frac{w}{4\rho}-\frac{1}{4}\right]\Phi(\rho)=0\ . \tag{3.13}$$

Equation (3.13) is completely similar to the corresponding radial wave function of the Schrödinger equation (see, e.g., Ref. 7, Eq. 16.7)

$$\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}-\frac{l(l+1)}{\rho^2}+\frac{\lambda}{\rho}-\frac{1}{4}\right]R=0\ . \tag{3.14}$$

The eigenvalues of K and $\mathscr{L}$ can be found as follows:

$$K^2=(\vec{\sigma}\cdot\vec{L}+1)^2=(\vec{L}+\tfrac{1}{2}\vec{\sigma})^2+\tfrac{1}{4}=\vec{J}^2+\tfrac{1}{4}\ , \tag{3.15}$$

$$\kappa=\pm(j+\tfrac{1}{2})\ , \tag{3.16}$$

where κ is the eigenvalue of K.

The eigenvalue of $\mathscr{L}$ is denoted by γ:

$$\mathscr{L}^2=K^2-(Ze^2)^2=(j+\tfrac{1}{2})^2-(Ze^2)^2\ , \tag{3.17}$$

$$\gamma=\pm[(j+\tfrac{1}{2})^2-(Ze^2)^2]^{1/2}\ . \tag{3.18}$$

Since κ can take both positive and negative values, we shall define $\tilde{\omega}$ to be the sign of κ:

$$\tilde{\omega}=\text{sign}(\kappa)\ . \tag{3.19}$$

Similarly we shall define $\tilde{\gamma}$ to be the sign of γ:

$$\tilde{\gamma}=\text{sign}(\gamma)\ . \tag{3.20}$$

Next we introduce $\lambda(\gamma)$ such that

$$\lambda(\gamma)=|\gamma|+\tfrac{1}{2}(\tilde{\gamma}-1)\ . \tag{3.21}$$

Then the solution of (3.13) is

$$\Phi_{\lambda(\gamma)}(\rho)=C_{\lambda(\gamma)}\rho^{\lambda(\gamma)}e^{-\rho/2}{}_1F_1(-\tfrac{1}{4}w+\lambda(\gamma)+1,2\lambda(\gamma)+2,\rho)\ , \tag{3.22}$$

where

$$C_{\lambda(\gamma)}=\frac{[\Gamma(\frac{1}{4}w+\lambda(\gamma)+1)\Gamma(\frac{1}{4}w-\lambda(\gamma)+1)]^{1/2}}{\Gamma(2\lambda(\gamma)+2)}\ . \tag{3.23}$$

For the confluent hypergeometric function to terminate, we must have

$$\tfrac{1}{4}w=\lambda+1+n'\ , \tag{3.24}$$

where

$$n'=\text{non-negative integer}=0,1,2,\ldots\ . \tag{3.25}$$

From Eqs. (3.24), (3.12), and (3.11) we obtain the eigenvalue of E:

$$\frac{Ze^2E}{(m^2-E^2)^{1/2}}=|\gamma|+\tfrac{1}{2}\tilde{\gamma}+\tfrac{1}{2}+n'\ , \tag{3.26}$$

or

$$E=\frac{m}{[1+(Ze^2)^2/(|\gamma|+\frac{1}{2}\tilde{\gamma}+\frac{1}{2}+n')^2]^{1/2}} \tag{3.27}$$

$$=\frac{m}{(1+(Ze^2)^2/\{[(j+\frac{1}{2})^2-Z^2e^4]^{1/2}+\frac{1}{2}\tilde{\gamma}+\frac{1}{2}+n'\}^2)^{1/2}}\ . \tag{3.28}$$

Equation (3.28) agrees with the usual energy spectrum of the Dirac Coulomb equation.

In order to obtain an explicit solution to the first-order Dirac Coulomb equation (3.1), we shall choose a special representation in which the operators ρ_3, K, and $\mathscr{L}$ have sharp eigenvalues. We shall choose the eigenvalues as follows:

$$\rho_3=-1\ , \tag{3.29}$$

$$\kappa=\tilde{\omega}(j+\tfrac{1}{2})\ , \tag{3.30}$$

$$\gamma=\tilde{\gamma}[(j+\tfrac{1}{2})^2-(Ze^2)^2]^{1/2}\ , \tag{3.31}$$

$$\tilde{\omega}=\tilde{\gamma}\ . \tag{3.32}$$

We would like to point out that Eq. (3.32) is not just a convention, but a necessary consequence of Eq. (3.29). The justification lies in the fact that when the Coulomb potential is turned off (or when $Z=0$), $\mathscr{L}=-\rho_3K$. Therefore $\tilde{\gamma}$ must be equal to $\tilde{\omega}$, when $\rho_3=-1$.

Then the spinors χ_κ^μ obey the eigenvalue problem

$$(\vec{\sigma}\cdot\vec{L}+1)\chi_\kappa^\mu=-\kappa\chi_\kappa^\mu \tag{3.33}$$

and have the following solution:

$$\chi_\kappa^\mu=\sum_\tau\left[\begin{array}{cc|c} l & \frac{1}{2} & j \\ \mu-\tau & \tau & \mu\end{array}\right]Y_l^{\mu-\tau}(\theta,\Phi)\chi_{1/2}^\tau \tag{3.34}$$

or, equivalently,

$$\chi_{-\kappa}^\mu=\sum_\tau\left[\begin{array}{cc|c} l & \frac{1}{2} & j \\ \mu-\tau & \tau & \mu\end{array}\right]Y_{l'}^{\mu-\tau}(\theta,\Phi)\chi_{1/2}^\tau\ , \tag{3.35}$$

where

$$l=j+\tfrac{1}{2}\tilde{\omega}\ ,\quad l'=j-\tfrac{1}{2}\tilde{\omega}\ . \tag{3.36}$$

Note that either (3.34) or (3.35) includes both cases for $\tilde{\omega}=\pm 1$. In other words, Eq. (3.35) is equivalent to Eq. (3.34) and each of them includes two equations.

Under the operation $\vec{\sigma}\cdot\hat{r}$, we have

$$(\vec{\sigma}\cdot\hat{r})\chi_\kappa^\mu=-\chi_{-\kappa}^\mu \,. \tag{3.37}$$

Since we have chosen the eigenvalue of ρ_3 to be -1, we can write the solution to (3.13) as

$$\Phi_{\rho_3=-1}=\Phi_-=\begin{bmatrix} 0 \\ \Phi_{\lambda(\gamma)}(\rho)\chi_\kappa^\mu \end{bmatrix} . \tag{3.38}$$

We have mentioned before that $\mathscr{L}$ is diagonalized by the transformation S, found by Biedenharn as given in Eq. (1.12). Thus the solution to the first-order Dirac Coulomb equation (3.1) or (3.2) is

$$\Psi=SO_+S^{-1}\Phi_- \tag{3.39}$$

where O_+ is defined in Eq. (3.4). We find

$$SO_+S^{-1}=\rho_2\vec{\sigma}\cdot\hat{r}\left[\frac{\partial}{\partial r}+\frac{1-\gamma\rho_3}{r}+\rho_3\frac{EZe^2}{\gamma}\right] -\rho_3\frac{E\kappa}{\gamma}+m \,. \tag{3.40}$$

Explicitly in ρ space the operator SO_+S^{-1} becomes

$$SO_+S^{-1}=\begin{bmatrix} -\dfrac{\kappa E}{\gamma}+m & -i\vec{\sigma}\cdot\hat{r}\left[\dfrac{\partial}{\partial r}+\dfrac{1+\gamma}{r}-\dfrac{EZe^2}{\gamma}\right] \\ i\vec{\sigma}\cdot\hat{r}\left[\dfrac{\partial}{\partial r}+\dfrac{1-\gamma}{r}+\dfrac{EZe^2}{\gamma}\right] & \dfrac{\kappa E}{\gamma}+m \end{bmatrix} . \tag{3.41}$$

Next we use the recurrence relation between $\Phi_{\lambda(\gamma)}$ and $\Phi_{\lambda(-\gamma)}$. This recurrence relation is a mathematical identity and is found to be

$$\left[\frac{d}{dr}+\frac{1+\gamma}{r}-\frac{EZe^2}{\gamma}\right]\Phi_{\lambda(\gamma)}=\mu\tilde{\gamma}\left[\left[\frac{EZe^2}{\mu\gamma}\right]^2-1\right]^{1/2}\Phi_{\lambda(-\gamma)}=i\tilde{\gamma}\left[m^2-\left[\frac{\kappa E}{\gamma}\right]^2\right]^{1/2}\Phi_{\lambda(-\gamma)} \,. \tag{3.42}$$

Equation (3.42) contains actually two equations, for $\pm\gamma$.

Then applying (3.41) to (3.38), and using (3.42), we obtain the decoupled solution to the first-order Dirac Coulomb equation as follows:

$$\Psi=N(\gamma)\begin{bmatrix} \left[m-E\dfrac{\kappa}{\gamma}\right]^{1/2}\Phi_{\lambda(-\gamma)}\chi_{-\kappa}^\mu \\ \tilde{\omega}\left[m+E\dfrac{\kappa}{\gamma}\right]^{1/2}\Phi_{\lambda(\gamma)}\chi_\kappa^\mu \end{bmatrix} , \tag{3.43}$$

where $N(\gamma)$ is a normalization constant whose value will be given in Eq. (3.45).

It is remarkable that the canonical transformation S, together with the recurrence relation (3.42), which is purely a mathematical identity, should give rise to the solution of the first-order Dirac Coulomb equation in the form of Eq. (3.43) which is so simple, symmetric, and may we add, elegant.

Let us add here that so far we have chosen to project out the $\rho_3=-1$ part, thereby making $\tilde{\gamma}=\tilde{\omega}$. Equation (3.43) then states that the lower component is the "large" component. If we have chosen the eigenvalue of ρ_3 to be $+1$, then we have to choose $\tilde{\gamma}=-\tilde{\omega}$. Equation (3.43) would still be valid, but the upper component will then be the "large" component.

It remains to find the normalization constant $N(\gamma)$. This can be done as follows. The spinor functions χ_κ^μ are already normalized. From (3.43) we obtain

$$N^2(\gamma)\int_0^\infty\left[\left[m-E\frac{\kappa}{\gamma}\right]\Phi_{\lambda(-\gamma)}{}^2+\left[m+E\frac{\kappa}{\gamma}\right]\Phi_{\lambda(\gamma)}{}^2\right]r^2dr=1 \,, \tag{3.44}$$

$$N(\gamma)=\left[\frac{4\mu^3}{n}\right]^{1/2}[(n-\gamma)!(n-\gamma-1)!]^{-1/2}[(n-\gamma)(n-\gamma+1)(m-\tilde{\omega}E\kappa/\gamma)+(m+\tilde{\omega}E\kappa/\gamma)]^{-1/2} \,, \tag{3.45}$$

where

$$n = w/4 . \quad (3.46)$$

The solution of the Dirac Coulomb equation for the continuum is similar to that of the bound state. Here we have

$$E^2 - m^2 > 0 . \quad (3.47)$$

Let us define

$$k' = (E^2 - m^2)^{1/2} . \quad (3.48)$$

Then

$$w/4 = Ze^2E/ik' = -iZe^2E/k' . \quad (3.49)$$

The solution to the second-order equation (3.13) becomes

$$\Phi_{\lambda(\gamma)} = c_{\lambda(\gamma)}(k'r)^{\lambda(\gamma)}e^{-k'r}{}_1F_1(\lambda(\gamma)+1-i\eta,\ 2\lambda(\gamma)+2,\ 2ik'r) , \quad (3.50)$$

where

$$c_\lambda = 2^\lambda e^{-\pi\eta/2}\,|\Gamma(\lambda+1+i\eta)|\,/\Gamma(2\lambda+2) , \quad (3.51)$$

$$i\eta = w/4 = -iZe^2E/k' . \quad (3.52)$$

The recurrence relation (3.41) can now be written as

$$\left[\frac{d}{dr} + \frac{1+\gamma}{r} - \frac{EZe^2}{\gamma}\right]\Phi_{\lambda(\gamma)} = -k'\tilde{\omega}\left[1 + \frac{Z^2e^4E^2}{k'^2\gamma^2}\right]^{1/2}\Phi_{\lambda(-\gamma)} = -i\tilde{\omega}\left[m^2 - \left[\frac{E\kappa}{\gamma}\right]^2\right]^{1/2}\Phi_{\lambda(-\gamma)} . \quad (3.53)$$

Finally the solution to the first-order Dirac Coulomb equation in the continuum is

$$\Psi = N(\gamma)\begin{bmatrix} \left[m - E\dfrac{\kappa}{\gamma}\right]^{1/2}\Phi_{\lambda(-\gamma)}\chi_{-\kappa}^{\mu} \\ \tilde{\omega}\left[m + E\dfrac{\kappa}{\gamma}\right]^{1/2}\Phi_{\lambda(\gamma)}\chi_{\kappa}^{\mu} \end{bmatrix} , \quad (3.54)$$

where $N(\gamma)$ is a normalization constant.

We shall now show that our solution (3.54) goes directly over to the plane-wave solution by turning off the Coulomb potential, i.e., by putting $Z=0$. Then (3.50) becomes

$$\Phi_{\lambda(\gamma)} = N(k'r)^\lambda e^{-k'r}{}_1F_1(\lambda+1, 2\lambda+2, 2ik'r) = N'\Gamma(\lambda+\tfrac{3}{2})r^{-1/2}2^{\lambda+1/2}J_{\lambda+1/2}(k'r) = N''j_{\lambda(\gamma)}(k'r) , \quad (3.55)$$

where $j_{\lambda(\gamma)}(k'r)$ is a spherical Bessel function.

If one defines

$$\tilde{\rho} = \text{eigenvalue of } \rho_3 \quad (3.56)$$

one obtains the plane-wave solution of the Dirac equation from (3.54),

$$\Psi = N(\gamma)\begin{bmatrix} (m - \tilde{\rho}E)^{1/2}j_{\lambda(-\gamma)}(k'r)\chi_{-\kappa}^{\mu} \\ \tilde{\omega}(m + \tilde{\rho}E)^{1/2}j_{\lambda(\gamma)}(k'r)\chi_{\kappa}^{\mu} \end{bmatrix} . \quad (3.57)$$

Equation (3.57) agrees with the traditional solution of the Dirac equation for the plane wave.

Thus we have reached the following conclusions.

1. The Dirac Coulomb equation for both the bound state and the continuum can be solved simply in the form of Eqs. (3.43) and (3.54), respectively.

2. This solution is very similar to the corresponding solutions of the Schrödinger and Klein-Gordon equations.

3. The plane-wave solution of the Dirac equation can be obtained from the Dirac Coulomb equation as a limiting case, when $Z=0$, i.e., when the potential is switched off.

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