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Absence of $N=\infty$ phase transition in $(d=1+1)$ Hamiltonian lattice gauge theory

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The Hamiltonian formulation of two-dimensional U(*N*) lattice gauge theory is shown to have no nonanalytic behavior even in the $N=\infty$ limit, in contrast to the Lagrangian formulation with Wilson's action and to the $(d=2+1)$ Hamiltonian.

I. INTRODUCTION

In order to implement the program of lattice gauge theory it is neccessary to investigate the critical behavior of the lattice theory, since phase transitions will determine the continuum-limit theory. One of the usual assumptions is that, while there exist many different ways to put a theory on the lattice, as long as these are in the same universality class, i.e., share the same symmetries, they should exhibit similar or (in the strongest statement of universality) identical critical behavior. This is, from a particle physicist's viewpoint, mostly just a version of the common belief that the particular way one regularizes a theory should not affect the physics.

A lattice field theory, and in particular a lattice gauge theory, can be formulated by either of two generally used methods (though others also exist)—the Lagrangian formulation of Wilson[1] and the time-continuum Hamiltonian approach developed by Kogut and Susskind.[2] Both methods have their advantages and particular areas of appropriateness. There also exists a well-known procedure to go from one formulation to the other, namely, the transfer matrix. Much work has been done[3] showing that where the type of question is such that both formulations can give an answer, the answers in general agree. What we shall show, however, is that in the case of two-dimensional lattice gauge theory the results are not always the same.

II. LAGRANGIAN FORMULATION WITH WILSON'S ACTION

We are concerned with the discovery[4,5] of a phase transition in the $N=\infty$ limit of U(*N*) lattice gauge theory, formulated in the symmetric Euclidean approach, and specifically using Wilson's action. For the reasons outlined above, we want to see what happens in the continuous-time Hamiltonian formulation of this model. Partly to establish notation, we begin with a brief review of the results from the Wilson symmetric space-time formulation.

The partition function of two-dimensional U(*N*) lattice gauge theory is defined by

$$Z=\int [DU]\exp[S_W(U)]\ , \tag{1}$$

where DU is the Haar measure invariant on the group U(*N*), and Wilson's action $S_W(U)$ is defined to be

$$S_W(U)=\sum_P \frac{1}{g^2}\mathrm{Tr}(U_P+U_P^\dagger)\ , \tag{2}$$

with U_P being the oriented product of the group elements in the *P*th plaquette and the sum is over all plaquettes.

The model greatly simplifies due to gauge invariance; indeed the problem reduces to an integral over a single unitary matrix, essentially one U(*N*) spin at a point, somewhat erroneously called the one-plaquette model. An easy way to see this is to consider the $A_0=0$ gauge, i.e., on all timelike links $U_\tau=1$. Then after a change of variables one finds

$$Z=\int \prod_n [dW_n]\exp\left[\sum_n \frac{1}{g^2}\mathrm{Tr}(W_n+W_n^\dagger)\right] =z^{V/a^2}\ , \tag{3}$$

where n labels the lattice sites, V is the volume of the system, a is the lattic spacing so that V/a^2 is the number of plaquettes, and

$$z=\int [dW]\exp\left[\frac{1}{g^2}\mathrm{Tr}(W+W^\dagger)\right]\ . \tag{4}$$

In order to evaluate Eq. (4) one rewrites[6] the integral in terms of the N eigenvalues α_i of W, defined by $W=TDT^\dagger$ where $D_{ij}=\delta_{ij}e^{i\alpha_j}$ and T is unitary. One obtains

$$z=\frac{1}{N!}\int_0^{2\pi}\prod_{i=1}^{N}\frac{d\alpha_i}{2\pi}J^2(\alpha_i)\exp\left[\frac{2}{g^2}\sum_{i=1}^{N}\cos\alpha_i\right], \quad (5)$$

where $J^2(\alpha_i)$ is the Jacobian coming from the change of variables and can be written

$$J^2(\alpha_i)=\prod_{i<j}4\sin^2\left[\frac{\alpha_i-\alpha_j}{2}\right]. \quad (6)$$

The integral (5) can be expressed[5,7] as a determinant of modified Bessel functions and has been evaluated numerically[5,8] for finite N.

In the $N\rightarrow\infty$ limit one can do things analytically[4,5] and it was found that a phase transition occurs at $\lambda\equiv g^2N=2$. This phase transition is characterized by having two different analytic functions for various quantities of interest above and below $\lambda=2$. The origin of the phase transition was shown to be whether random, unitary matrices or those close to $W=I$ dominate the integration. Furthermore, following the somewhat misleading[9] Ehrenfest classification, Gross and Witten[4] called it a third-order phase transition since the derivative of the specific heat is what shows the discontinuity.

III. HAMILTONIAN FORMULATION

The standard method for obtaining the Hamiltonian of a lattice theory from the Lagrangian formulation is via the transfer matrix.[10,3] We will follow the usual approach[10] using the $A_0=0$ gauge, although the same result can be obtained using the modified procedure suggested by Wadia.[11] The sum over plaquettes in the action (2) simplifies to a sum over links when we fix $A_0=0$, and we then distinguish between spacelike and timelike links:

$$\begin{aligned}S&=\beta\sum_{n,x}\mathrm{Tr}[U_x(n)U_x^{-1}(n+\hat{\tau})+\mathrm{H.c.}-2I]\\&=\sum_{n_1,x}\left[\beta a\sum_{n_0}\mathrm{Tr}[U_x(n)U_x^{-1}(n+\hat{\tau})+\mathrm{H.c.}-2I]\right], \quad (7)\end{aligned}$$

where $\beta=1/2g^2\tau$, $n=(n_0,n_1)$ labels sites, τ denotes the lattice spacing in the time direction, and we have explicitly included a factor of twice the identity matrix for normalization. One should note that because we are in only two-dimensional space-time, the usual spacelike coupling of four spins in a plaquette is absent. This absence of an interchain coupling is what reduces the model to the trivial one-dimensional statistical mechanics system of classical spin chains with nearest-neighbor interactions. We shall see that it is just this simplification which is responsible for the absence of a phase transition.

To construct the Hamiltonian is a standard exercise, here even simpler than usual. We merely indicate a few of the steps since it will be helpful to refer to them later. The transfer matrix is obtained from (7),

$$\langle U'|T|U\rangle=\exp\left[-\beta a\sum_l \mathrm{Tr}(U_l'^{-1}U_l+\mathrm{H.c.}-2I)\right] \quad (8)$$

with $|U'\rangle=R_l|U\rangle$ and R_l a unitary rotation operator on the lth link (here only in the x direction). When we parametrize the group elements in the standard form $U=e^{is_j\Lambda_j}$, with Λ_j the group generators, $R=e^{is\cdot L_l}$ where L_l^j are conjugate momenta to s_j and differential operators in the group parameters. For the transfer matrix T one obtains

$$T=\prod_l\left\{\int\left[\prod_j ds_j\right]J(s)e^{is\cdot L_l}\exp[\beta a 2\,\mathrm{Tr}(\cos\Lambda\cdot s-I)]\right\}, \quad (9)$$

where $J(s)$ is the Jacobian for the change to the s_j variables. When we take the τ-continuum limit, $\tau\rightarrow 0$, then $\beta\rightarrow\infty$ and $\beta\tau$ remains fixed at $1/2g^2$. In this limit the integral is dominated by s_j small and slowly varying so that we expand the cosine to lowest order,

$$2\,\mathrm{Tr}(\cos\Lambda\cdot s-I)=-s^2+O(s^4). \quad (10)$$

Now the Gaussian integrations give

$$T\approx\exp(-\tau H) \quad (11)$$

with

$$H=\frac{g^2}{2a}\sum_l L_l^2=-\frac{g^2}{2a}\Delta=\frac{a^3}{2}\sum_l \vec{E}_l^{\,2}, \quad (12)$$

and L_l^2 is the quadratic Casimir operator for the group, which is also minus the group Laplace-Beltrami operator Δ, and can also be expressed as the "electric field" squared. Equation (12) is of course the $(1+1)$-dimensional version of the Kogut-Susskind Hamiltonian,[2] and could also have been easily derived starting directly from there. However, since we are interested in studying possible differences between formulations, it is important to see that the Hamiltonian is in fact derivable from the Lagrangian formulation.

What strikes one immediately about the Hamiltonian (12) is its simplicity. Restricting to the gauge-invariant sector, it is after all nothing but the Laplacian over the group which in the space of class functions can be written, up to a constant, in terms of the eigenvalues of the group as introduced above,

$$\Delta=\frac{1}{J}\sum_{i=1}^{N}\frac{\partial^2}{\partial\alpha_i{}^2}(J\ ) , \tag{13}$$

with J as defined in Eq. (6). Hence the eigenvalue problem for the Hamiltonian (12) is simply (setting $a=1$)

$$H\Psi=-\frac{g^2}{2}\frac{1}{J}\sum_{i=1}^{N}\frac{\partial^2}{\partial\alpha_i{}^2}(J\Psi)=E\Psi , \tag{14}$$

or, using $\Phi=J\Psi$,

$$-\frac{g^2}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial\alpha_i{}^2}\Phi=E\Phi . \tag{15}$$

This is, of course, a separable problem, with

$$-\frac{g^2}{2}\frac{\partial^2}{\partial\alpha^2}\phi(\alpha)=\epsilon\phi(\alpha) , \tag{16}$$

and (16) is trivially exactly soluble with plane waves. Thus there is certainly no phase transition for finite N.

We now want to consider what happens as $N\to\infty$. It certaintly looks like it would be extremely improbable for anything of a nonanalytic nature to occur even when $N=\infty$ with such a simple Hamiltonian as (15). Indeed there is no phase transition, and out of the no doubt many ways to see this, we feel it is instructive to focus on a comparison with the $(2+1)$-dimensional one-plaquette Hamiltonian model analyzed by Wadia[11] following the method of Brézin *et al.*[12]

In the ground state the wave function Ψ_0 is a symmetric function of the coordinates α_i, so that the scaled wave function

$$\Phi_0(\alpha_1,\ldots,\alpha_N)=\prod_{i<j}2\sin\left[\frac{\alpha_i-\alpha_j}{2}\right]\times\Psi_0(\alpha_1,\ldots,\alpha_N) , \tag{17}$$

is totally antisymmetric and represents a separable many-fermion problem. In fact we have a noninteracting Fermi gas where each "particle," and this is the crucial point, is *free*. This is in contrast to the case studied by Wadia[11] where the noninteracting fermions were each in a central potential since there the single-particle Hamiltonian was

$$H_W=-2g^2\frac{\partial}{\partial\alpha^2}+\frac{2}{g^2}(1-\cos\alpha) . \tag{18}$$

Therefore in that case there was a Fermi level which could be greater or less than the potential barrier, and this is precisely what gave rise to having two distinct analytic functions in the $N=\infty$ limit and so a phase transition. In our case there is no potential, hence the system is always in the strong-coupling phase and there is no phase transition. The ground-state energy E_0 is given exactly for all N by

$$N^2E_0=\frac{g^2}{2}\int_{-N/2}^{N/2}dp\,p^2 ; \tag{19}$$

so

$$E_0=\frac{g^2N}{24} . \tag{20}$$

If one prefers, one can of course consider this value for E_0 [Eq. (20)] as the constant mentioned in connection with Eq. (13) and use it to shift E_0 to zero.

IV. CONCLUDING REMARKS

We have seen that the same model defined on the one hand by Wilson's Lagrangian method and using his form of the action shows a so-called third-order phase transition in the $N=\infty$ limit in two space dimensions, while on the other hand the model as formulated in the Kogut-Susskind Hamiltonian framework in one-latticized-space—one-continuous-time dimension shows no such phase transition. At first this may seem alarming since, as noted above, we expect the same physics regardless of regularization. However, is this phase transition physics? We remind the reader that the $N=\infty$ phase transition, as shown by Gross and Witten,[4] is not associated with a zero in the β function nor, therefore, with an infinite correlation length. The usual arguments about universality (which anyway sometimes appears to be violated in two dimensions[3]) are not so easily extended to this situation. So perhaps one should not worry too much about this discrepancy from the point of view of universality, although it is still somewhat disturbing. Nevertheless, it does imply that this $N=\infty$ phase transition is only an artifact of the particular lattice method used. Furthermore, one may make the argument that since the Hamiltonian version is already halfway to the full continuum limit and moreover is in some sense closer to the quantum mechanics of the ground state, the features it shows about a theory are perhaps more to be trusted.

It is also relevant to note that, since our work was completed, a couple of related articles[13,14] have come to our attention showing that even within the context of Wilson's Lagrangian formulation, if one uses a different form for the action than his, namely Manton's action[13] or a generalized Villain action based on the "heat kernel,"[14] there is also no $N=\infty$ phase transition. [The authors of Ref. 14 are also aware that the $(1+1)$-dimensional Hamiltonian is singularity free.] These results are not surprising in light of our investigation since both these forms of the action are closely related to the Hamiltonian formulation in the following sense. In deriving the Hamiltonian from the transfer matrix, we saw that one expands the cosine term of Wilson's action and keeps only the quadratic term (10). It is this procedure that essentially avoids the interactions which give rise to the nonanalytic behavior seen using Wilson's action. This, however, is not an approximation as far as the Hamiltonian is concerned since throwing away these terms is dictated by the τ-continuum limit. In the case of Manton's or the generalized Villain action there are no terms to neglect, for if one writes the action in terms of the eigenvalues α_i as in (5), then instead of the $\cos\alpha_i$ of Wilson's action, Manton's action has only α_i^2 and the generalized Villain action has $(\alpha_i+2\pi n_i)^2$. So both are already the Gaussian "approximation" of the Wilson action. But, to be sure, they are no more actual approximations than the Hamiltonian is. All are perfectly acceptable[15] ways to put the theory on a lattice and all have the same naive continuum limit. There is no particular reason to prefer the Wilson action over the other actions or over the Hamiltonian. Indeed, as we (above) and various others including the authors of Refs. 13 and 14 have discussed, there exist some reasons for tending to favor the non-Wilson-type formulations. In this regard it is worth noting that continuum QCD_2 has no $N=\infty$ phase transition.[16] Thus it gives one ground for caution in accepting the conjecture[4] that this $N=\infty$ phase transition found in one version of two-dimensional lattice gauge theory but not in any of the others so far considered, including the Hamiltonian, has something to do with the abrupt change seen in, for example, the string tension in four-dimensional SU(2) or SU(3) lattice gauge theory. Nevertheless, as discussed elsewhere,[9] the situation once one leaves the simplicity of the two-dimensional one-plaquette world becomes much less clear and there is evidence on both sides.

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