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Gauge-invariant variational methods for Hamiltonian lattice gauge theories

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This paper develops variational methods for calculating the ground-state and excited-state spectrum of Hamiltonian lattice gauge theories defined in the $A_0=0$ gauge. The scheme introduced in this paper has the advantage of allowing one to convert more familiar tools such as mean-field, Hartree-Fock, and real-space renormalization-group approximations, which are by their very nature gauge-noninvariant methods, into fully gauge-invariant techniques. We show that these methods apply in the same way to both Abelian and non-Abelian theories, and that they are at least powerful enough to describe correctly the physics of periodic quantum electrodynamics (PQED) in $2+1$ and $3+1$ space-time dimensions. This paper formulates the problem for both Abelian and non-Abelian theories and shows how to reduce the Rayleigh-Ritz problem to that of computing the partition function of a classical spin system. We discuss the evaluation of the effective spin problem which one derives for PQED and then discuss ways of carrying out the evaluation of the partition function for the system equivalent to a non-Abelian theory. The explicit form of the effective partition function for the non-Abelian theory is derived, but because the evaluation of this function is considerably more complicated than the one derived in the Abelian theory no explicit evaluation of this function is presented. However, by comparing the gauge-projected Hartree-Fock wave function for PQED with that of the pure SU(2) gauge theory, we are able to show that extremely interesting differences emerge between these theories even at this simple level. We close with a discussion of fermions and a discussion of how one can extend these ideas to allow the computation of the glueball and hadron spectrum.

I. INTRODUCTION

Quantum chromodynamics may well be the only satisfactory candidate we have for a theory of the strong interactions; nevertheless, no satisfactory treatment of the most basic aspects of the theory, e.g., confinement, the glueball spectrum, the hadron spectrum, etc., has been given to date. Attempts to analyze the theory from the point of view of continuum perturbation theory, even including instanton effects, fail to clarify the physics of confinement. Lattice calculations, which make the physics of confinement clear at strong coupling, founder when one attempts to extract the physics of the weak-coupling regime.[1] Nonperturbative methods, such as real-space renormalization-group techniques, have not been able to deal successfully with the requirement that successive truncation steps must keep one within the set of gauge-invariant states. In this paper we present a formalism for carrying out gauge-invariant variational calculations for the ground state of any lattice gauge theory defined in the $A_0=0$ gauge, which holds out the promise of improving upon this situation. The virtues of this formalism are as follows:

1. It provides a general way of converting any variational scheme to a gauge-invariant one without losing the ability to compute.
2. It shares with perturbation theory the virtue of being directly applicable to the weak-coupling regime (this is the regime of physical interest if one wants to make a correspondence to the continuum).
3. It can be demonstrated, at least for the case of $Z(2)$ gauge theories[2] and periodic lattice quantum electrodynamics (PQED), that application of this method to improve either the mean-field or Hartree-Fock analysis of these models allows one to obtain new and better results which cannot be

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obtained using these methods alone.

4. The method is readily generalized from the case of Abelian theories to that of non-Abelian theories.

5. The method in principle provides a way of carrying out a nonperturbative computation for the glueball spectrum in a pure gauge theory, and the hadron spectrum in the case of theories with fermions.

6. It is possible that the method can be extended to provide implementable nonperturbative computational schemes for continuum gauge theories.

We begin by presenting a general formalism for dealing with gauge theories based upon continuous gauge groups. We will then show how to combine this formalism with a variational technique, such as the Hartree-Fock approximation, to obtain results which are not obtainable from the variational method alone. To demonstrate how this works we discuss the physics of PQED in $2+1$ and $3+1$ dimensions and show how one establishes that PQED in $2+1$ dimensions exhibits confinement for all nonzero values of the coupling constant. We conclude with a discussion of the extension of the method to non-Abelian gauge theories. A description of the way in which this material is divided among the different sections of this paper follows.

In order to orient the reader unfamiliar with lattice gauge theories Sec. II begins with a discussion of the general problem and then explains the idea behind our gauge-invariant variational scheme. In Sec. III we turn to a discussion of the physics of lattice PQED. Here we establish notation, discuss the way in which one implements our technique within the context of a general Hartree-Fock variational scheme, and explain how one goes about arguing whether the theory confines or does not confine. The main result of this section is the reduction of the problem of computing a variational estimate of the ground-state energy to the computation of the partition function for a d-dimensional statistical-mechanical system. Section IV is less general and specifically addresses the question of how one carries out the evaluation of such partition functions. There are two reasons for including this discussion. First, and most important, we wish to compare the results of the general Hartree-Fock analysis with those obtained from the simpler mean-field approach. This comparison will show that the mean-field analysis can be systematically improved to allow a straightforward computation of the string tension. Although this technique of analysis is not really required for dealing with the Abelian theory, it holds out the promise of simplifying calculations for the non-Abelian system considerably. Second, from the pedagogical point of view, we wish to show that this sort of problem is, to a large degree, amenable to analytic methods of analysis. The arguments presented in this section make use of results established in earlier work by Drell *et al.*[3] and so the discussion will not be self-contained; however, the simple example of the physics of a single plaquette will be explained in detail. Since this example contains most of the features of the more complicated problem, knowledge of the previous work will not be required in order to understand the bulk of the discussion. Section V explains the extension of this method to the case of non-Abelian gauge theories. The Hamiltonian formulation of the theory is presented in detail and then the generalization of the mean-field and Hartree-Fock approximations to the specific case of an SU(2) gauge theory are explained. The correspondence between the discussion of the Abelian and non-Abelian theories is made explicit, and the formalism for carrying out a computation of the string tension for a non-Abelian theory is set forth. In the concluding section of the paper we summarize the results obtained to date and discuss the directions in which this idea can be developed.

II. THE PROBLEM OF GAUGE-INVARIANT STATES

The rules for formulating locally gauge-invariant Hamiltonian theories in $d+1$ dimensions are chosen to coincide with those which one would obtain by transcribing a continuum Abelian gauge theory canonically quantized in the $A_0=0$ gauge to a lattice. Such a theory has gauge generators, $G(\vec{i})$, associated with every site "$\vec{i}$" of the d-dimensional spatial lattice. These generators commute with the Hamiltonian H and, by assumption, with all physical observables. Their existence reflects the fact that setting $A_0=0$ restricts the degrees of freedom sufficiently to make canonical quantization possible, but the theory remains invariant with respect to arbitrary time-independent gauge transformations.

Since all gauge transformations commute with both the Hamiltonian and all physical observables, the Hilbert space of the theory divides into an infinite of number noncommunicating sectors, defined by the condition that the states of any one

sector span an irreducible representation of the gauge group. The fact that these sectors do not get mapped into one another by any physical observable is what is meant by the statement that gauge invariance is a superselection rule. To understand the meaning of the different sectors for the case of the Abelian theory one notes that quantizing the theory in the $A_0=0$ gauge yields all but one of the Maxwell equations as Heisenberg equatins of motion. The missing equation relating the divergence of the electric field, $\vec{\nabla}\cdot\vec{E}(\vec{i})$, to the matter charge density $\rho(\vec{i})$ is not a Heisenberg equation of motion, and in fact does not hold for all states in the Hilbert space. It is true, however, that the operator $G(\vec{i})=\vec{\nabla}\cdot\vec{E}(\vec{i})-\rho(\vec{i})$ commutes with the Hamiltonian. The $G(\vec{i})$'s are nothing but the generators of local time-independent gauge transformations and thus they and the Hamiltonian can be simultaneously diagonalized. The presence of nonzero eigenvalues for a $G(\vec{i})$ measures the existence of a classical background charge distribution. Thus, the usual quantization of QED in this gauge requires that we restrict attention to the sector of gauge-invariant states, i.e., those states annihilated by the generators $G(\vec{i})$. For the Yang-Mills theory, as for the Abelian theory, it is the sector of gauge-invariant states for which the source-free Yang-Mills equations hold.

The necessity of searching for the vacuum of the theory in the sector of gauge-invariant states greatly complicates the task of discussing the weak-coupling limit of a gauge theory. In the $g\to 0$ limit the usual perturbative expansion is in terms of free fields, and eigenstates of the free-field Hamiltonian are not gauge invariant; hence keeping track of effects due to gauge invariance is at best cumbersome. There is a way to modify any perturbative or variational scheme so as to enable us to calculate with gauge-invariant states alone; namely, one need only multiply any approximation to the ground state of the theory by the operator which projects it onto its locally gauge-invariant part. In general, an arbitrary state will have a nonvanishing projection onto a gauge-invariant state. Furthermore, as one varies over parameters defining the state, its projection will vary over some submanifold of the family of gauge-invariant states. Hence, one need not restrict attention to gauge-invariant trial wave functions, if one extremizes the ratio

$$\mathcal{E}=\frac{\langle\psi|HP|\psi\rangle}{\langle\psi|P|\psi\rangle} \tag{2.1}$$

instead of $\langle\psi|H|\psi\rangle$, where P is the projection operator onto the sector of gauge-invariant states and we have used the fact that $PHP=HP^2$ and $p^2=P$.

Precisely this technique was applied by Boyanovsky *et al.*[2] to the mean-field analysis of $Z(2)$ gauge theories in $2+1$ and $3+1$ dimensions. These authors showed that whereas mean-field theory incorrectly predicts that $Z(2)$ theory exhibits a first-order transition in both $2+1$ and $3+1$ dimensions, the gauge-invariant mean-field calculation correctly predicts a second-order transition for the theory in $2+1$ dimensions and a first-order transition in $3+1$ dimensions. Modifying the variational calculation to include restriction to gauge-invariant states produces a qualitative improvement in the results obtained even by mean-field theory.

In the remaining sections of this paper we will show that this same method readily generalizes to the case of gauge theories based upon continuous symmetry groups, and that one can develop a computational scheme for carrying out variational calculations based upon a much more general class of trial wave functions than that provided by mean-field theory. The idea of projecting a state onto its gauge-invariant part is an obvious one; what is surprising is that for a wide class of variational wave functions it leads to computations which can be carried out.

III. PQED: GENERAL FORMALISM

A. Notation

Formulations of lattice gauge theories distinguish between gauge fields and matter fields in that gauge fields are associated with links of the lattice and matter fields with vertices. The Hamiltonian consists of two sorts of terms: the first, proportional to a sum over links of the squares of gauge generators, and the second, a sum over plaquettes of exponentials in the "magnetic field" variables. Before beginning our discussion of the specific case of PQED we must establish a general notation for labeling of vertices, links, and plaquettes, and variables associated with these fundamental objects.

In general we will be discussing Hamiltonian lattice gauge theories in $d+1$ dimensions, where $d=2$ or 3. Vertices of the lattice will be labeled by d-tuples of integers, $\vec{i}=(i_1,\ldots,i_d)$. Since each link joins two adjacent points we will label any link $\mathcal{L}$ by an integer $\alpha=1,\ldots,d$ and a lattice point $\vec{i}$,

e.g., $\mathscr{L}\equiv(\vec{i},\alpha)$. This notation means that $\mathscr{L}$ is the link joining the points $\vec{i}$ and $\vec{i}+\hat{n}_\alpha$, where $\hat{n}_\alpha$ stands for the unit vector associated with the d-tuple of integers which has a 1 in the αth place and zeros everywhere else. A variable associated with a link will be denoted as $\theta(\vec{i},\alpha)$ or $\theta_{\mathscr{L}}$, whenever no confusion can arise.

In order to label variables associated with plaquettes we must adopt a convention for naming such objects. Since plaquette terms in the Hamiltonian involve forming ordered products of variables associated with the links bounding a plaquette, one also needs to specify an orientation for each plaquette. One way to denote a plaquette, together with its orientation, is the by symbol $\square\equiv(\vec{i},\alpha,\beta)$, where this symbol means the plaquette obtained by traversing the links $\mathscr{L}^1\equiv(\vec{i},\alpha)$, $\mathscr{L}^2\equiv(\vec{i}+\hat{n}^\alpha,\beta)$, $\mathscr{L}^3\equiv(\vec{i}+\hat{n}_\beta,\alpha)$, and $\mathscr{L}^4\equiv(\vec{i},\beta)$, in the order in which they are given. For the case $d=2$ there is only one positively oriented plaquette associated with each point $\vec{i}$, and so we can simplify our notation and denote every plaquette $(\vec{i},\alpha,\beta)$ by the symbol $\square(\vec{i})$ or just $\square$ when no confusion can arise. In the case $d=3$, there are three positively oriented plaquettes associated with each point $\vec{i}$. To simplify notation we will adopt the usual right-hand rule and subscript each plaquette so that each component of the vector $\square(\vec{i})$ stands for the plaquette whose normal points in the direction of the unit vector $\hat{n}_\alpha$.

B. General formulation of the problem

Periodic (or compact) QED (i.e., PQED) (Ref. 4) is a U(1) lattice gauge theory defined by the Hamiltonian

$$H=\frac{g^2}{2}\sum E_{\mathscr{L}}{}^2+\frac{1}{g^2}\sum[1-\cos(B_\square)]\,, \tag{3.1}$$

where $E_{\mathscr{L}}$ is the electric field operator for link $\mathscr{L}$ and $B(\square)$ is the magnetic field operator associated with the plaquette $\square$. An explicit realization of these operators is given by defining the Hilbert space of the theory to be the set of periodic square-integrable functions of angle variables $\phi_{\mathscr{L}}$, where we assume there is one angle variable $\phi_{\mathscr{L}}$ associated with each link of the lattice. The operators $E_{\mathscr{L}}$ and $B_\square$ are defined as

$$E_{\mathscr{L}}=-i\frac{\partial}{\partial\phi_{\mathscr{L}}} \tag{3.2}$$

and

$$B_\square=(\nabla\times\phi)_\square\,, \tag{3.3}$$

where $(\nabla\times\phi)_\square$ stands for the lattice curl of the variables $\phi_{\mathscr{L}}$ associated with the plaquette $\square$, eg.,

$$(\nabla\times\phi)_\square=\phi_{\mathscr{L}^1}+\phi_{\mathscr{L}^2}-\phi_{\mathscr{L}^3}-\phi_{\mathscr{L}^4}\,, \tag{3.4}$$

where $\mathscr{L}^1$ through $\mathscr{L}^4$ are the links associated with $\square$. The U(1) gauge generators of the theory are defined by

$$G(\vec{i})=\vec{\nabla}\cdot\vec{E}(\vec{i})\,, \tag{3.5}$$

the lattice divergence of the link operators being defined in the usual way,

$$\vec{\nabla}\cdot\vec{E}(\vec{i})=\sum_{\mathscr{L}\in\mathscr{L}^+}E_{\mathscr{L}}-\sum_{\mathscr{L}\in\mathscr{L}^-}E_{\mathscr{L}}\,, \tag{3.6}$$

where $\mathscr{L}^+$ stands for the set of links $\mathscr{L}^+=\{(\vec{i},\alpha)\}$ and $\mathscr{L}^-$ stands for the set of links $\mathscr{L}^-=\{(\vec{i}-\hat{n}\alpha,\alpha)\}$.

A general state in the Hilbert space of this system can be expanded in terms of the eigenstates of the electric field operators as follows:

$$|\Psi\rangle=\sum\left[\exp\left[i\sum m_{\mathscr{L}}\phi_{\mathscr{L}}\right]\psi(\ldots,m_{\mathscr{L}},\ldots)\right]\,, \tag{3.7}$$

where the variables $m_{\mathscr{L}}$ are integers associated with the link $\mathscr{L}$.

C. A simple class of variational wave functions

For simplicity we focus on a family of variational wave functions which allow us to carry out all computations analytically. This permits us to establish contact with earlier work on the subject of Hamiltonian PQED (Ref. 3) and to discuss the physics of confinement in a way which readily generalizes the non-Abelian theories. The class of wave functions we will consider is the set of periodic Gaussians in the link variables $\phi_{\mathscr{L}}$. The most general wave function of this class has the form

$$|\Psi\rangle=\sum\left[\exp\left[i\sum m_{\mathscr{L}}\phi_{\mathscr{L}}\right]\Gamma\left[\frac{1}{\sqrt{2}}m_{\mathscr{L}}\right]\right]\,, \tag{3.8}$$

where we have defined $\Gamma(m_{\mathscr{L}})$ to be the generalized Gaussian function of the link variables,

$$\Gamma(m_{\mathscr{L}})=\exp\left[-\sum m_{\mathscr{L}}\Delta(\mathscr{L},\mathscr{L}')m_{\mathscr{L}'}\right]\,. \tag{3.9}$$

In general the values of the function $\Delta(\mathscr{L},\mathscr{L}')$, can be taken as variational parameters. One form of this function which will be of interest to us in subsequent discussions is the single-site mean-field

form of the wave function

$$\Delta_{\rm mf}(\mathscr{L},\mathscr{L}')=\delta_{\alpha\beta}\delta_{\vec{i}\,\vec{m}}\frac{1}{\gamma}\ , \tag{3.10}$$

where we have taken $\mathscr{L}$ and $\mathscr{L}'$ to be $(\vec{i},\alpha)$ and $(\vec{m},\beta)$, respectively. Equation (3.10) can be viewed as the diagonal part of a more general function which does not vanish for $\vec{i}\neq\vec{m}$. Such a function can be generally represented by its Fourier transform as

$$\Delta_{\rm mom}(\mathscr{L},\mathscr{L}')=\frac{\delta_{\alpha\beta}}{V}\sum\exp[i\vec{k}\cdot(\vec{i}-\vec{m})]\frac{1}{c_{\vec{k}}}\ , \tag{3.11}$$

where the $c_{\vec{k}}$ are variational parameters and the links $\mathscr{L}$ and $\mathscr{L}'$ are taken to be $(\vec{i},\alpha)$ and $(\vec{m},\beta)$, respectively. Equation (3.10) corresponds to the extreme case $c_{\vec{k}}=$const. In general, it is natural to assume that the $c_{\vec{k}}$'s are functions of the "frequencies"

$$\Omega_{\vec{k}}=\left[d-2\sum\cos(k_\alpha)\right]^{1/2} \tag{3.12}$$

which are eigenvalues of the operator $[-\nabla^2]^{1/2}$. When we apply this approximation to the case of PQED we will find that in the weak-coupling limit the variables $c_{\vec{k}}\to\Omega_{\vec{k}}$, whereas in the strong-coupling limit $c_{\vec{k}}\to$const i.e., the mean-field form, becomes a good approximation. A combination of these two extreme sorts of behavior is necessary to accurately describe the interpolation between these two regimes.

D. Computing gauge-invariant expectation values

As noted, the wave functions given in (3.8) are not the wave functions of gauge-invariant states. Hence, we must project them onto their gauge-invariant component. This is easily accomplished by operating on them with the projection operator

$$P_g=\prod P_g(\vec{i})\ ,$$

where

$$P_g(\vec{i})=\left[\frac{1}{2\pi}\int d\alpha(\vec{i})\exp[i\alpha(\vec{i})\cdot G(\vec{i})]\right]. \tag{3.13}$$

If we define $Z=\langle\psi\,|\,P_g\,|\,\psi\rangle$, then for the class of wave functions defined by (3.8) through (3.11) we have

$$Z=\prod\left[\frac{1}{2\pi}\int d\alpha(\vec{i})\right]\prod\left[\int d\phi_{\mathscr{L}}\right]\psi(\ldots\phi_{\mathscr{L}},\ldots)^*\psi(\ldots,\phi_{\mathscr{L}}+\vec{\nabla}_{\mathscr{L}}\vec{\alpha},\ldots)\ , \tag{3.14}$$

where $\vec{\nabla}_{\mathscr{L}}\vec{\alpha}$ is the difference between the α parameters on the two ends of the link $\mathscr{L}=(\vec{i},\alpha)$. There are options open to us at this stage. One is to explicitly carry out the ϕ integrations and do the $m_{\mathscr{L}}$ sums. This gives Z and the expectation value of HP_g in terms of the partition function of a classical system whose degrees of freedom are given by the α parameters which define the gauge transformations. It is not useful for us to pursue this point in this paper but in other cases there could well be an advantage, for certain ranges of parameters, to define an effective action as

$$L(\alpha(\vec{i}))=\ln\left[\prod\left[\int d\phi_{\mathscr{L}}\right]\psi(\ldots,\phi_{\mathscr{L}},\ldots)^*\psi(\ldots,\phi_{\mathscr{L}}+\vec{\nabla}_{\mathscr{L}}\vec{\alpha},\ldots)\right] \tag{3.15}$$

and then try to evaluate the integral, (3.15), by a stationary-phase approximation. For now it will be more convenient to carry out both the ϕ and α integrations, leaving us with the problem of evaluating a set of discrete sums over the integer variables $m_{\mathscr{L}}$.

Carrying out the ϕ integrations for the wave functions defined by (3.11) we obtain

$$Z=\prod\left[\frac{1}{2\pi}\int d\alpha(\vec{i})\right]\sum\left\{\exp\left[i\sum(m_{\mathscr{L}}+\nabla_{\mathscr{L}}\alpha)-\sum m_{\mathscr{L}}\Delta(\mathscr{L},\mathscr{L}')m_{\mathscr{L}'}\right]\right\}. \tag{3.16}$$

By explicitly carrying out the α integrations one obtains constraints on the configurations of $\{m_{\mathscr{L}}\}$ which make nonvanishing contributions to (3.16); i.e., (3.16) becomes

$$Z=\sum\left[\Gamma(m_{\mathscr{L}})\prod\delta(\vec{\nabla}\cdot\vec{m}(\vec{i}))\right], \tag{3.17}$$

where $\Gamma(m_{\mathscr{L}})$ was defined in Eq. (3.9). The notation $\delta(\vec{\nabla}\cdot\vec{m}(\vec{i}))$ means, as before, that we only include configurations of the variables $m_{\mathscr{L}}$ for which the lattice divergence

$$\vec{\nabla}\cdot\vec{m}(\vec{i})=\sum_{\mathscr{L}\in\mathscr{L}^+}m_{\mathscr{L}}-\sum_{\mathscr{L}\in\mathscr{L}^-}m_{\mathscr{L}}$$

vanishes. In the case of the (2+1)-dimensional theory this condition can be removed by introducing a set of integer-valued plaquette variables $L_\Box$ and defining

$$m_{\mathscr{L}}=(\vec{\nabla}_\Box\times\vec{L})_{\mathscr{L}}\ , \tag{3.18}$$

where the plaquette curl of the L variables is defined to be

$$m_{\mathscr{L}}=L_\Box(\vec{i})-L_\Box(\vec{i}-\hat{n}_1)\ \text{for}\ \mathscr{L}=(\vec{i},2)$$

and

$$m_{\mathscr{L}}=L_\Box(\vec{i})-L_\Box(\vec{i}-\hat{n}_2)\ \ \text{for}\ \mathscr{L}=(\vec{i},1)\ . \tag{3.19}$$

The sum over the $L_\Box$ variables being unconstrained,[5] we find Z is given by the expression

$$Z=\sum\Gamma[(\vec{\nabla}_\Box\times\vec{L})_{\mathscr{L}}] \tag{3.20}$$

and the expectation value $\langle\psi\,|\,HP_g\,|\,\psi\rangle$ is given by

$$\mathscr{E}=\frac{1}{Z}\left[\frac{g^2}{2}\sum\langle E_{\mathscr{L}}{}^2\rangle-\frac{1}{g^2}\sum\langle\cos(B_\Box)\rangle\right]+\frac{1}{g^2}(\text{volume})\ , \tag{3.21}$$

where

$$\langle E_{\mathscr{L}}{}^2\rangle=\sum\left[(\vec{\nabla}_\Box\times\vec{L})_{\mathscr{L}}{}^2\Gamma[(\vec{\nabla}_\Box\times\vec{L})]\right] \tag{3.22}$$

and

$$\langle\cos(B_\Box)\rangle=\sum\Gamma\{[\vec{\nabla}_\Box\times(\vec{L}-\vec{S})]_{\mathscr{L}}\}\Gamma[(\vec{\nabla}_\Box\times\vec{S})_{\mathscr{L}}]\ , \tag{3.23}$$

where the shift function $\underline{S}_\Box$ is defined to be

$$\vec{S}_{\Box(\vec{i})}=\tfrac{1}{2}\delta_{\Box(\vec{i}),\Box(\vec{m})}\ . \tag{3.24}$$

The shift function appears in the formula for the expectation value of $\cos(B_\Box)$ because this operator is

$$\tfrac{1}{2}[\exp(i\vec{\nabla}_{\mathscr{L}}\times\vec{\phi})+\exp(-i\vec{\nabla}_{\mathscr{L}}\times\vec{\phi})]$$

and the operator $\exp(\mp i\phi_{\mathscr{L}})$ acting on a state of the form specified in (3.5) changes the factor $\exp(im_{\mathscr{L}}\phi_{\mathscr{L}})$ to $\exp[i(m_{\mathscr{L}}\mp1)\phi_{\mathscr{L}}]$; or, in other words, the exponential of the curl of the ϕ's about a given plaquette shifts the L value associated with that plaquette by one unit.

E. Introducing sources

The preceding section discussed the general formulation of the problem of projecting a Hartree-Fock wave function onto its gauge-invariant part. In order to discuss the question of confinement we need to be able to introduce static sources into the problem. This means that if we wish to discuss the force between two charges of opposite sign in PQED we must deal with states for which $\vec{\nabla}\cdot\vec{E}$ is either ∓1 at two points and zero otherwise. One obtains such states by operating on a general Hartree-Fock wave function with the projection operator

$$P_{\text{sources}}(\vec{p}_1,\vec{p}_2)=\left[\prod_{\vec{i}\neq\vec{p}_1,\vec{p}_2}P_g(\vec{i})\right]P_{g+}(\vec{p}_1)P_{g-}(\vec{p}_2)\ , \tag{3.25}$$

where

$$P_{g\mp}\equiv\frac{1}{2\pi}\int\exp\{i\alpha(\vec{i})\cdot[G(\vec{i})\pm1]\}\ . \tag{3.26}$$

In principle, since the system with sources is no longer translationally invariant, one must redo the variational calculation for the parameters $\Delta(\mathscr{L},\mathscr{L}')$ for each of these sectors of the gauge theory. This is, however, unnecessary whenever there are only a finite number of sources. The reason for this is that no finite number of sources can modify the part of the ground-state energy which diverges like the volume. Since the parameters $\Delta(\mathscr{L},\mathscr{L}')$ are determined by extremizing the ground-state energy density the changes in them due to any finite number of sources must vanish like 1/(volume) in the limit (volume)$\to\infty$. Following the same procedure as one followed in treating the source free case, one can show that the normalization factor $Z_{\text{sources}}=\langle\psi\,|\,P_{\text{sources}}\,|\,\psi\rangle$ and the energy $\mathscr{E}_{\text{sources}}=\langle\psi\,|\,HP_{\text{sources}}\,|\,\psi\rangle$ are given by expressions identical to those for Z and $\mathscr{E}$ except that the condition that the divergence of the integers $\vec{m}(\vec{i})$ vanish at every vertex, must be replaced by the statement that it vanishes at every vertex except $\vec{p}_1$ and $\vec{p}_2$ where it must be $+1$ and -1, respectively. It follows from this that evaluating $\mathscr{E}_{\text{sources}}$ provides an upper bound on the ground-state energy of the sector with two oppositely charged sources in the same way that $\mathscr{E}$ provides an upper bound on the ground-state energy of the source-free sector of the theory. Within the spirit of the approximation one can determine whether or not PQED exhibits linear confinement for a given value of the coupling constant by taking the difference, $\mathscr{E}_{\text{sources}}-\mathscr{E}$, and seeing if it grows linearly with separation or goes to a constant. Of course, since our energies are derived from variational calculations, finding a non vanish-

ing string tension by this method is no proof that the theory truly exhibits such behavior. Nevertheless, experience has shown that when results obtained in this way can be compared to exact results the variational calculation seems to consistently give good answers.

This concludes our discussion of the general formalism for carrying out gauge-invariant Hartree-Fock computations for Abelian gauge theories. The main result of this discussion is that the variational computation for the ground state in the presence or absence of static sources can be recast as a d-dimensional statistical-mechanics problem. In the next section we turn to the practical problem of evaluating partition functions of this sort, and extracting information about the physics of the original quantum system.

IV. EVALUATING PARTITION FUNCTIONS

A. The theory of a single plaquette

1. *Basic formulas*

In order to work out in detail an instructive example of the methods presented in the preceding section, we will restrict attention to a world made up of exactly four links arranged to form a single square plaquette. The four vertices of this simple lattice will be labeled $\vec{1}=(0,0)$, $\vec{2}=(1,0)$, $\vec{3}=(1,1)$, and $\vec{4}=(0,1)$. Associated with each of the four links of the lattice, $\mathscr{L}_1\equiv(\vec{1},1)$, $\mathscr{L}_2\equiv(\vec{2},2)$, $\mathscr{L}_3\equiv(\vec{4},1)$, and $\mathscr{L}_4\equiv(\vec{1},2)$, are angular variables $\theta_{\mathscr{L}1}$, $\theta_{\mathscr{L}2}$, and $\theta_{\mathscr{L}4}$ and their associated electric field variables, $E_{\mathscr{L}}=-i\partial/\partial\theta_{\mathscr{L}}$.

The Hamiltonian of the single-plaquette universe is

$$H=\frac{g^2}{2}(E^2_{\mathscr{L}1}+E^2_{\mathscr{L}2}+E^2_{\mathscr{L}3}+E^2_{\mathscr{L}4}) -\frac{1}{g^2}\cos(B_\square)\,, \tag{4.1}$$

where the magnetic field operator is defined to be

$$B_\square=\theta_{\mathscr{L}1}+\theta_{\mathscr{L}2}-\theta_{\mathscr{L}3}-\theta_{\mathscr{L}4} \tag{4.2}$$

and where a constant $1/g^2$ has been dropped from the Hamiltonian. A complete set of states for the Hilbert space of the theory is given by the set of all functions

$$|m_{\mathscr{L}1},m_{\mathscr{L}2},m_{\mathscr{L}3},m_{\mathscr{L}4}\rangle=\exp[i(m_{\mathscr{L}1}\theta_{\mathscr{L}1}+m_{\mathscr{L}2}\theta_{\mathscr{L}2}+m_{\mathscr{L}3}\theta_{\mathscr{L}3}+m_{\mathscr{L}4}\theta_{\mathscr{L}4})]\,, \tag{4.3}$$

where the variables $m_{\mathscr{L}}$ run over the positive and negative integers.

The four gauge generators of the single-plaquette system are

$$G(\vec{1})=E_{\mathscr{L}1}+E_{\mathscr{L}4}\,,\quad G(\vec{2})=E_{\mathscr{L}2}-E_{\mathscr{L}1}\,,\quad G(\vec{3})=-(E_{\mathscr{L}2}+E_{\mathscr{L}3})\,,\quad \text{and } G(\vec{4})=E_{\mathscr{L}3}-E_{\mathscr{L}4}\,. \tag{4.4}$$

In terms of these generators, the most general position-dependent gauge transformation is given by the operator

$$U(\alpha(\vec{1}),\alpha(\vec{2}),\alpha(\vec{3}),\alpha(\vec{4}))=\exp\left[i\sum_{\vec{i}=\vec{1}}^{\vec{4}}\alpha(\vec{i})\cdot G(\vec{i})\right]\,. \tag{4.5}$$

It follows from the canonical commutation relations that application of U to a state $\psi(\theta_{\mathscr{L}1},\theta_{\mathscr{L}2},\theta_{\mathscr{L}3},\theta_{\mathscr{L}4})$ yields the state $(U\psi)$, where

$$(U\psi)(\ldots,\theta_{\mathscr{L}},\ldots)=\psi[\ldots,\theta_{\mathscr{L}}+\alpha(\vec{j}+\hat{n}_\beta)-\alpha(\vec{j}),\ldots]\,, \tag{4.6}$$

where we have assumed $\mathscr{L}=(\vec{j},\beta)$.

The most general state of the Hilbert space is of the form

$$|\psi\rangle=\sum f(\ldots,m_{\mathscr{L}},\ldots)|\ldots,m_{\mathscr{L}},\ldots\rangle\,. \tag{4.7}$$

The projection of this state onto its gauge-invariant part is accomplished by operating upon it with the projection operator $P_g=P_g(\vec{1})P_g(\vec{2})P_g(\vec{3})\,P_g(\vec{4})$; hence

$$P_g\,|\psi\rangle=\prod\left[\frac{1}{2\pi}\int d\alpha(\vec{j})\right]\sum(f(\ldots,m_{\mathscr{L}},\ldots)\exp[im_{\mathscr{L}}(\theta_{\mathscr{L}}+\nabla_{\mathscr{L}}\alpha)])\,. \tag{4.8}$$

The partition function is defined to be the norm of $Pg\,|\psi\rangle$ and is a function of "f". Taking the norm of

(4.8), recalling that $P_g{}^2=P_g$ we obtain

$$Z(f)\equiv\langle\psi\,|\,P_g\,|\,\psi\rangle=\prod\left[\frac{1}{2\pi}\int d\alpha(\vec{j})\right]\sum\left\{f(\ldots,m_{\mathscr{L}},\ldots)^*f(\ldots,m_{\mathscr{L}},\ldots)\exp\left[i\sum\alpha(\vec{j})\vec{\nabla}\cdot\vec{m}(\vec{j})\right]\right\}, \quad (4.9)$$

where the lattice divergence of the m's is

$$\vec{\nabla}\cdot\vec{m}(\vec{1})=m_{\mathscr{L}^1}+m_{\mathscr{L}^4}\,,$$
$$\vec{\nabla}\cdot\vec{m}(\vec{2})=m_{\mathscr{L}^2}-m_{\mathscr{L}^1}\,,$$
$$\vec{\nabla}\cdot\vec{m}(\vec{3})=-(m_{\mathscr{L}^2}+m_{\mathscr{L}^3})\,, \quad (4.10)$$

and

$$\vec{\nabla}\cdot\vec{m}(\vec{4})=m_{\mathscr{L}^3}-m_{\mathscr{L}^4}\,.$$

Carrying out the $\alpha(\vec{j})$ integrations, leads to the relations

$$m_{\mathscr{L}^1}=m_{\mathscr{L}^2}=-m_{\mathscr{L}^3}=-m_{\mathscr{L}^4}=L_{\square} \quad (4.11)$$

which allows us to rewrite $Z(f)$ as

$$Z(f)=\sum f^*f(L) \quad (4.12)$$

for an arbitrary function "f". Note, we have used the relations (4.11) and written $f(L)$ for $f(L,L,-L,-L)$.

It follows from (4.12) that there is no difference between the mean-field approximation and the more general Hartree-Fock approximation to the ground state of the single-plaquette system after gauge projection. This fact is just another way of saying that the single-plaquette system admits only one gauge-invariant combination of the variables $\theta_{\mathscr{L}}$. Since the general Hartree-Fock approximation, as we defined it in the preceding section, corresponds to choosing a general quadratic form in the variables $m_{\mathscr{L}}$, it follows that in this approximation the partition function is given by

$$Z(\gamma)=\sum\exp\left[-\frac{4L^2}{\gamma}\right]. \quad (4.13)$$

Proceeding in the same way we obtain for the expectation value of the Hamiltonian in an arbitrary Hartree-Fock state,

$$\langle\psi\,|\,HP_g\,|\,\psi\rangle=\sum\left\{2g^2L^2\exp\left[-\frac{4L^2}{\gamma}\right]-\frac{1}{g^2}\exp\left[-\frac{4(L+\frac{1}{2})^2}{\gamma}\right]\exp\left[-\frac{1}{\gamma}\right]\right\}+\frac{Z}{g^2}\,. \quad (4.14)$$

Given (4.13) and (4.14) we can determine the variational parameter γ by minimizing the ratio

$$\mathscr{E}(\gamma)=\frac{\langle\psi\,|\,HP_g\,|\,\psi\rangle}{\langle\psi\,|\,P_g\,|\,\psi\rangle}\,.$$

In order to project a trial wave function into a sector corresponding to a given distribution of static sources, one multiplies by a different operator. In the case of a positive charge at $\vec{1}$ and a negative charge at $\vec{2}$ the operator of interest is

$$P_{\text{sources}}(\vec{1},\vec{2})=P_{g+}(\vec{1})P_{g-}(\vec{2})P_g(\vec{3})P_g(\vec{4})\,. \quad (4.15)$$

Applying this operator to the state $|\psi\rangle$ and computing the normalization factor Z_{sources} one obtains the general result

$$Z_{\text{sources}}=\sum f^*f(L+1,L,-L,-L)\,. \quad (4.16)$$

If one assumes that the function "f" is a product over links of independent Gaussians in the variables $m_{\mathscr{L}}$ (4.16) becomes

$$Z_{\text{sources}}=\exp\left[-\frac{3}{4\gamma}\right]\sum\exp\left[-\frac{(4L+1)^2}{4\gamma}\right]. \quad (4.17)$$

If, on the other hand, one assumes the most general Hartree-Fock wave function with $\Delta(\mathscr{L},\mathscr{L}')=\delta_{\alpha\beta}\,\Delta(\vec{i}-\vec{j})$, one obtains

$$Z_{\text{sources}}=\exp\left[-\frac{(6\Delta_0+\Delta_1)}{8}\right]\times\sum\exp\left[(-4\Delta_0+2\Delta_1)\frac{(4L+1)^2}{16}\right], \quad (4.18)$$

where we have defined $\Delta(\vec{0})=\Delta_0$ and $\Delta(\hat{n}_1)=\Delta(\hat{n}_2)=\Delta_1$. While overall normalization factors differ, if one identifies the parameter γ^{-1} with the

combination $-4\Delta_0+2\Delta_1$, then the partition functions are exactly the same for both the mean-field and general Hartree-Fock wave functions. Since the normalization factor is L independent it plays no role in ratios, and so it can be ignored. Examination of (4.17) reveals only one important difference between the partition function for the source-free problem and the problem with sources at the points $\vec{1}$ and $\vec{2}$; namely, that for the case with sources the argument L is shifted by $\frac{1}{4}$. This comes from completing the square in the exponent. We will see in the sections to follow that this is a general result, as is the fact that the shift parameters are the same for the mean-field and general Hartree-Fock wave functions.

2. Evaluating Z for weak and strong coupling

Evaluating (4.13) and (4.14), and the analogous formulas for the situation in which there are sources at $\vec{1}$ and $\vec{2}$, is particularly straightforward for $g \gg 1$, so we will begin with this case. Examination of (4.14) reveals that for large g^2 the energy is essentially given by the first term in the ratio

$$\mathscr{E}(\gamma)=\frac{\sum\left[2g^2L^2\exp\left[-\frac{4L^2}{\gamma}\right]\right]}{\sum\exp\left[-\frac{4L^2}{\gamma}\right]} \tag{4.19}$$

and so we would expect that in order to minimize the energy for large g the parameter γ must be chosen to be quite small. In this case (4.13) can be well approximated by its first two terms, i.e.,

$$Z(\gamma)=1+2\exp\left[-\frac{4}{\gamma}\right]+\cdots, \tag{4.20}$$

and $\epsilon(\gamma)$ can, to leading order in $\exp(-1/\gamma)$, be written as

$$\mathscr{E}(\gamma)=\frac{4g^2\exp\left[-\frac{4}{\gamma}\right]-\frac{2}{g^2}\exp\left[-\frac{2}{\gamma}\right]}{1+2\exp\left[-\frac{4}{\gamma}\right]}+\frac{1}{g^2}. \tag{4.21}$$

We can check our assumption about the size of γ for large g by minimizing $\epsilon(\gamma)$ as defined in (4.21) and verifying that the value of γ which extremizes $\epsilon(\gamma)$ is small. Taking the derivative of (4.21) with respect to γ we obtain

$$\exp\left[\frac{2}{\gamma}\right]\simeq 4g^4 \tag{4.22}$$

for $g \gg 1$.

Given the relationship between γ and g for a situation with no static sources we can carry out the same exercise for the case of the situation with static sources at $\vec{1}$ and $\vec{2}$. Forgetting the overall normalization factor we have, to leading order in $\exp(-1/\gamma)$,

$$Z_{\text{sources}}=\exp\left[-\frac{1}{4\gamma}\right]+\exp\left[-\frac{9}{4\gamma}\right]+\cdots$$

and

$$\mathscr{E}_{\text{sources}}=\frac{g^2}{2}\exp\left[-\frac{1}{4\gamma}\right]+\frac{3g^2}{2}\exp\left[-\frac{9}{4\gamma}\right]-\frac{1}{g^2}\exp\left[-\frac{5}{4\gamma}\right]\Big/Z_{\text{sources}}. \tag{4.23}$$

Minimizing $\mathscr{E}(\gamma)$ sources with respect to the parameter γ yields

$$\exp\left[\frac{1}{\gamma}\right]\simeq 2g^4 \tag{4.24}$$

which differs somewhat from the source-free case. In either case, we see that as $g\to\infty$, $\gamma\to 0$ as $\ln(g)$. As already mentioned, when we discuss the case of an infinite lattice the variational parameters for the source-free case and the case with sources must be the same, unlike what happens for one plaquette. The reason the values of γ come out different in the case of a single plaquette is that finite-volume effects play a significant role.

Obviously, all sorts of information can be extracted at this point; but, we will limit ourselves to a discussion of the ground-state expectation values of the operators $E_{\mathscr{L}}$. In the source-free case we have

$$\langle E_{\mathscr{L}}\rangle=\sum L\exp\left[-\frac{4L^2}{\gamma}\right]\Big/Z$$

and

$$\langle E_{\mathscr{L}^1}\rangle=\langle E_{\mathscr{L}^2}\rangle=-\langle E_{\mathscr{L}^3}\rangle=-\langle E_{\mathscr{L}^4}\rangle \tag{4.25}$$

which vanishes. However, in the case of sources we have

$$\langle E_{\mathscr{L}^1}\rangle-1=\langle E_{\mathscr{L}^2}\rangle=-\langle E_{\mathscr{L}^3}\rangle=-\langle E_{\mathscr{L}^4}\rangle$$

and

$$\langle E_{\mathscr{L}^2}\rangle=\sum L\exp\left[-\frac{3}{4\gamma}\right]\exp\left[-\frac{(4L+1)^2}{4\gamma}\right]\Big/Z_{\text{sources}} \tag{4.26}$$

from which it follows that at strong coupling, up to exponentially small corrections, all the flux goes down the shortest line joining the two sources. This is, of course, the one-plaquette version of confinement.

We now turn to the study of the evaluation of Z, $\mathscr{E}(\gamma)$, and the expectation values of the electric fields in the presence of sources in the limit $g\to 0$. The reason for carrying out this calculation in detail is to introduce the tricks needed in the more general case to evaluate the weak-coupling results, and to see explicitly how the field in the presence of weak sources goes over to the Coulomb configuration.

We see from the preceding discussion that as $g\to 0$, $1/\gamma$ must also tend to zero. This is because for small g, exciting high m values costs very little. Evaluating (4.13) or (4.17) appears to be difficult for small $1/\gamma$, since in that case one has to keep a great many terms in each sum. There is a way, however, to recast (4.21) and (4.2) into a form which is easily evaluated in the limit of small $1/\gamma$. To do this we make use of the Poisson identity

$$\sum_M f(M)=\sum_N \int d\phi\, f(\phi)\exp(i2\pi N\phi)\ . \tag{4.27}$$

Substituting (4.27) and (4.13) and performing the ϕ integration we obtain

$$Z(\gamma)=\frac{(\pi\gamma)^{1/2}}{2}\sum\exp\left[-\frac{\pi^2\gamma N^2}{4}\right]. \tag{4.28}$$

At this stage it is useful to define the more general function

$$Z(\gamma,\eta)=\sum\exp\left[-\frac{4(L+\eta)^2}{\gamma}\right]=\frac{(\pi\gamma)^{1/2}}{2}\sum\exp\left[-\frac{\pi^2\gamma N^2}{4}\right]\exp(-i2\pi\eta N)\ . \tag{4.29}$$

The expectation value of the Hamiltonian can be defined in terms of $Z(\gamma,\eta)$ as

$$\langle H\rangle=\frac{g^2\gamma^2}{2}\frac{\partial}{\partial\gamma}Z(\gamma,0)+\frac{1}{g^2}\left[1-\exp\left[-\frac{1}{\gamma}\right]Z(\gamma,\tfrac{1}{2})\right]. \tag{4.30}$$

Using the leading $N=0$ and 1 terms we find

$$\mathscr{E}\simeq\frac{g^2\gamma}{4}+\frac{1}{g^2}[1-\exp(-1/\gamma)]\ . \tag{4.31}$$

This result coincides also with what we would have gotten by treating L as a continuous variable in (4.13) and (4.14). Minimizing (4.31) with respect to γ we find that $\gamma\simeq 2g^{-2}$ in the weak-coupling limit.

One can carry out analogous calculations in the presence of sources. In this case $\langle H\rangle$ retains the same functional form but both $Z(\gamma,0)$ and $Z(\gamma,\frac{1}{2})$ get replaced by

$$Z_{\text{sources}}=\exp\left[-\frac{3}{4\gamma}\right]Z(\gamma,\tfrac{1}{4})\ .$$

To leading order this is a shift in the continuous variable L and therefore leads to $\mathscr{E}_{\text{sources}}=\mathscr{E}$ in the limit $g\to 0$ Keeping the next-to-leading-order terms in the calculation we find that $\mathscr{E}_{\text{sources}}$ has an additional, exponentially small, term so that

$$\mathscr{E}_{\text{sources}}-\mathscr{E}\simeq\frac{\pi^2-4}{g^2}\exp\left[-\frac{\pi^2}{2g^2}\right]. \tag{4.32}$$

This nonanalytic dependence upon the coupling constant g is a well-known property of the Mathieu problem. If one rewrites the problem as that of a particle in a periodic potential, then in the limit $g\to 0$ we are studying the "tight-binding region" and these nonanalytic effects come from "tunneling corrections" to the energy.

Let us now turn to the evaluation of the electric fields. In the source-free case we see from (4.25)

that the expectation value of the electric field vanishes. However, when sources are present the L distribution gets shifted as shown in (4.26). Using the expression for $Z(\gamma,\eta)$ we can write

$$\langle E_{\mathscr{L}^2}\rangle = -\frac{\gamma}{8}\frac{\partial}{\partial\eta}\ln Z(\gamma,\eta)\Big|_{\eta=1/4} \equiv -\tfrac{1}{4} \quad (4.33)$$

and all other values of $\langle E_{\mathscr{L}}\rangle$ follow from (4.24). In the strong-coupling limit, where we can use the L expansion directly, $Z\to\exp(-4\eta^2/\gamma)$, $\langle E_{\mathscr{L}^2}\rangle$ vanishes and $\langle E_{\mathscr{L}^1}\rangle\to 1$. This is the expected result, i.e., that in the strong-coupling limit the string of flux joining the two external charges chooses to have the shortest possible length. In the weak-coupling limit, one must use the N expansion given in (4.28). This leads to

$$\langle E_{\mathscr{L}^2}\rangle = -\tfrac{1}{4}+\frac{\pi}{g^2}\exp\left[-\frac{\pi^2}{2g^2}\right] = 1-\langle E_{\mathscr{L}^1}\rangle\ . \quad (4.34)$$

Hence, for small g, we see that the expectation value of the fields on the four links is that which one would expect to obtain by solving the Coulomb problem in the four-link universe. By the Coulomb problem in the four-link universe, we mean that configuration of classical fields $\langle E_{\mathscr{L}}\rangle$, for which the energy

$$\mathscr{E}_{\text{Coulomb}} = \frac{g^2}{2}(\langle E_{\mathscr{L}^1}\rangle^2+\langle E_{\mathscr{L}^2}\rangle^2+\langle E_{\mathscr{L}^3}\rangle^2 +\langle E_{\mathscr{L}^4}\rangle^2) \quad (4.35)$$

is minimized subject to the conditions

$$\begin{aligned}&\langle E_{\mathscr{L}^1}\rangle+\langle E_{\mathscr{L}^4}\rangle=1\ ,\\ &\langle E_{\mathscr{L}^2}\rangle-\langle E_{\mathscr{L}^1}\rangle=-1\ ,\\ &\langle E_{\mathscr{L}^3}\rangle+\langle E_{\mathscr{L}^2}\rangle=0\ ,\end{aligned} \quad (4.36)$$

and

$$\langle E_{\mathscr{L}^3}\rangle-\langle E_{\mathscr{L}^4}\rangle=0\ .$$

Equation (4.36) is satisfied if we let

$$|\langle E_{\mathscr{L}^2}\rangle| = |\langle E_{\mathscr{L}^3}\rangle| = |\langle E_{\mathscr{L}^4}\rangle| = x$$

and

$$|\langle E_{\mathscr{L}^1}\rangle| = |1-x|\ . \quad (4.37)$$

This allows us to rewrite the Coulomb energy as

$$\mathscr{E}_{\text{Coulomb}} = \frac{g^2}{2}[(1-x)^2+3x^2]\ . \quad (4.38)$$

Taking the minimum of (4.38) with respect to x yields $x_{\min}=\frac{1}{4}$; hence, three-quarters of the flux flows along the straight line joining the charges and the remaining one-quarter of the flux flows along the longer route between the vertices $\vec{1}$ and $\vec{2}$.

From this we see that whereas at strong coupling the flux is essentially confined to the straight line joining the two external charges, as the coupling tends to zero the flux spreads out into the appropriate Coulomb configuration, up to terms of order $\exp[-(\text{const})/g^2]$. The effects of these exponentially small terms is to slightly strengthen the field along the line joining the two charges and slightly weaken the field which spreads out to the other links. These are the terms which are responsible for the linear confinement exhibited by the $(2+1)$-dimensional theory at weak coupling, and we will sketch the treatment of this problem in the next section.

B. Partition functions and string tension for infinite lattices

1. The string and its radiation field

To treat PQED on an infinite lattice, in both $2+1$ and $3+1$ dimensions, one proceeds in essentially the same way; however, all computations are more complicated. This calculation was already carried out by Drell *et al.*[3] using a different approach which was suitable for PQED but which could not be generalized to non-Abelian theories. In this section we will develop our treatment of PQED and bring it to a stage at which it can be directly compared to Ref. 3, so that the rest of the calculation can be drawn from there. Since, for the case of QCD we are most interested in establishing confinement for all couplings, we will concentrate on showing that our methods are strong enough to establish this phenomenon for the case of PQED. For this reason the detailed discussion to follow will quickly be restricted to the case of PQED in $2+1$ dimensions. We will, however, establish the connection between our projection-operator formulation of the problem and the methods used by Drell *et al.* in sufficient detail to enable the interested reader to refer to their work to see how the same technique can be used to establish the existence of a deconfining transition for the $(3+1)$-dimensional theory.

Our first aim is to recast the problem in the presence of sources into an equivalent form in which the sums over link variables are uncon-

strained and external variables representing the appearance of the string joining the charges appears explicitly.

The general problem which confronts us in the presence of arbitrary charge distributions is the evaluation of the partition function

$$Z_{\text{gen}}=\sum \exp[-\sum m_{\mathscr{L}^1}\Delta(\mathscr{L}_1,\mathscr{L}_2)m_{\mathscr{L}^2}] \times \prod [\delta(\vec{\nabla}\cdot\vec{m})(\vec{j})-\rho(\vec{j})] , \tag{4.39}$$

where the charge distribution $\rho(\vec{j})$ is taken to be an arbitrary integer-valued function. The special case $\rho(\vec{j})=0$ for all $\vec{j}$ is the source-free situation, and the case $\rho(\vec{p}_1)=-\rho(\vec{p}_2)=1$ and $\rho(\vec{j})=0$ for all $\vec{j}\neq\vec{p}_1$ or $\vec{p}_2$ describes the situation with an oppositely charged pair of external sources. We already observed in passing from the constrained sum (3.17) to (3.20) that one could eliminate the constraint in the source free case in $2+1$ dimensions. What we now wish to do is show that one can do the same for the situation with sources. For simplicity let us consider the case of two sources one at $\vec{p}_1$ and one at $\vec{p}_2$. In this case the constraint on the $\vec{m}$ configurations is that $\vec{\nabla}\cdot\vec{m}$ is zero everywhere except at the points $\vec{p}_1$ and $\vec{p}_2$ where it is $+1$ and -1, respectively. To eliminate the constraints appearing in the sum we observe that if we have a configuration of $\vec{m}$'s satisfying these constraints then it can always be rewritten as

$$m_{\mathscr{L}}=E_{\text{string}}(\mathscr{L})+m'_{\mathscr{L}} , \tag{4.40}$$

where $E_{\text{string}}(\mathscr{L})$ is zero for all links except along a line joining $\vec{p}_1$ and $\vec{p}_2$, along which it is either $+1$ or -1 (depending upon the location of $\vec{p}_1$ and $\vec{p}_2$ and the way in which the line of links joining them is chosen), and where the configuration $m'_{\mathscr{L}}$ satisfies the condition $\vec{\nabla}\cdot\vec{m}'(\vec{j})=0$. If we choose $\vec{p}_1$ and $\vec{p}_2$ to be two points on the 1 axis with $\vec{p}_1$ lying to the left of $\vec{p}_2$, then the line joining these two points can be chosen to lie along the x axis. In this case $E_{\text{string}}(\mathscr{L})$ is plus one for every link $\mathscr{L}$ lying between $\vec{p}_1$ and $\vec{p}_2$, and is zero otherwise. Since every configuration of $m'_{\mathscr{L}}$'s satisfying the source conditions can be written as a given configuration E_{string} plus a source-free configuration, it follows from our discussion for the source-free case that

$$Z_{\text{sources}}(\vec{p}_1,\vec{p}_2)=\sum \exp[-(\vec{\nabla}\times\vec{L}+\vec{E}_{\text{string}})_{\mathscr{L}}\Delta(\mathscr{L},\mathscr{L}')(\vec{\nabla}\times\vec{L}+\vec{E}_{\text{string}})_{\mathscr{L}'}] , \tag{4.41}$$

where now the sum over the integer variables $L(\vec{j},\alpha)$ is unconstrained.

Equation (4.39) can be further simplified if we observe, as is proven in Ref. 3, that the function $\vec{E}_{\text{string}}(\mathscr{L})$ can be written as the lattice gradient of a scalar function $\phi_{\text{string}}(\vec{j})$ plus the curl of a plaquette function $\vec{\epsilon}(\square)$, i.e.,

$$\vec{E}_{\text{string}}(\mathscr{L})=-\nabla_{\mathscr{L}}\phi(\vec{j})+(\nabla_{\square}\times\vec{\epsilon})_{\mathscr{L}} . \tag{4.42}$$

Substituting this expression into (4.41), using the fact that for $\Delta(\mathscr{L},\mathscr{L}')=\delta_{\alpha\beta}\Delta(\vec{i}-\vec{j})$ there is no cross between the gradient of a scalar and a curl, we obtain

$$Z_{\text{sources}}(\vec{p}_1,\vec{p}_2)=Z'\sum \exp\{-[\vec{\nabla}_{\square}\times(\vec{L}+\vec{\epsilon})]_{\mathscr{L}}\Delta(\mathscr{L},\mathscr{L}')[\vec{\nabla}_{\square}\times(\vec{L}+\vec{\epsilon})]_{\mathscr{L}'}\} \tag{4.43}$$

which is the infinite volume generalization of the fractional shifts which appeared in the case of the one-plaquette universe. Z' is the part of the partition function which depends on the scalar function ϕ_{string} alone. This is the source of the Coulomb term in the expression for the energy of the state. The presence of this term plays no role in the question of linear confinement and does not have any interest for us at this time.

The field ϕ_{string} is the unique solution to the lattice Coulomb problem $\Delta^2\phi_{\text{string}}=\rho(\vec{j})$. Since $\nabla_{\mathscr{L}}\phi_{\text{string}}$ is the Coulomb field of the pair of static charges the plaquette function $\epsilon_{\square}$ is the vector potential of the radiation field configuration which must be added to the Coulomb field of the pair of charges in order to focus it into a string joining them. For a given E_{string} the function $\epsilon_{\square}$ is uniquely defined. Since E_{string} is introduced as a mathematical artifice to remove constraints on a sum of integers, we can shift it around at will so long as the location of the endpoints of the "string" stay fixed at $\vec{p}_1$ and $\vec{p}_2$, respectively. Such a shifting of the string corresponds to changing the plaquette function $\epsilon_{\square}$ by integers. Following Drell *et al.*, we will use the convention that the function $\epsilon_{\square}$ will take values in the range $(-\frac{1}{2},\frac{1}{2})$. If the sources lie on an axis of the lattice, we will also assume that the string is drawn along the straight

link (shortest path) joining the two charges.

We can now restate the problem of determining the parameters $\Delta(\mathscr{L},\mathscr{L}')$ in the presence of arbitrary charge distributions as that of minimizing the energy of the system for a general partition function $Z(\epsilon)$. The vacuum state of the theory is given by setting $\epsilon_{\square}=0$, while the case of two static charges is obtained by choosing the $\epsilon_{\square}$ configurations discussed above. The manipulations which allow us to rewrite the problem in terms of an $\epsilon_{\square}$ configuration and Coulomb field are unique to PQED, and depend upon the simple form of Gauss's law for the Abelian theory. The situation is not so simple for the non-Abelian case and we know of no straightforward generalization of the discussion we will now present.

C. Field-theoretic techniques

For simplicity we restrict discussion to the case of the (2 + 1)-dimensional theory, for which the partition function can be rewritten as

$$Z(\epsilon)=\sum \exp[-(\vec{\nabla}\times(\vec{L}+\vec{\epsilon}))_{\mathscr{L}}\Delta(\mathscr{L},\mathscr{L}')[\vec{\nabla}\times(\vec{L}+\vec{\epsilon})]_{\mathscr{L}'}] \,. \tag{4.44}$$

Using the representation of (3.11) and the notation of (3.12) we have

$$\Delta(\mathscr{L},\mathscr{L}')=-\frac{1}{V}\sum \exp[i\vec{k}\cdot(\vec{i}-\vec{j})/c_{\vec{k}}] \,, \tag{4.45}$$

where V stands for the volume of the lattice. Now, by invoking the Poisson identity, we can replace the integer-valued plaquette variables L by continuous fields $\phi(\square)$. In terms of these fields the partition function can be written as

$$Z(\epsilon)=\prod \left\{ \int d\phi_{\square} \left[1+2\sum \cos(2\pi N_{\square}\phi_{\square})\right] \right\} \exp\left[-\sum (\phi+\epsilon)_{\square}\Delta_1(\square,\square')(\phi+\epsilon)_{\square'}\right] \,, \tag{4.46}$$

where we have integrated by parts and defined the new plaquette function

$$\Delta_1(\square,\square')=\nabla_{\mathscr{L}}\times\Delta(\mathscr{L},\mathscr{L}')\times\nabla_{\mathscr{L}'} \,.$$

This is the representation which coincides with Eq. (4.32) of Ref. 3. Since the variables $\phi_{\square}$ are continuous fields we can shift the ranges of integration and absorb these factors to rewrite Z as

$$Z(\epsilon)=\prod \left[\int d\phi_{\square} \left[1+2\sum \cos[2\pi N_{\square}(\phi_{\square}-\epsilon_{\square})]\right] \right] \exp\left[-\sum \phi_{\square}\Delta_1(\square,\square')\phi_{\square'}\right] \,. \tag{4.47}$$

The Hamiltonian can now be rewritten as

$$\langle H\rangle=\prod \left[\int d\phi_{\square} \left[1+2\sum \cos[2\pi N_{\square}(\phi-\epsilon)_{\square}]\right] \right] \Psi^* H'\Psi \,, \tag{4.48}$$

where

$$\Psi\equiv\exp(-\tfrac{1}{2}\sum \phi_{\square}\Delta_1(\square,\square')\phi_{\square'}) \tag{4.49}$$

and where the operator H' is defined as

$$H'=\mathscr{E}_{\text{Coulomb}}+\frac{g^2}{2}\sum (\nabla_{\mathscr{L}}\phi)^2+\frac{1}{g^2}\sum \left[1-\tfrac{1}{2}[\exp(\partial/\partial\phi_{\square})+\exp(-\partial/\partial\phi_{\square})]\right] \,. \tag{4.50}$$

It should be noted at this point that we are rather committing an abuse of notation in using the symbols ϕ and Ψ at this point, since they do not stand for the objects introduced in Sec. III. We have done this in order to establish the connection with the notation of Ref. 3 and the normal quantum mechanics of a system of many degrees of freedom. This abuse of notation will not carry over to the following sections of the paper.

In the weak-coupling limit one saturates the magnetic term in the Hamiltonian which means that one can expand the $\cosh(\partial/\partial\phi_{\square})$ as a power series in its argument keeping only the first two terms. This leads to an approximate form for H'; i.e.,

$$H'=\mathscr{E}_{\text{Coulomb}}+H_1 \,,$$

where

$$H_1=\frac{g^2}{2}\sum(\nabla_{\mathscr{L}}\phi)^2-\frac{1}{2g^2}\sum\frac{\partial^2}{\partial\phi_{\square}{}^2}\ . \qquad (4.51)$$

H_1 is the Hamiltonian of a free massless scalar field ϕ; hence, it is natural to take Ψ as the wave function of the ground state of this system of coupled harmonic oscillators. This means that Ψ can be represented as in (4.49) with

$$\Delta_1(\square,\square')=\frac{g^2}{V}\sum\exp[i\vec{k}\cdot(\vec{i}-\vec{j})]\omega_{\vec{k}}\ ,$$

where V stands for the volume of the lattice, $\vec{i}$ and $\vec{j}$ are integers labeling the centers of the plaquettes, and

$$\omega_{\vec{k}}\equiv\Omega_{\vec{k}}=\left[4-2\sum\cos(k_\alpha)\right]^{1/2}\ , \qquad (4.52)$$

Comparing this result with the general form for Δ_{mom} we see that in the weak-coupling limit $c_{\vec{k}}\to\Omega/g^2$.

The reader will have realized that in the preceding discussion we have implicitly neglected all the contributions of the terms $\cos(2\pi N_{\square}\phi_{\square})$. This is allowed because they are negligible to leading order in g. We already saw this effect when we explicitly carried out the one-plaquette calculations. In that case the corresponding N expansion Eq. (4.28), was very well approximated by the $N=0$ term. Using harmonic oscillator techniques, it is straightforward to show that any $N=1$ term of (4.46) contributes a term of the form

$$\prod\left[\int d\phi\right]\Psi^*(\phi)\cos[2\pi(\phi_{\square}-\epsilon_{\square})]\Psi(\phi)$$
$$=Z_0\cos(2\pi\epsilon_{\square})\exp\left[-\frac{\pi^2}{Vg^2}\sum\frac{1}{\omega_{\vec{k}}}\right] \qquad (4.53)$$

which is exponentially small compared to the $N=0$ term, Z_0. Using this expression one can write the ground-state energy as

$$E(\epsilon)=\mathscr{E}_{\text{Coulomb}}+\tfrac{1}{2}\sum\omega_{\vec{k}}+\frac{1}{8\pi^2g^2Z(\epsilon)}\sum(\pi^2-4)\frac{\partial^2Z(\epsilon)}{\partial\epsilon^2}\ . \qquad (4.54)$$

The last term in this equation represents the $N=1$ correction to both the kinetic and potential terms of the Hamiltonian. It has a negligible effect on the energy density but it plays an important role in establishing the existence of the string tension.

Starting from the equivalent of (4.54) Drell *et al.* show that

$$\mathscr{E}_{\text{sources}}-\mathscr{E}_{\text{vacuum}}=\mathscr{E}_{\text{Coulomb}}+\frac{\pi^2-4}{4\pi^3}\mu^2D \qquad (4.55)$$

for the problem of two static sources separated by a distance D along the x axis. The "mass" parameter μ^2 is given by

$$\mu^2=4\pi^2\exp\left[-\frac{\pi^2}{g^2V}\sum\frac{1}{\omega_{\vec{k}}}\right]\ . \qquad (4.56)$$

It may be understood as a correction to $\omega_{\vec{k}}\to\omega_{\vec{k}}+\mu^2/g^2$ induced in the ϕ propagator by the factors $\cos(2\pi N\phi)$. The factor μ^2 represents the existence of a nonperturbative screening effect in the ϕ-field theory whose existence turns out to be crucial for confinement; i.e., it is only because μ^2 is different from zero that a nonvanishing string tension persists in the weak-coupling limit. The screening of the ϕ field keeps the effect of the ϵ parameters from dissipating and thus the focusing of the field persists to order $\exp(-\text{const}/g^2)$; hence, even at weak coupling, there is a string of electric flux joining the two static charges.

At this point it is interesting to ask what would happen if we did not adopt the general Hartree-Fock form of our wave function but rather persisted in using the mean-field form down into the region of weak coupling. This amounts to choosing the case $c_{\vec{k}}=\text{const}$ and $\omega_{\vec{k}}\simeq\Omega_{\vec{k}}{}^2$. We would then find that we not only obtained a poor value for the ground-state energy density for values of $g<1$, but also there would be a value of g, below which the string tension vanished. The reason this occurs is that inserting $\omega_{\vec{k}}\simeq\Omega_{\vec{k}}{}^2$ in (4.56) results in a logarithmic infrared divergence of the exponent leading to $\mu^2=0$. The disappearance of the string tension implies the existence of an apparent phase transition in the mean-field approximation. Indeed, the partition function in the mean-field approximation is

$$Z_{\text{mf}}=\sum\exp\left[-\sum[\vec{\nabla}\times(\vec{L}+\vec{\epsilon})]^2/\gamma\right] \qquad (4.57)$$

and this is just the partition function of the X-Y model in two spatial dimensions. This model is known to exhibit a Kosterlitz-Thoules (KT) phase transition at a finite value of γ.

Although one must conclude from this analysis that the mean-field approximation gives the wrong answer for PQED in $2+1$ dimensions, it is interesting to ask how wrong it is, and if one can use it as the basis for a systematic treatment of the problem. If we compare the strong tension as

computed in the mean-field and Hartree-Fock approximation, Fig. 1, we see that in the region below the KT phase transition the real string tension is also small. Since the difference between these two calculations is that in the Hartree-Fock calculation we allow the variational parameters $\Delta(\mathscr{L},\mathscr{L}')$ (or alternatively the parameters c_k) to be arbitrary, we see that if we generalize the mean-field approximation by allowing a $\Delta(\mathscr{L},\mathscr{L}')$ to be nonvanishing for a finite number of links we obtain a sequence of partition functions to evaluate which interpolates between the mean-field calculation and the full Hartree-Fock approximation. As shown in Fig. 1, we expect these finite range theories will also be expected to give a KT phase transition, but the location of this transition should move towards $g^2=0$ as the range of the function $\Delta(\mathscr{L},\mathscr{L}')$ increases. Hence, if one is only interested in doing a good job in computing the string tension for $1 \gg g^2 \gg {g_0}^2$, then one should be able to carry this out by performing a cluster expansion about the Kosterlitz-Thouless approximation to the partition function. Of course, in the case of PQED, such an approach is uninteresting since one is able to deal with the general Hartree-Fock case without any serious problems. However, as we will see in the next section, for the case of the non-Abelian gauge theory, dealing directly with the Hartree-Fock approximation might prove quite difficult, and the procedure just outlined may prove to be the only feasible one.

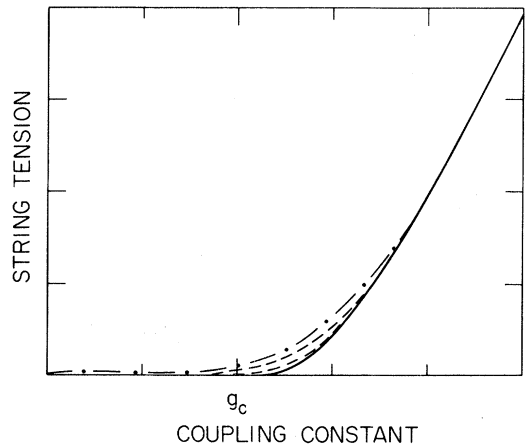

FIG. 1. Graph of the string tension in (2+1)-dimensional PQED as it would be calculated in mean-field (solid curve), perturbative Hartree-Fock (dot-dash curve), and finite-range Hartree-Fock approximations (dashed curves).

V. NON-ABELIAN GAUGE THEORIES

A. A review of the general formalism

Extension of the formalism presented in the preceding sections to non-Abelian gauge theories is straightforward. Unfortunately, the evaluation of the resulting partition functions and the expectation value of the ground-state energy with and without static sources is not as easy as it is in the Abelian theory. We will not, at this time, give any results for SU(2) or SU(3) gauge theories relating to the string tension, spectrum, etc. What we will do in this section is present the general formalism for the case of an SU(2) gauge theory, and outline the treatment of the mean-field and Hartree-Fock versions of the variational calculation. As we will show, significant differences between the SU(2) gauge theory and PQED emerge from very simple considerations.

Before going on to a discussion of the projection-operator formalism we will discuss the general formulation of non-Abelian lattice gauge theories in a way which emphasizes finite, as opposed to infinitesimal, gauge transformations. Our treatment will be brief, but we will try to keep the intuitive notions clear by emphasizing the parallels between the treatment of the Abelian and non-Abelian theories, as well as pointing out differences.

1. Reformulating the Abelian theory in abstract language

The Hilbert space of the Abelian gauge theory was defined to be the space of periodic functions of the link variables $\theta_{\mathscr{L}}$. This set of functions is acted upon by a gauge group, which in the case of the Abelian theory is a product of a U(1) group for each vertex $\vec{i}$. The U(1) group is the set of complex numbers of unit modulus and group multiplication is just the usual multiplication of complex numbers. Hence, we can identify the variable θ with the complex number $\exp(i\theta)$. Under this identification, we can think of the Hilbert space for each link of the lattice gauge theory as being the set of functions from U(1) into the complex numbers. The Hilbert space of the full theory is then generated by taking the product over links of these spaces. Alternatively, we can think of it as the space of complex valued functions from the direct product $\mathrm{U}(1)\times\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$, where the product has a U(1) factor for each link in the lattice. In order to extend this notion to a gauge

theory based upon an arbitrary Lie group G we replace U(1) by G in the preceding definitions; i.e., the Hilbert space associated with each link of the non-Abelian lattice gauge theory based upon the Lie group G is the space of square-integrable functions from the group G into the complex numbers, where integration is done with respect to the usual Haar measure on the group.

Now that we have defined the Hilbert space for the arbitrary gauge theory, the next step is to establish the way in which an arbitrary gauge transformation acts upon any function. Referring back to (4.5) and (4.6) we see that the effect of the arbitrary position-dependent gauge transformation defined by a gauge function $\alpha(\vec{j})$ is to take $\psi(\ldots,\theta_{\mathscr{L}},\ldots)$ to $\psi(\ldots,\theta_{\mathscr{L}}+\alpha(\vec{j}+\hat{n}_\beta)-\alpha(\vec{j}),\ldots)$, for $\mathscr{L}=(\vec{j},\beta)$. Following the conventions just established we can rewrite the periodic function $\psi(\ldots,\theta_{\mathscr{L}},\ldots)$ as $\psi(\ldots,u_{\mathscr{L}},\ldots)$, where $u_{\mathscr{L}}$ is the U(1) element $\exp(i\theta_{\mathscr{L}})$. It then follows that shifting the argument $\theta_{\mathscr{L}}$ by a gauge transformation corresponds multiplying the group element $u_{\mathscr{L}}$ by a phase factor, $\exp\{i[\alpha(\vec{j}+\hat{n}_\beta)-\alpha(\vec{j})]\}$ which is itself a product of group elements. Hence, we can rewrite the formula for gauge transforming a wave function as

$$\psi(\ldots,u_{\mathscr{L}},\ldots)\to\psi(\ldots,g^{-1}(\alpha(\vec{j}))u_{\mathscr{L}}\times g(\alpha(\vec{j}+\hat{n}_\beta)),\ldots)\,, \quad (5.1)$$

where $g(\alpha(\vec{j}))$ stands for the group element $\exp[i\alpha(\vec{j})]$. Obviously, for the case of the Abelian theory the order in which we write the factors is irrelevant, but for the non-Abelian theory the order matters.

When we generalize (5.1) to the case of a non-Abelian gauge group it is not particularly convenient to continue to think of the gauge transformation $g(\alpha(\vec{j}))$ as being given by a function $\alpha(\vec{j})$, and one tends to drop the α label entirely. There is no loss of generality if we define the arbitrary gauge transformation to be given by specifying a group element $g(\vec{j})$ for each vertex $\vec{j}$. In this case, an arbitrary function $\psi(\ldots,R_{\mathscr{L}},\ldots)$, where $R_{\mathscr{L}}$ stands for the group element of the link $\mathscr{L}$, transforms under a gauge transformation specified by the $g(\vec{j})$'s as follows:

$$\psi(\ldots,R_{\mathscr{L}},\ldots)\to\phi(\ldots,g^{-1}(\vec{j})R_{\mathscr{L}}g(\vec{j}+\hat{n}_\beta),\ldots)\,. \quad (5.2)$$

Now the fact that the gauge transformation associated with the left multiplication of $R_{\mathscr{L}}$ is by g^{-1} and right multiplication is by g is forced by the condition that the product of two gauge transformations is a gauge transformation.

We should observe that the formula (5.2) plays an important role in theory of group representation. The space of square-integrable functions from the group SU(2) to the complex numbers is a representation of the group SU(2)×SU(2), called the regular representation of the group. Its importance lies in the fact that, for compact groups, it contains every irreducible representation of the group a number of times equal to the dimension of the representation. The reason the group SU(2)×SU(2) acts on the regular representation is that SU(2) can act on the space of functions in two ways, either by left or right translation; i.e., for a given function $f(h)$ we can define the left translation of f to be

$$(L_g f)(h)=f(g^{-1}h) \quad (5.3)$$

and the right translation of $f(h)$ can be defined as

$$(R_g f)(h)=f(hg)\,. \quad (5.4)$$

Given the definitions (5.3) and (5.4) it is easy to check that

$$L_{gt}=L_gL_t\,,$$

$$R_{gt}=R_gR_t\,,$$

and

$$L_gR_t=R_tL_g \quad (5.5)$$

for arbitrary $g,t\in$SU(2). A general result of harmonic analysis is that every function over the group can be expanded uniquely in terms of the basis functions of the irreducible representations of SU(2)×SU(2) (or $G\times G$ for the general case). For the special case of SU(2) this means that every function $f(g)$ can be expanded as

$$f(h)=\sum a^j_{lm}D^{(j)}_{lm}(h)\,, \quad (5.6)$$

where the D functions are nothing but the representation matrices of the spin-j representation of SU(2). Under left translations the D matrices transform as

$$(L_gD^{(j)}_{lm})(h)=\sum D^{(j)}_{lp}(g^{-1})D^{(j)}_{pm}(h)$$

and under right translations

$$(R_gD^{(j)}_{lm})(h)=\sum D^{(j)}_{lp}(h)D^{(j)}_{pm}(g)\,. \quad (5.7)$$

From these equations we see that each matrix element of the representation matrix $D^{(j)}_{lm}(h)$ is a function over the group, and that these $(2j+1)^2$ functions transform among themselves under the action

of SU(2)×SU(2).

This concludes the definition of our general Hilbert space, and the definition of the way arbitrary gauge transformations act upon our space of states. Let us now turn to the question of generalizing our formulas for the arbitrary mean-field and Hartree-Fock wave functions. Recall that the arbitrary mean-field wave function was defined to a product over links of the same wave function for each link; i.e.,

$$\phi_{\rm mf}(\ldots,R_{\mathscr{L}},\ldots)=\prod_{\mathscr{L}}\psi(R_{\mathscr{L}})\ . \tag{5.8}$$

For the specific case of the U(1) theory we chose for $\psi(\theta_{\mathscr{L}})$ the Gaussian form

$$\psi(\theta)=\sum\exp(-m^2/2\gamma)\exp(im\theta)\ . \tag{5.9}$$

In order to generalize this to the case of the arbitrary compact Lie group we need to rewrite this form in slightly more general notation.

Since the group U(1) is Abelian, all of its irreducible representations are one dimensional, and are given by the functions $\chi^m(\theta)\equiv\exp(im\theta)$. We have already noted that U(1) acts upon these functions by multiplication by a phase factor. In other words, the representation matrices of the group are 1×1 matrices and so the group characters (i.e., the traces of the representation matrices) and the representation matrices themselves, coincide. Moreover each representation of U(1) is characterized by an integer m; the characters $\chi^m(g)$ are an orthogonal set of functions over the group, and the eigenvalues of the Casimir operator of U(1), $L^2=(-i\partial/\partial\theta)^2$, are m^2.

Adopting this notation we can rewrite our single-link mean-field wave function in a way which immediately generalizes to the non-Abelian case. To be specific, we write $\psi(R_{\mathscr{L}})$ as

$$\psi(R_{\mathscr{L}})=\sum\exp\left[-\frac{L^2}{2\gamma}\right]\chi^{(m)}(R_{\mathscr{L}}) \tag{5.10}$$

Obviously, this form of the wave function immediately generalizes to the case of SU(2), in which case we write

$$\psi(R_{\mathscr{L}})=\sum\exp\left[-\frac{\vec{\mathrm{J}}\cdot\vec{\mathrm{J}}}{2\gamma}\right](2j+1)\chi^{(j)}(R_{\mathscr{L}})$$

$$=\sum\exp\left[-\frac{j(j+1)}{2\gamma}\right](2j+1)\chi^{(j)}(g)\ , \tag{5.11}$$

where $\vec{\mathrm{J}}$ stands for the three generators of SU(2), $\vec{\mathrm{J}}\cdot\vec{\mathrm{J}}$ is the quadratic Casimir operator of SU(2) whose eigenvalues are $j(j+1)$, and $(2j+1)$ is a normalization factor introduced to simplify later formulas.

Generalizing the Hartree-Fock wave function to the non-Abelian case is equally simple. All one has to do is define it to be

$$\phi(\ldots,R_{\mathscr{L}},\ldots)=\sum\exp\left[-\sum J_m(\mathscr{L})\Delta(\mathscr{L},\mathscr{L}')J_m(\mathscr{L}')\right]\prod[(2j+1)\chi^{(j)}(R_{\mathscr{L}})]\ , \tag{5.12}$$

where $\Delta(\mathscr{L},\mathscr{L}')$ is, as before, an arbitrary set of variational parameters, and where the subscripts "m" indicate that one is to sum over the three generators of SU(2).

The final concept which needs to be defined in order to be able to write down the Hamiltonian of an arbitrary non-Abelian gauge theory is the notion of the U operators associated with a link. These link operators are the lattice equivalent of the path-ordered exponentials

$$[U(\text{path})]_{\alpha\beta}=\{P[\exp(i\oint dx^\mu\vec{\tau}\cdot\vec{\mathrm{A}}_\mu)]\}_{\alpha\beta} \tag{5.13}$$

and like these operators tranform under gauge transformations like the (n,n) representations of SU(2)×SU(2) (or $G\times G$), where the representation "n" is determined by which matrix representations of the generators $\vec{\tau}$ we use in (5.13). Given the discussion presented in the preceding paragraphs it is easy to find a set of link operators which transform as an irreducible representation of SU(2) ×SU(2) under arbitrary gauge transformations; all one has to do is define

$$[U^{(j)}_{\alpha\beta}\psi](R_{\mathscr{L}})=D^{(j)}_{\alpha\beta}(g)\psi(R_{\mathscr{L}})\ , \tag{5.14}$$

where the functions $D^{(j)}_{\alpha\beta}(R_{\mathscr{L}})$ are the matrix elements of the spin-j representation of SU(2). Under an arbitrary gauge transformation by elements g_1 and g_2 the functions $D^{(j)}_{\alpha\beta}(R_{\mathscr{L}})$ transform as

$$D^{(j)}_{\alpha\beta}(g_1{}^{-1}R_{\mathscr{L}}g_2)=D^{(j)}_{\alpha\nu}(g_1{}^{-1})D^{(j)}_{\nu\mu}(R_{\mathscr{L}})D^{(j)}_{\mu\beta}(g_2) \tag{5.15}$$

and so we see that under the same transformation

$$U^{(j)}_{\alpha\beta}\rightarrow D^{(j)}_{\alpha\nu}(g_1{}^{-1})U^{(j)}_{\nu\mu}D^{(j)}_{\mu\beta}(g_2)\ . \tag{5.16}$$

Adopting this definition we can form gauge-

invariant expressions in the U's associated with the links of a plaquette, $\square$, by forming the ordered products of the form

$$\sum U_{\alpha\beta}(R_{\mathscr{L}'})U_{\beta\mu}(R_{\mathscr{L}^2})U^{-1}{}_{\mu\nu}(R_{\mathscr{L}^3})U^{-1}{}_{\nu\alpha}(R_{\mathscr{L}^4}) , \tag{5.17}$$

where $\mathscr{L}^1, \ldots, \mathscr{L}^4$ are the four links associated with the plaquette $\square$ taken in the same order we used in the example of the one-plaquette universe. Replacing the operators $U_{\alpha\beta}(R_{\mathscr{L}})$ by their definitions in terms of D matrices, we see that a sum over products of operators of this form, when acting on a function of the group variables, is equivalent to multiplying that function by

$$\chi^{(j)}(R_{\mathscr{L}^1}R_{\mathscr{L}^2}R_{\mathscr{L}^3}{}^{-1}R_{\mathscr{L}^4}{}^{-1}) , \tag{5.18}$$

where $\chi^{(j)}(R)$ is the character of the spin-j representation of SU(2) and is equivalent to

$$\chi^{(j)}(R)=\mathrm{Tr}[D^{(j)}(R)] . \tag{5.19}$$

3. The Hamiltonian of the general non-Abelian theory

The Kogut-Susskind Hamiltonian for the SU(2) gauge theory can be written in a form which parallels that of PQED; namely,

$$H=\frac{g^2}{2}\sum E_{\mathscr{L}}{}^2-\frac{1}{g^2}\sum \chi^{(1/2)}(R_{\square}) , \tag{5.20}$$

where $R_{\square}$ stands for the plaquette ordered product of group elements defined in (5.19), and the notation $E_{\mathscr{L}}{}^2$ stands for a sum over the links of the quadratic Casimir operator constructed from the sum of squares of left-handed gauge generators for that link. Note that it follows from our definition of the gauge transformations that the sum of the squares of the left-handed gauge generators is the same as that of the right-handed gauge generators. Hence, the form of the Hamiltonian appearing in (5.20) is quite general.

The interaction term in (5.20) has been chosen, following Kogut and Susskind, to be given as the operator which multiplies any wave function by the character of the fundamental representation. This is not the most general thing one can do; for example, one could generalize this expression to involve an arbitrary sum over characters. It is commonly believed that the Hamiltonian as defined in (5.20) confines charges in the fundamental representation (i.e., $j=\frac{1}{2}$) but that this is no longer true if $\chi^{(1/2)}$ is replaced by $\chi^{(1)}$.

3. *Defining projection operators*

Having defined the Hilbert space of our theory, the Hamiltonian, and the space of trial wave functions it remains for us to give an explicit expression for the projection of a state onto its source-free (or gauge-invariant) part, and onto a state having sources at a finite number (usually two) points. These formulas, too, are analogous to the formulas for the Abelian case.

To derive the general form for the projection of a function over SU(2) onto its spin-j part, we first expand the function in terms of D^j functions; e.g.,

$$f(g)=\sum a^j_{lm}D^j_{lm}(g) . \tag{5.21}$$

Using the orthogonality relation for D^j functions we obtain

$$a^j_{lm}=(2j+1)\int dg\, f(g)D^{*(j)}_{lm}(g) . \tag{5.22}$$

Defining the projection of $f(g)$ onto its spin-j part to be

$$f^{(j)}(g)\equiv\sum_{m=-j}^{j} a^j_{lm}(g)D^{(j)}_{lm}(g) \tag{5.23}$$

we obtain

$$\begin{aligned} f(g)&=(2j+1)\sum\int dh\, f(h)D^{*(j)}_{lm}(g)\\ &=d_j\int dh\, f(h)\chi^{(j)}(g^{-1}h)^*\\ &=d_j\int dh' f(gh')\chi^{(j)}(h')^* , \end{aligned} \tag{5.24}$$

where d_j is the normalization factor $(2j+1)$. It follows from this discussion that in order to project a function of the link variables onto its gauge-invariant part we integrate it against the constant function, and to project onto a configuration with sources we integrate it against the appropriate character. To be exact, the gauge-invariant projection of a function $\psi(\ldots,R_{\mathscr{L}},\ldots)$ is

$$(P_g\psi)(\ldots,R_{\mathscr{L}},\ldots)=\prod\left[\int dg(\vec{i})\right]\psi(\ldots,g^{-1}(\vec{i})R_{\mathscr{L}}g(\vec{i}+\hat{n}),\ldots) \tag{5.25}$$

and the projection onto a state with spin-j sources at $\vec{p}_1$ and $\vec{p}_2$ is

$$P_{\text{sources}}=d_j{}^2\prod\left[\int dg(\vec{i})\right]\psi(\ldots,g^{-1}(\vec{i})R_{\mathscr{L}}g(\vec{i}+\hat{n}),\ldots)\chi^{(j)}[g^{-1}(\vec{p}_1)]^*\chi^{(j)}[g^{-1}(\vec{p}_2)]^* \tag{5.26}$$

B. SU(2) mean-field theory

Equation (5.11) defines a mean-field approximation to the SU(2) gauge theory. This is by no means the most general mean-field wave function which one can choose, but it does provide a simple one-parameter family of variational wave functions and has reasonable behavior in both the strong and weak-coupling limits. If one takes $\gamma \to 0$, one obtains the exact ground-state wave function for the infinite coupling-constant limit of the gauge theory. If one takes $\gamma \to \infty$ one obtains a δ function over the group parameters, which maximizes the expectation value of all $\chi^{(1/2)}(R)$ as it must in the limiting of vanishing coupling constant. The small vibrations around the weak-coupling point are not treated as well in the mean-field approximation as in the Hartree-Fock approximation, but one must pay a price for having only a single variational parameter.

To derive the effective statistical-mechanics problem equivalent to this class of variational wave functions, we must compute the partition function

$$Z \equiv \langle \psi \mid P_g \mid \psi \rangle = \prod \left[\int dR_{\mathscr{L}} \right] \prod \left[\int dg(\vec{i}) \right] \psi(\ldots,R_{\mathscr{L}},\ldots)^* \psi(\ldots,g^{-1}(\vec{i})R_{\mathscr{L}}g(\vec{i}+\hat{n}a),\ldots) \, . \tag{5.27}$$

Since the mean-field wave function is a product of independent link wave functions, carrying out the $R_{\mathscr{L}}$ integrations is quite straightforward. Each $R_{\mathscr{L}}$ integration gives a factor

$$\zeta_{\mathscr{L}} = \int dR_{\mathscr{L}} \sum \exp\left[-\frac{j(j+1)}{2\gamma} \right] \exp\left[-\frac{l(l+1)}{2\gamma} \right] (2j+1)(2l+1)\chi^{(j)}(R_{\mathscr{L}})^* \chi^{(l)}[g^{-1}(\vec{i})R_{\mathscr{L}}g(\vec{i}+\hat{n})] \, . \tag{5.28}$$

Substituting the definition of the character as a trace of $D^{(j)}$ functions, using the fact that the $D^{(j)}$ functions are the representation functions of the group, and using the orthogonality relations among the $D^{(j)}$ functions we obtain

$$\zeta_{\mathscr{L}} = \exp\left[-\frac{j(j+1)}{\gamma} \right] (2j+1)\chi^{(j)}[g^{-1}(\vec{i})g(\vec{i}+\hat{n}a)] \tag{5.29}$$

and so

$$Z = \prod \int dg(\vec{i}) \prod \zeta_{\mathscr{L}} \, . \tag{5.30}$$

Having carried out the $R_{\mathscr{L}}$ integrations we have reduced the evaluation of the partition function Z, or normalization factor for the projection of the mean-field state, to the evaluation of sums of integrals over the group variables. The explicit structure of each integrand appearing in the sum is determined by specifying an integer $j_{\mathscr{L}}$ for each link in the lattice. We will now carry out the group integrations explicitly and convert this formula for Z into a partition function defined in terms of sums over configurations which are specified by giving three integers per link. More specifically, Z will be shown to have the general form

$$Z = \sum \prod C_{\vec{i}}[(j,m,m')_{\mathscr{L}}] \prod Q(j_{\mathscr{L}},m_{\mathscr{L}},m''_{\mathscr{L}}) \, , \tag{5.31}$$

where $C_{\vec{i}}[(j,m,m')_{\mathscr{L}}]$ is a factor associated with every vertex of the lattice and is a function of only those integer variables $j_{\mathscr{L}}$, $m_{\mathscr{L}}$, and $m'_{\mathscr{L}}$ belonging to links attached to the vertex $\vec{i}$. The functions $Q(j_{\mathscr{L}},m_{\mathscr{L}},m'_{\mathscr{L}})$ are link dependent factors. For the case of the mean-field wave function Q will be a function of $j_{\mathscr{L}}$ alone and is given by

$$Q(j_{\mathscr{L}}) = (2j+1)\exp\left[-\frac{j(j+1)}{\gamma} \right] , \tag{5.32}$$

where we have, for simplicity of notation, dropped the dependence on the link label on the right-hand side of (5.32). The vertex factor $C_{\vec{i}}$ is a factor which expresses the constraints imposed by gauge invariance and is different for the cases of two and three spatial dimensions. The $Q_{\vec{i}}$ factors come directly from the weighting factors appearing in the expression for $\zeta_{\mathscr{L}}$, (5.29), and nothing more needs to be said about them. Deriving the form of the factors $C_{\vec{i}}$ is a bit more complicated and the remainder of this section will be devoted to deriving their explicit form for the case of two spatial dimensions. The generalization to three spatial dimensions is straightforward, and will not be presented here.

Since the $C_{\vec{i}}$ factors are associated with a single vertex of the lattice they can be derived by fixing attention upon that part of the partition function which involves the variable $g(\vec{i})$. Isolating that term in (5.27) we find, for the case $d=2$, that we need to evaluate a sum over expressions of the

form

$$V_{\vec{i}} = \int dg(\vec{i})\zeta_{\mathscr{L}^1}\zeta_{\mathscr{L}^2}\zeta_{\mathscr{L}^3}\zeta_{\mathscr{L}^4} , \tag{5.33}$$

where we have taken $\mathscr{L}^1=(\vec{i},1)$, $\mathscr{L}^2=(\vec{i},2)$, $\mathscr{L}^3=(\vec{i}-\hat{n}_1,1)$, and $\mathscr{L}^4=(\vec{i}-\hat{n}_2,2)$, respectively. We now proceed by rewriting $\zeta_{\mathscr{L}}$ as

$$\zeta_{\mathscr{L}}=\Upsilon(j)\mathrm{Tr}(D^{(j)}[g^{-1}(\vec{i})]D^{(j)}[g(\vec{i}+\hat{n}_a)]) \tag{5.34}$$

where $\Gamma(j)$ is the weighting factor in (5.29), and we expand the trace as

$$\mathrm{Tr}[D^{(j)}(g^{-1}h)]=\sum D^{(j)}_{mm'}(g^{-1})D^{(j)}(h)_{m'm} . \tag{5.35}$$

Note that in expanding the trace we have introduced two more integer fields per link, i.e., m and m', where we have adopted the convention of always ordering the indices m and m' so that the matrix associated with the inverse of a group element is subscripted in the order m,m' and the matrix associated with the second group element always is subscripted in the order m',m. With these conventions each term in (5.33) is specified by giving three integers for each link attached to each vertex, and for each one of these terms it is a simple matter to carry out the group integrations. Dropping all factors associated with vertices other than $\vec{i}$ we define

$$C_{\vec{i}}[(j,m,m')_{\mathscr{L}}]=\int dg(\vec{i})D^{(j_1)}_{m^1m'^1}[g(\vec{i})^{-1}]D^{(j_2)}_{m^2m'^2}[g(\vec{i})^{-1}]D^{(j_3)}_{m^3m'^3}[g(\vec{i})]D^{(j_4)}_{m^4m'^4}[g(\vec{i})] . \tag{5.36}$$

We can now use the identity

$$D^{(j)}_{m'm}(g)D^{(k)}_{p'p}(g)=\sum\langle n,i' \mid j,m',k,p'\rangle\langle j,m,k,p \mid n,i\rangle D^{(1)}_{i'i(g)} \tag{5.37}$$

and its complex conjugate to rewrite (5.36) as

$$C_{\vec{i}}[(j,m,m')_{\mathscr{L}}]=\sum\frac{1}{2J+1}\langle j_1,m'^1,j_2,m'^2 \mid J',M'\rangle\langle J,M \mid j_1,m^1,j_2,m^2\rangle$$
$$\times\langle J',M' \mid j_3,m'^3,j_4,m'^4\rangle\langle j_3,m^3,j_4,m^4 \mid J,M\rangle . \tag{5.38}$$

It follows immediately from the properties of the Clebsch-Gordan coefficients that

$$m'^1+m'^2=m'^3+m'^4 \text{ and } m^1+m^2=m^3+m^4$$

which can be rewritten as

$$\nabla\cdot m'=0 \text{ and } \nabla\cdot m=0 . \tag{5.39}$$

Hence, the two sets of integer fields, m and m', separately satisfy divergence conditions of the sort encountered in the treatment of PQED in two dimensions, and the constraints on their sums can be removed in the same way. In this case removing these constraints amounts to introducing two integer-valued variables L and L' for each plaquette and defining

$$m=\nabla_\square\times L \text{ and } m'=\nabla_\square\times L . \tag{5.40}$$

The only nontrivial constraint is the one on the j fields, and this is automatically taken care of since the Clebsch-Gordan coefficients vanish for j's not satisfying the appropriate constraints.

This completes our derivation of the form of the partition function Z for a mean-field approximation to the $d=2$ SU(2) gauge theory. This partition function cannot be treated as easily as the partition function for the Abelian theory, and the general problem of evaluating this sort of function is under study. To date it is not at all clear how much of the general problem can be handled by analytic methods, and how much requires the use of a computer. What is clear is that even if one has to go to Monte Carlo techniques to evaluate this function, the fact that one has reduced the problem by one dimension by working with the Hamiltonian formalism, and the fact that for most ranges of γ only a few j values per link contribute to the energy of the ground state should allow us to work on significantly larger lattices than one can deal with in the Euclidean formalism.

Introducing sources into our problem only modifies things in a simple way; namely, it introduces a limited number of distinguished vertices for which the vertex factor is more complicated. This comes about because one projects the mean-field wave function against a character $\chi^{(1)}[g(\vec{j})]$ for any vertex $\vec{j}$ at which a source is located, instead of against the constant function. If one loosely summarizes the formula derived for Z by saying that only those configurations of j's, m's, and m''s which add up to zero angular momentum at each vertex contribute to the partition function, then in

the case of sources of spin $\frac{1}{2}$, only configurations which add up to spin $\frac{1}{2}$ at the source locations and zero otherwise contribute to those partition functions.

Although no calculations of this sort have been carried out by us, it is expected that for the case of $d=3$ the mean-field approximation to the SU(2) theory will incorrectly predict a deconfining phase transition, in analogy to what happens for the case of PQED in $2+1$ dimensions. It is our hope that, as for the case of $(2+1)$-dimensional PQED, this will be remedied by going over to the Hartree-Fock approximation. Unfortunately, dealing with the Hartree-Fock wave function is considerably more difficult for the non-Abelian theory than it is for the Abelian theory, and we will close this section with a disucssion of what we know about this problem.

C. The Hartree-Fock approximation to the SU(2) gauge theory

The Hartree-Fock approximation for the non-Abelian gauge theory is defined in (5.12). Note that even before gauge projecting, by using the perturbation theory form of $\Delta(\mathscr{L},\mathscr{L}')$ in the definition of the differential operator and letting it act on a sum of products of characters, we already include effects related solely to the fact that the group is compact. While gauge projecting the perturbative wave function would force this anyhow, this method of incorporating perturbation theory into our formalism simplifies subsequent manipulations considerably. Unfortunately, even with these tricks the evaluation of the equivalent partition function (and energy) for the Hartree-Fock approximation is considerably more difficult than for the mean-field case. The problem of finding the best way in which to carry out this evaluation is under study. We will limit the discussion in this section to outlining the derivation of the partition function in order to exhibit the new difficulties encountered in the Hartree-Fock case. Our approach will be the brute-force procedure alluded to at the end of the discussion of the Abelian theory. It may well be true that this technique will be superceded by better computational techniques in the near future, however this analysis is very interesting. We will show that a qualitative difference between the Abelian and non-Abelian gauge theories emerges even before we carry out an evaluation of the partition function. As in the preceding section, since no explicit evaluation of energies, etc., will be presented we will focus on the partition function alone. Analogous formulas for the expectation value of the energy, string tension, etc. can be derived in exactly the same manner.

The principal problem encountered in evaluating the partition function for the Hartree-Fock wave function comes from the fact that the differential operators appearing in the exponential in (5.12) do not commute with local gauge transformations, nor with one another on the same link. This fact makes it impossible to carry out the gauge integrations in closed form. However, one can evaluate this expression as an expansion about the mean-field partition function, and that is what we will do in this section. We begin by rewriting (5.12) as

$$\phi_{\mathrm{HF}}(\ldots,R_{\mathscr{L}},\ldots)=\sum\exp\left[-\sum(\vec{\mathrm{J}}\cdot\vec{\mathrm{J}})_{\mathscr{L}}\Delta(\vec{\mathrm{O}})\right]\exp\left[-\sum J_m(\mathscr{L})\Delta(\mathscr{L},\mathscr{L}')J_m(\mathscr{L}')\right]\prod\chi^{(j)}(R_{\mathscr{L}})\,, \tag{5.41}$$

where we are able to break the exponential into the product of two terms because the operator $\vec{\mathrm{J}}\cdot\vec{\mathrm{J}}$ is a Casimir operator of SU(2) and commutes with all of the operators $J_m(\mathscr{L})$. The first term in (5.41) acting upon the product of characters produces the mean-field wave function, hence if we expand the second exponential is

$$\exp\left[-\sum J_m(\mathscr{L})\Delta(\mathscr{L},\mathscr{L}')J_m(\mathscr{L}')\right]=1-\sum J_m(\mathscr{L})\Delta(\mathscr{L},\mathscr{L}')J_m(\mathscr{L}')+(1/2!)\left[-\sum\cdots\right]^2+\cdots\,, \tag{5.42}$$

then we can derive the partition function corresponding to the Hartree-Fock representation as a sum of terms, each of which is the mean-field average of a product of operators. This is of course a crude procedure, and there is reason to believe that one can do much better than this in the future. However, we have already seen in our discussion of the physics of PQED in $2+1$ dimensions, that one can go a long way towards analyzing the physics of the theory by means of such an expansion. We believe that in order to obtain some intuition for the physics of confinement in $3+1$ dimensions, it will be fruitful to begin by analyzing the differences between the Abelian and non-Abelian gauge theory in this approximation.

Using the expansion given in (5.42) we can rewrite (5.41) as

$$\phi_{\mathrm{HF}}(\ldots,R_{\mathscr{L}},\ldots)=\sum\prod\exp[-j_{\mathscr{L}}(j_{\mathscr{L}}+1)\Delta(0)]\chi^{(j)}(R_{\mathscr{L}})$$
$$+\sum\left\{\sum\Delta(\mathscr{L},\mathscr{L}')\mathrm{Tr}[T_a^{(j)}D^{(j)}(R_{\mathscr{L}})]\mathrm{Tr}[T_a^{(1)(1)}(R_{\mathscr{L}'})]\right\}\prod\chi^{(k)}(R_{\mathscr{L}''})+\ldots\,, \tag{5.43}$$

where we have used the fact that the character of a representation is simply the trace of the D function and the fact that the left generator of SU(2)×SU(2) acting upon that representation is equivalent to multiplying the D function to the left by the matrix representation of the appropriate generator of SU(2).

In order to compute the Hartree-Fock partition function we need to compute the overlap integral

$$Z_{\mathrm{HF}}=\int dg(\vec{\mathrm{i}})\int dR_{\mathscr{L}}\phi_{\mathrm{HF}}(\ldots,R_{\mathscr{L}},\ldots)^*\phi_{\mathrm{HF}}(\ldots,g^{-1}(\vec{\mathrm{i}})R_{\mathscr{L}}g(\vec{\mathrm{i}}+\hat{n}_\alpha),\ldots)\,. \tag{5.44}$$

Substituting (5.43) into (5.44) we obtain Z_{HF} as a sum of terms. The first contribution is the mean-field partition function which we have already discussed. The remaining contributions have a structure which is different from those obtained in the mean-field approximation. One difference is obvious from the form of (5.43); namely, that these contributions to the partition function appear with link factors involving matrix elements $(T_a^{(j)})_{\mu\nu}$ on some of the links. The second, more interesting, difference is that when computing the effects of terms of this sort the condition that one only sums over configurations of link variables for which the angular momenta at a vertex add up to zero is no longer correct. It must be replaced by the condition that one sums over configurations which add up to angular momentum zero or one. Of course, the vertex factors for the different configurations are different. We close this section by outlining the derivation of this result.

In order to derive the general structure of the expansion of the Hartree-Fock partition function about the mean-field case, we only need to focus on two typical terms which appear in the expansion of (5.44). The first of these terms is of the form

$$\prod\left[\int dR_{\mathscr{L}}\right]\prod\left[\int dg(\vec{\mathrm{i}})\right]\prod\left[\chi^{(j)}(R_{\mathscr{L}})^*\right]\sum\Delta(\mathscr{L},\mathscr{L}')\mathrm{Tr}[T_b^{(j)}D^{(j)}(R_{\mathscr{L}})]^*\mathrm{Tr}[T_b^{(k)}D^{(k)}(R_{\mathscr{L}'})]^*$$
$$\times\prod\left[\chi^{(j')}[g^{-1}(\vec{\mathrm{i}})R_{\mathscr{L}''}g(\vec{\mathrm{i}}+\hat{n}_\rho)]\right]\,. \tag{5.45}$$

Now one can do the $R_{\mathscr{L}}$ integrations. As we already saw when we evaluated the mean-field partition function this leads to Kronecker δ's, and so (5.45) becomes

$$\prod\left[\int dg(\vec{\mathrm{i}})\right]\sum\Delta(\mathscr{L},\mathscr{L}')\tau_b(\mathscr{L}')\,,$$

where the factors $\tau_b(\mathscr{L})$ are defined to be

$$\tau_b(\mathscr{L})=\mathrm{Tr}\{D^{(j)}[g^{-1}(\vec{\mathrm{j}})]T_b^{(j)}D^{(j)}[g(\vec{\mathrm{j}}+\hat{n}\alpha)]\} \tag{5.46}$$

for $\mathscr{L}=(\vec{\mathrm{j}},\beta)$. At this point one proceeds to evaluate the integrals over the variables $g(\vec{\mathrm{i}})$. As in the mean-field case, we do this by expanding out the expressions for characters of products of group elements as sums over products of D functions, and then we focus on one vertex at a time. In the case of (5.46) this leads to the same set of vertex factors derived in the mean-field case; however, two of the link factors are modified due to the fact that $T_b^{(j)}$ matrices appear in the traces in (5.46). One then sums over all insertions of this sort.

The second sort of term one encounters in evaluating the Hartree-Fock wave function as an expansion about a mean-field configuration is of the form

$$\prod\left[\int dR_{\mathscr{L}}\right]\prod\left[\int dg(\vec{\mathrm{i}})\right]\prod\left[\chi^{(j)}(R_{\mathscr{L}})^*\right]\sum\Delta(\mathscr{L},\mathscr{L}')\tau_b'(R_{\mathscr{L}})\tau'{}_b(R_{\mathscr{L}'})\prod\left[\chi^{(j')}[g^{-1}(\vec{\mathrm{i}})R_{\mathscr{L}}g(\vec{\mathrm{i}}+\hat{n}_\rho)]\right]\,,$$

where the factors $\tau_b'(R_{\mathscr{L}})$ are defined by

$$\tau_b'(R_{\mathscr{L}})=\mathrm{Tr}\{T_b^{(j)}D^{(j)}[g^{-1}(\vec{\mathrm{j}})R_{\mathscr{L}}g(\vec{\mathrm{j}}+\hat{n}_\alpha)]\}\,. \tag{5.47}$$

As before one now carries out the $R_{\mathscr{L}}$ integrations and obtains

$$\prod \left[\int dg(\vec{i}) \right] \sum \Delta(\mathscr{L},\mathscr{L}')\tau'_b(\mathscr{L})\tau'_b(\mathscr{L}') ,$$

where the link factors $\tau'_b(\mathscr{L})$ are

$$\tau'_b(\mathscr{L})=\mathrm{Tr}\{T_b^{(j)}D^{(j)}[g^{-1}(\vec{j})g(\vec{j}+\hat{n}_\alpha)]\} . \tag{5.48}$$

Comparing (5.48) with (5.46) we see that there is an important difference in these two expressions in that the matrices $T_b^{(j)}$ and $T_b^{(k)}$ appear in different places in the trace. In order to compare the equations and carry out the explicit integrations over the $g(\vec{i})$ variables it is convenient to write the traces appearing in (5.48) as

$$\mathrm{Tr}\{T_b^{(j)}D^{(j)}[g^{-1}(\vec{i})g(\vec{i}+\hat{n}_\alpha)]\}=D^{(1)}[g(\vec{i})]_{cb}\mathrm{Tr}\{D^{(j)}[g(\vec{i})]T_c^{(j)}D^{(j)}[g(\vec{i}+\hat{n}_\alpha)]\} , \tag{5.49}$$

where the factor $D^{(1)}[g(\vec{i})]$ comes from the fact that the SU(2) generators transform as the basis vectors of a spin-1 representation of the group under the transformation $T_a \rightarrow D(g)T_aD^{-1}(g)$. Substituting (5.49) into (5.48) we see that this sort of term modifies both the link factors and the "vertex factors" which one must use in evaluating this contribution to the Hartree-Fock partition function. The link factors are changed in the same way as they were changed when we evaluated (5.45) and (5.46); namely, by the introduction of the matrix $T_b^{(j)}$. It is the change in the vertex factors which is new, and it comes about because of the extra factor of $D^{(1)}[g(\vec{i})]$ appearing in (5.49). In all of the preceding cases we did the $g(\vec{i})$ integrations and derived the condition that only J,m,m' configurations for which the sum of the angular momenta at a vertex added up to zero could contribute to the partition function. However, when we carry out the same procedure for (5.48) we see, substituting (5.49), that this condition is changed at the vertices $\vec{1}$ and $\vec{m}$. In this case there is one additional $D^{(1)}$ function appearing at the vertices $\vec{i}$ and $\vec{m}$ and so, in order for the link plus this extra vertex angular momentum to add up to zero, the link momenta must add up to spin-1. This feature of the expansion of the non-Abelian Hartree-Fock wave function about its mean-field part has no parallel in the case of the Abelian theory. It has the effect of introducing a new kind of charged medium into the effective statistical-mechanics problem, and it is our hope that the effect of this medium will be to wipe out the deconfining phase transition which one expects to find in the mean-field approximation to the (3+1)-dimensional gauge theory.

VI. SUMMARY AND CONCLUSIONS

A. Pure gauge theories

In this paper we presented a formalism for carrying out nonperturbative gauge-invariant calculations for both Abelian and non-Abelian Hamiltonian lattice gauge theories. One virture of this technique is that the formulation of the method is the same for both the Abelian and non-Abelian theory, and when applied to the Abelian theory it simplifies previous formulations of the Hamiltonian problem without losing any of the results. This method differs from other mean-field approaches[6] in that incorporates the requirement of gauge invariance exactly in all space-time dimensions, and can be used outside of the framework of the mean-field approximation. In particular, although our focus in this paper was on application of these methods to approximations which could be written out analytically, it is a simple matter to incorporate these techniques into numerical procedures such as real-space renormalization-group calculations. While this holds out the promise of allowing us to accurately calculate the behavior of the string tension for couplings of order unity, we have already seen that one has reason to hope that reasonable results might be obtained by carrying out the generalized Hartree-Fock calculation which we have described; that is, a Hartree-Fock calculation wherein the function $\Delta(\mathscr{L},\mathscr{L}')$ is treated as the set of variational parameters. As we have seen, this kind of calculation would give reasonable results for PQED in 2+1 dimensions. To our minds, the most exciting possibility at this time is that a calculation of the gauge-projected perturbation-theoretic wave function will lead to an understanding of the physics of confinement in the weak-coupling regime. As we indicated at the end of the last section, we believe this for two reasons: first, because this statement proved to be true for PQED, and second, because a comparison of the expansion of the Hartree-Fock wave function for PQED and the SU(2) gauge theory revealed, at the crudest level, significant differences between the gauge-projected structure of the perturbative wave functions. Obviously, we believe that one of the most important things to do at this time is to actively pursue the Hartree-Fock calculation for the non-Abelian theory.

B. Introducing fermions

The introduction of fermions presents no new difficulties in either the mean-field or Hartree-Fock formulation of these methods. In the mean-field approximation one chooses the fermion wave function to be a product over sites of the same single-site wave function, whereas in the Hartree-Fock approximation one uses a generalization of the fermionic free-field wave function. While, in the case of systems with a large number of degrees of freedom, this leads to additional complexity the reduction of the combined boson-fermion problem to an effective statistical-mechanical partition function is quite manageable.

C. Computing the hadron spectrum

Since the introduction of fermions into this scheme seems to be an extension of the techniques we have discussed in this paper, one very interesting aspect of this sort of nonperturbative computational scheme is that it permits us to formulate the problem of computing the spectrum of quark bound states in a concrete way. The key point to be made is that having computed a variational form of the ground state one can consider carrying out a variational calculation for the lowest-lying states of nontrivial flavor; i.e., one could gauge project states of the form

$$|\Phi_{\text{hadron}}\rangle = \sum f_{\alpha\beta}(\vec{j}_1 - \vec{j}_2)\Psi^*(\vec{j}_1)_\alpha \Psi(\vec{j}_2)_\beta |\Phi_{\text{vacuum}}\rangle \ . \tag{6.1}$$

The variational parameters in this problem are first the "bound-state" wave function $f_{\alpha\beta}(\vec{j})$, and then the parameters appearing in the trial wave function $|\Phi_{\text{vacuum}}\rangle$. Since the parameters appearing in the vacuum part of the wave function are determined by the computation of the vacuum energy density, it follows that the only "free parameter" in the problem is the function $f_{\alpha\beta}(\vec{j})$; hence, performing the variational calculation for the energy of this state will yield an equation for this fixed-time "bound-state" wave function. A similar calculation is in principle possible for the "glue-ball" spectrum of the theory with or without fermions, since one could take the perturbation-theory state of two gluons in a global color-singlet state of nonvanishing three-momentum. The nonvanishing three-momentum forces this state to be orthogonal to the vacuum state determined in the original calculation. By plotting the spectrum as a function of the three-momentum minus the vacuum energy one should be able to estimate the glueball mass. Obviously, all of these calculations are subject to the usual caveats related to doing variational calculations and taking differences of upper bounds, but one can only try and see how well the method works.

D. Conclusion

This discussion briefly summarizes those aspects of the formalism presented in this paper which we feel merit the most study at this time. There will be a great deal of work involved in learning how best, in practice, to carry out the evaluation of the partition functions of the effective statistical-mechanics problems. At this time it is not possible for us to say what the best balance between analytical and numerical methods will be, nor can we say with certainty that the results one will be able to obtain will be accurate enough to compare with experiment. Nevertheless, we are heartened by the way this problem works out in the case of PQED. Even if one is forced in the end to use Monte Carlo methods for doing the statistical-mechanics sums, the fact that one is always working in one dimension less than the corresponding Euclidean problem, and that one is doing discrete sums rather than continuous integrations should make it possible to deal with more complicated systems on larger lattices.

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[5]The same sort of thing is possible for PQED in $3+1$ dimensions, except that one has to introduce integer variables L for each plaquette, which amounts to going to the dual lattice. However, in this case the sum of the L's is not unconstrained even though we are now only dealing with gauge-invariant variables essentially because the Maxwell theory automatically satisfies the condition $\vec{\nabla}\cdot\vec{B}=0$, which in this case becomes the condition that for each cube the oriented sum of the L's over the six faces of the cube must vanish. This point is explained in Ref. 3 and the way of dealing with it is completely discussed. Since we are only interested in discussing the $(2+1)$-dimensional case from this point on we will not go into the fine points of this discussion and the introduction of the dual lattice in this paper.

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