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Instability of flat space at finite temperature

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The instabilities of quantum gravity are investigated using the path-integral formulation of Einstein's theory. A brief review is given of the classical gravitational instabilities, as well as the stability of flat space. The Euclidean path-integral representation of the partition function is employed to discuss the instability of flat space at finite temperature. Semiclassical, or saddle-point, approximations are utilized. We show how the Jeans instability arises as a tachyon in the graviton propagator when small perturbations about hot flat space are considered. The effect due to the Schwarzschild instanton is studied. The small fluctuations about this instanton are analyzed and a negative mode is discovered. This produces, in the semiclassical approximation, an imaginary part of the free energy. This is interpreted as being due to the metastability of hot flat space to nucleate black holes. These then evolve by evaporation or by accretion of thermal gravitons, leading to the instability of hot flat space. The nucleation rate of black holes is calculated as a function of temperature.

I. INTRODUCTION

Gravity, unlike the other fundamental forces of nature, is universally attractive and cannot be screened. This property of gravity, to which we owe our ability to detect this incredibly weak interaction, is the source of many instabilities.

The instability of gravity already appears in classical Newtonian theory. As Jeans[1] showed, a static, homogeneous nonrelativistic fluid is unstable under long-wavelength gravitational perturbations.[2] Consider a nonviscous fluid of mass density ρ, pressure p, and velocity $\vec{v}$ that satisfies the equation of continuity and the Navier-Stokes equation

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0\ ,$$
$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{1}{\rho}\vec{\nabla}p-\vec{g}\ , \tag{1.1}$$

where $\vec{g}$ is the gravitational field, given by

$$\vec{\nabla}\times\vec{g}=0\ ,\quad \vec{\nabla}\cdot\vec{g}=-4\pi G\rho\ . \tag{1.2}$$

One now considers small perturbations $\delta\rho$, δp, $\delta\vec{v}$, and $\vec{g}$ about the static, homogeneous nongravitating fluid of constant density ρ and pressure p. One finds that they are governed by the equation

$$\frac{\partial^2\delta\rho}{\partial t^2}-V_s^2\nabla^2\delta\rho=4\pi G\rho\delta\rho\ , \tag{1.3}$$

where V_s is the speed of sound in the fluid ($V_s^2=\partial P/\partial\rho$). Note that the right-hand side of Eq. (1.3) has the form of a "mass term," but with the wrong sign. Therefore the solutions of this equation

$$\delta\rho=C\exp(i\vec{k}\cdot\vec{x}-i\omega t)\ ,$$
$$\omega=(\vec{k}^2V_s^2-4\pi G\rho)^{1/2} \tag{1.4}$$

will grow exponentially if the wave number k is less than k_J, where

$$k_J=(4\pi G\rho/V_s^2)^{1/2}\ . \tag{1.5}$$

This instability is due to the attractive nature of gravity, which, in contrast to the damping of charge density fluctuations in a plasma, "antiscreens" mass density fluctuations thus leading to their amplification.

The same instability occurs if we consider a gas in isothermal equilibrium in a finite volume. Suppose that we have a spherical ball of perfect gas that is in isothermal equilibrium under its own gravitational field.[3,4] At each point, the equation of state is

$$p=\rho T/m \ , \tag{1.6}$$

where p is the pressure, ρ the mass density, T the temperature, and m is the mass of each molecule. The equation of hydrostatic equilibrium for such a system is

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dp}{dr}\right]=-4\pi G\rho \ , \tag{1.7}$$

where r is the distance from the center of the cloud. At the origin, $r=0$, $\rho'(r)=0$. The final boundary condition is the value of the pressure at the outermost edge of the cloud which must be specified. These conditions determine the solution to (1.6) and (1.7). Suppose we surround the cloud by a spherical box of variable radius R, with external pressure P, containing N molecules, and whose walls are fixed at temperature T. Then it follows that

$$\left[\frac{\partial P}{\partial R}\right]_{N,T}=-\frac{P(8\pi PR^4-Gm^2N^2)}{R^2(4\pi PR^2-NT)} \tag{1.8}$$

along the curve of equilibrium.

For R sufficiently large, then $(\partial P/\partial R)<0$ as is usually expected. However, if R is decreased, then the pressure increases to a maximum of

$$P=\frac{Gm^2N^2}{8\pi R^4} \ . \tag{1.9}$$

Thereafter an instability sets in since $(\partial p/\partial R)>0$. Any small fluctuation in the gas that decreases the pressure will decrease the volume of the box. This will decrease the pressure further, and so the system is unstable.

It is essentially this mechanism that causes stars to condense out of clouds of interstellar gas. This instability is terminated when a cloud becomes sufficiently hot that nuclear reactions are initiated and support the cloud against further collapse. Again the basic reason for this instability is due to attractive nature of gravitation.

In the general theory of relativity, as formulated by Einstein, gravitational instabilities are even more severe, since the spacetime manifold is warped by the presence of matter. Gravitational collapse can give rise to singularities in the fabric of spacetime. Thus, for example, a star with a mass greater than the Oppenheimer-Volkoff limit[5] (currently estimated to be about 1.4 $M_\odot$)[6] cannot support itself against gravitational collapse. After collapsing to a white dwarf (a degenerate gas of electrons) and then to a neutron star (a degenerate gas of neutrons), it will continue to collapse.

It seems that this collapse will lead first to the formation of an event horizon and then to a spacetime singularity.[7] Thus, a black hole is formed as a direct result of this type of instability. (It could be that a spacetime singularity is formed without an event horizon: a naked singularity. Such behavior is ruled out by the cosmic censorship hypothesis.[8] However, this conjecture has not been proved.) The type of singularity associated with a black hole is relatively mild. If however we assume that the universe is well described by a Friedmann-Robertson-Walker model with a density larger than the critical density of 2×10^{-29} g cm^{-3}, which has not been observationally ruled out, then the universe itself is unstable in the sense that it too must undergo collapse leading to a spacelike singularity to the future of all observers.

Given the inevitable instabilities of gravity one might worry about the stability of the ground state of quantum gravity. One is accustomed to regarding Minkowski space as the ground state, or vacuum, of quantum gravity. Small perturbations about this vacuum are certainly stable; however one might find that flat space is quantum mechanically unstable. This would occur if the "potential" for gravity had the form given in Fig. 1, so that the metastable vacuum A would decay by tunneling through a barrier to some configuration B. This concern is nontrivial since there is no way to define a local energy density in gravity whose positivity ensures stability. Moreover the Einstein action is not positive definite, even when continued

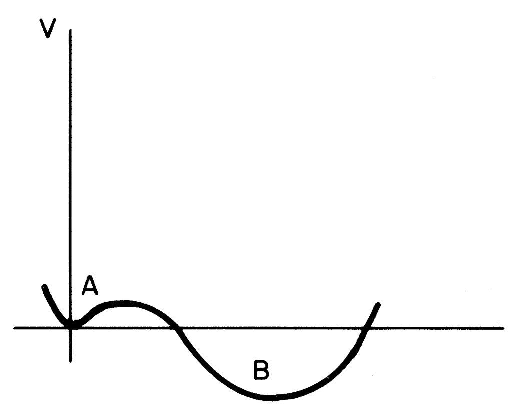

FIG. 1. The form of the potential for gravity at zero temperature that would hold were there "instanton bounces" in gravity.

to Euclidean space (i.e., the curvature can be arbitrarily large and positive or negative). Thus the Euclidean functional integral formalism of quantum gravity[9] is not well defined. One might be worried that this implies the nonexistence of a ground state.

It is remarkable that the issue of the stability of flat space was only settled recently. First Schoen and Yau[10] proved that there are no instanton bounces, i.e., solutions of Einstein's equations with Euclidean metric that asymptotically approach flat space, which would represent the tunneling through a barrier. (We shall return to discuss the meaning of such instantons in Sec. II.) This implies that flat space is stable under processes that can be treated by semiclassical (or WKB) methods. Second, Schoen and Yau[11] proved the long-outstanding conjecture that the total energy (defined in terms of the asymptotic behavior of the gravitational field) of asymptotically flat manifolds (which satisfy Einstein's equations in the presence of matter that itself has positive energy in any frame) is positive semidefinite, and that only Minkowski space has zero energy.[12] This is the positive-energy theorem, which since energy is conserved seems to preclude the possibility of flat space decaying by any mechanism.

In this paper we explore the instabilities of Minkowski space at a finite temperature. Here, unlike the case of zero temperature, we find two distinct sources of instability. These imply that hot flat space is unstable and will decay. This result obtains for any finite, however small, temperature.

One source of instability is expected on the basis of the classical Jeans analysis. Hot flat space is not empty. It contains a gas of gravitons in thermal equilibrium, which are a source for the gravitational field. For nonvanishing temperature, flat space contains "matter." Thus one expects a Jeans instability to occur for large-wavelength density fluctuations of the thermal gravitons. We shall show how this instability arises when one uses functional integral techniques to examine the partition function for quantum gravity.

The second source of instability is the nucleation of black holes. This is a quantum effect, which cannot be understood on the classical level. We discovered this effect by investigating the contribution of Euclidean instantons (i.e., the Euclidean section of the Schwarzschild metric) to the functional integral. These instantons have been discussed by many authors, notably Hawking who attempted to use them to deduce the thermodynamic properties of black holes.[13] Our interpretation of their meaning is somewhat different. We found that there exist small fluctuations about these instantons that decrease the action.[14] These give rise to an imaginary part in the free energy of flat space, which is to be interpreted as a finite lifetime for decay. The decay proceeds by thermal (= quantum) fluctuations nucleating black holes of radius $R=\hbar c/4\pi kT$ and mass$=\hbar c^3/8\pi GkT$ (where T is the temperature). These then either expand or contract by absorption or emission of gravitational radiation.

We shall be discussing throughout the paper the quantum version of Einstein's theory. As is well known, this theory is problematical at high energies or small distances due to its nonrenormalizability. However we believe that our considerations should be insensitive to possible short-distance modifications of the theory (e.g., R^2 or $R^{ab}R_{ab}$ terms in the action, supergravity, etc.) as long as we restrict the temperature T to be small (in natural units, $kT << m_P c^2$). We shall use units in which $\hbar=c=k=1$, and often will express masses, inverse lengths, and temperatures in units of the Planck mass $m_P \equiv (\hbar c/G)^{1/2}$ or $m_P=2.18\times 10^{-5}$ g. When discussing spacetime, we shall use the signature $(-+++)$.

The main purpose of this paper is to use the Euclidean functional integral formalism to explore the instabilities of gravity. It is evident that gravity must exhibit instabilities by the arguments of this section. The Jeans instability, of course, has been known since 1902. Also, it is clear that a second instability should exist. If we imagine a box whose walls are kept at constant temperature, then it will certainly be filled with thermal gravitons. Now if we assume that this system is in some sense ergodic, then all points in phase space will eventually be reached from any reasonable initial state, for example, from a configuration of thermal gravitons. In particular the spontaneous formation of a black hole (as opposed to collapse via the Jeans mechanism) would appear to be a possible process. If this process can happen, it will happen. However, this argument does not yield the rate for such events to occur. One of the main results of our work is the calculation of this rate from the Euclidean formulation of gravity.

Black-hole formation, via nucleation or via the Jeans instability, renders the canonical ensemble ill defined. This is because, even classically, some trajectories run into the boundaries of the allowed region of phase space. Such a catastrophe happens when a black hole starts to grow and engulfs the box in which the system is contained. One way out of this impasse would be to use the microcanonical ensemble to discuss the thermodynamic properties of gravity. We do not know how to carry out such a program.

Although this paper deals with the instabilities of self-gravitating systems its conclusions bear on an issue of much greater importance. In the functional integral formulation of quantum gravity the following question arises. Should one, when summing over path histories of spacetime, include the contributions of all possible metrics, independent of their topology? In ordinary field theories (e.g., gauge theories) the issue of which topological classes of field configurations must be included in the path integral is resolved by energy considerations. Thus for example instantons must be included in the Euclidean path integral (for Yang-Mills theories) in order to construct the correct ground state. Here one can show that this must be so by continuously deforming the naive vacuum into a widely separated instanton-antiinstanton pair with a finite cost in action. Thus one can smoothly construct a configuration that has a nonzero topological charge in any given finite volume.

In gravity, however, energy considerations are notoriously problematic. Furthermore the topology of spacetime is not additive—one can add handles to a manifold (with a positive increase of the Euler character) but there are no corresponding "antihandles." In fact there is no convincing argument that one *must* include anything but continuous deformations of flat space in the path integral. It would be very desirable to investigate a set of physical situations in which this issue arises. Given that one has little hope, with present techniques, of going beyond semiclassical approximations this means that one must come up with very special situations. These must have the property that they give rise to boundary conditions for the Euclidean functional integral that allow for the existence of finite-action gravitational saddle points (instantons) with nontrivial topology. The only case we know of where this condition is met is the example of hot flat space discussed below.

In the case of the canonical ensemble the boundary conditions for the Euclidean functional integral for the partition function do indeed allow two kinds of saddle points. There is, of course, the topologically trivial flat-space saddle point, but also the Schwarzschild instanton with nonzero Euler character. Thus we can address the important issue of whether one should or should not include strange topologies in the path integral in the context of a well-defined calculation.

As described below, we find that the effect of summing over Schwarzschild instantons is to produce a totally reasonable physical process, namely, the nucleation at finite temperature of black holes. If we were to ignore these topologically nontrivial manifolds there would be no mechanism, in the semiclassical limit, of producing this expected process. We therefore present this calculation as evidence that one *should* include nontrivial topologies in the path integral, and that no strange effects need emerge.

The remainder of the article is arranged as follows. In Sec. II we discuss various quantum-mechanical systems at finite temperatures, by exploring the functional integral representation of the partition function. We discuss the stability of various systems. We show how many of the features of the finite-temperature behavior can be calculated using semiclassical approximations. We then examine the issue of stability in gravitation. In Sec. III, we show that the functional integral can be used to explore the gravitational vacuum at zero temperature. Perturbation theory is discussed, and we conclude that, at least semiclassically, the theory is stable. In Sec. IV, we extend our treatment to perturbations of flat space at finite temperature. We show how the Jeans instability emerges in the language of quantum field theory. In Sec. V, we discuss the role of gravitational instantons in the finite-temperature case. We discover that there is a further instability associated with the nucleation of black holes. This process cannot be described by perturbations of flat spacetime. Finally, we discuss some of the physical consequences of this instability in Sec. VI.

II. TOY MODELS

In this section we consider quantum theories describing a single degree of freedom. In particular, we discuss how certain features of the finite-temperature behavior of a system may be extracted from the Euclidean function integral formalism. Most of this material is, or should be, well known.[15] Our intention is to remind the reader of various points which will prove useful when we apply familiar techniques in an unfamiliar context.

The finite-temperature theory is defined as that given by the canonical ensemble. Thus, the density matrix, $\exp(-\beta\hat{H})$ ($\beta\equiv 1/T$, T is the temperature) represents the equilibrium behavior of the system that is weakly coupled to an external heat bath. All thermodynamic quantities may be extracted from the partition function

$$Z\equiv \mathrm{Tr}[\exp(-\beta\hat{H})] . \tag{2.1}$$

For example, the free energy $\mathscr{F}\equiv -\beta^{-1}\ln Z$. The expected value of any observable $\mathscr{O}$ is given by

$$\langle \mathscr{O}\rangle = \mathrm{Tr}[\mathscr{O}\exp(-\beta\hat{H})]/Z . \tag{2.2}$$

Functional integral representations may be derived for these quantities by repeatedly inserting a complete set of states. If the Hamiltonian has the standard form $\hat{H}=\frac{1}{2}\hat{p}^2+V(\hat{x})$, then the partition function is given by

$$\begin{aligned} Z &= \lim_{N\to\infty}\int \frac{dp_1}{2\pi\hbar}\cdots\frac{dp_N}{2\pi\hbar}dx_1\cdots dx_N \left\langle p_N \left| \left[1-\frac{\epsilon}{\hbar}\hat{H}\right]\right| x_N\right\rangle \langle x_N | p_{N-1}\rangle \\ &\quad\times\left\langle p_{N-1}\left|\left[1-\frac{\epsilon}{\hbar}\hat{H}\right]\right| x_{N-1}\right\rangle\cdots\left\langle p_1\left|\left[1-\frac{\epsilon}{\hbar}\hat{H}\right]\right|x_1\right\rangle\langle x_1|p_N\rangle \\ &= \lim_{N\to\infty}\int\left[\frac{dp_i}{2\pi\hbar}\right](dx_i)\exp\left[-\frac{\epsilon}{\hbar}\sum_i[\tfrac{1}{2}p_i{}^2+V(x_i)+ip_i(x_{i+1}-x_i)/\epsilon]\right] \\ &= \lim_{N\to\infty}\int\frac{(dx_i)}{(2\pi\hbar\epsilon)^{1/2}}\exp\left\{-\frac{\epsilon}{\hbar}\sum_i\left[\frac{1}{2}\left[\frac{x_{i+1}-x_i}{\epsilon}\right]^2+V(x_i)\right]\right\} \\ &\equiv\int_{x(0)=x(\beta\hbar)}\mathscr{D}X(t)\exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}dt[\tfrac{1}{2}\dot{x}(t)^2+V(x(t))]\right] . \end{aligned} \tag{2.3}$$

Here $\epsilon\equiv\beta\hbar/N$ and $x_{N+1}\equiv x_1$. The argument of the exponential is $(-1/\hbar)$ times the Euclidean action, $S_E\equiv\int_0^{\beta\hbar}dt[\frac{1}{2}\dot{x}^2+V(x)]$, and the integral is over all periodic trajectories with period $\beta\hbar$.

If we rescale time $t\to\beta\hbar\tau$, then the action becomes

$$\frac{1}{\hbar}S_{\rm cl}=\beta\int_0^1 d\tau\{\tfrac{1}{2}[x(\tau)/\partial\tau]^2/(\beta\hbar)^2+V(x(\tau))\} .$$

This shows that the action of any nonstatic trajectory becomes arbitrarily large as $(\beta\hbar)\to 0$. Therefore, for small $(\beta\hbar)$ the integral is highly peaked about static trajectories, and in the $(\beta\hbar)\to 0$ limit the quantum partition function reduces to the classical result

$$Z_{\rm cl}=\int dx\exp[-\beta V(x)] .$$

This limit may be regarded as either the classical ($\hbar\to 0$, β fixed) limit, or the high-temperature ($\beta\to 0$, $\hbar$ fixed) limit; the important fact is that the temperature becomes arbitrarily large compared to the spacing between quantum levels.

Typically of more interest is the semiclassical limit ($\hbar\to 0$, $\beta\hbar$ fixed). This is equivalent to the weak-coupling, $g^2\to 0$, limit if we replace $V(x)$ by the rescaled potential $(1/g^2)V(gx)$. In this limit the integrand is highly peaked about trajectories which minimize the classical action. These are periodic trajectories obeying the Euclidean equations of motion

$$-\ddot{x}(t)+V'(x(t))=0 . \tag{2.4}$$

Suppose that the potential has the simple form shown in Fig. 2, with $V(x)$ having a single minimum with positive curvature:

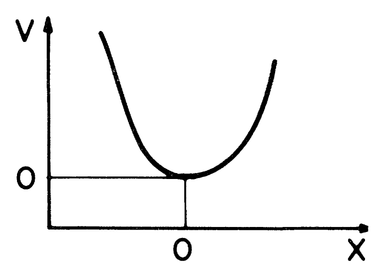

FIG. 2. An example of a potential for which there is no instability.

$$V(x) \approx \tfrac{1}{2}\omega^2 x^2 + O(x^3) .$$

Then the only classical trajectory obeying (2.4) is simply $x(t)=0$. Expanding about this trajectory yields

$$Z = \int_{\text{periodic}} \mathscr{D}x(t) \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} dt[\tfrac{1}{2}(\dot{x}^2+\omega^2 x^2) + O(x^3)]\right\}$$

$$= [\det{}_+(-\partial_t{}^2+\omega^2)]^{-1/2}[1+O(\hbar)]$$

$$= \left[\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}\right][1+O(\hbar)] . \tag{2.5}$$

(Here, $\det_+(-\partial_t{}^2+\omega^2)$ indicates the determinant of $(-\partial_t{}^2+\omega^2)$ on the interval $[0,\beta h]$ with periodic boundary conditions. See Appendix A for the evaluation of such functional determinants.) This, of course, is the standard result for a harmonic oscillator. Higher-order terms in the saddle-point expansion around $x(t)=0$ yield an asymptotic series in powers of $\hbar$.

Now consider a potential of the form shown in Fig. 3. $x=0$ is now only a local minimum of the potential. The genuine equilibrium thermodynamic behavior of the theory obviously depends crucially on the behavior of the potential for $x>b$. A particle initially placed to the left of the barrier will have a finite probability of escaping. However, if the associated lifetime is very long, then it makes sense to speak of the metastable quasiequilibrium state describing particles confined within the potential well. Two different effects contribute to the finite lifetime of this state, quantum-mechanical tunneling through the barrier, and classical thermal excitation over the barrier. At sufficiently low temperatures tunneling will dominate, while at high temperatures thermal excitation will dominate.

To calculate the thermodynamic properties of this metastable state, one may begin by expanding the functional integral (2.3) about the local minimum $x(t)=0$. This yields a free energy

$$\mathscr{F} = \tfrac{1}{2}\hbar\omega\{1+(2/\beta\hbar\omega)\ln[1-\exp(-\beta\hbar\omega)]\} + O(\hbar^2) . \tag{2.6}$$

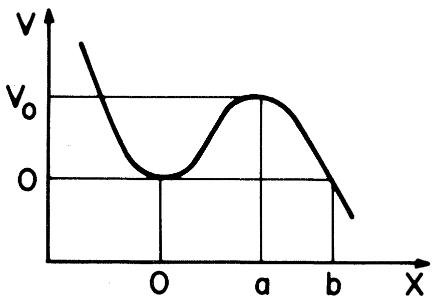

FIG. 3. An example of a potential that exhibits an instability. A particle with zero energy at $x=0$ can tunnel into region $x>b$.

$\mathscr{F}$ is real to all orders in the perturbative expansion around $x=0$: no sign of any instability is found. However, $x(t)=0$ is no longer the only periodic solution of the Euclidean equations of motion (2.4). Since these equations are identical to the usual classical equations of a point particle moving in the potential $-V(x)$, it is trivial to see that other periodic solutions exist which describe a particle oscillating back and forth under the barrier. The period of these trajectories is given by

$$\tau(E) = 2\int_{x_1}^{x_2} dx\{2[V(x)-E]\}^{1/2} , \tag{2.7}$$

where $E \equiv V(x) - \tfrac{1}{2}\dot{x}^2$ is the conserved energy. If $V(x) \sim V_0 - \tfrac{1}{2}\omega_0 x^2$ near $x=a$, then $\tau(E)$ varies from $+\infty$ down to $2\pi/\omega_0$ as E varies from 0 to V_0. Since the only trajectories which contribute to the functional integral are those with period $\beta\hbar$, one finds that for temperatures $0 \le kT \le \hbar\omega_0/2\pi \equiv \beta_0{}^{-1}$ there is another extrema of the functional integral given by the periodic trajectory $x=\bar{x}(t)$ for which $\tau(E)=\beta\hbar$. These solutions are commonly called "bounces" (see Fig. 4).

At temperatures above the critical temperature $\beta_0{}^{-1}$ the periodic bounce degenerates to the static solution, $\bar{x}(t)=a$, which simply sits at the top of the barrier.

One may now try to evaluate the contribution to the functional integral coming from the neighborhood of the bounce. Expanding in $\delta \equiv x - \bar{x}(t)$, one finds

$$\delta Z = \exp\{-S_E[\bar{x}(t)]/\hbar\} \int [\mathscr{D}\delta(t)] \exp\left[-\frac{1}{2\hbar}\int_0^{\beta\hbar} dt\, \delta[-\partial_t{}^2 + V''(\bar{x}(t))]\delta\right] + O(\delta^3)$$

$$= \exp\{-S_E[\bar{x}(t)]/\hbar - \tfrac{1}{2}\ln\det{}_+[-\partial_t{}^2 + V''(\bar{x}(t)) + O(\hbar)]\} , \tag{2.8}$$

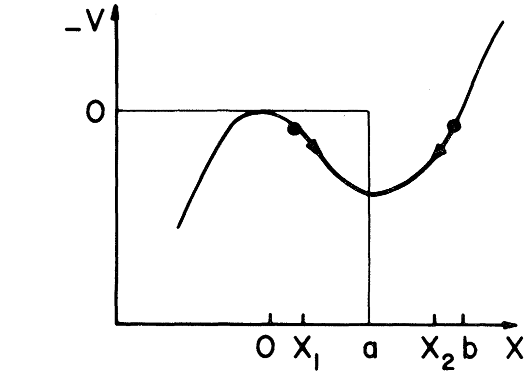

FIG. 4. A plot of $-V$, V being defined as in Fig. 3. A particle can oscillate between x_1 and x_2, the so-called "bounce" solution.

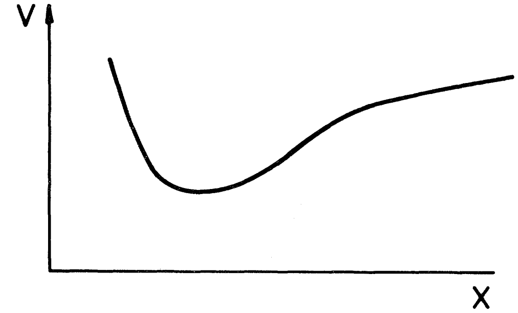

FIG. 5. A potential from which the potential shown in Fig. 3 might be obtained by some suitable analytic continuation.

where

$$S_E[\bar{x}(t)]=\begin{cases}\beta\hbar E+2\int_{x_1}^{x_2}dx\{2[V(x)-E]\}^{1/2}, & \beta>\beta_0\\ \beta\hbar V_0\ , & \beta<\beta_0\ .\end{cases}$$

Although formally correct, this result is ill defined. The quadratic operator $\hat{\mathscr{M}}\equiv-\partial_t{}^2+V''(\bar{x}(t))$ is not positive definite. For $\beta<\beta_0$, $\hat{\mathscr{M}}$ equals $-\partial_t{}^2-\omega_0{}^2$ and the lowest eigenvector, $\delta(t)$ = const, has a negative eigenvalue $-\omega_0{}^2$. For $\beta>\beta_0$, $\delta(t)=\dot{\bar{x}}(t)$ is a zero mode of $\hat{\mathscr{M}}$, $\hat{\mathscr{M}}\dot{\bar{x}}=\partial_t[-\ddot{\bar{x}}+V'(\bar{x}(t))\,]=0$. Since $\dot{\bar{x}}(t)$ changes sign there must be a lower eigenvector with a negative eigenvalue. This shows that $\bar{x}(t)$ is only a saddle point, not a minimum, of $S_E[x(t)]$ and consequently the Gaussian integral leading to (2.8) was actually divergent.

These negative modes could have been predicted from the outset if we had been more precise about the meaning of the metastable state. A careful definition always requires a process of analytic continuation. Suppose we began with a potential of the form shown in Fig. 5, where $x=0$ is the global minimum. The functional integral is perfectly well defined and $\mathscr{F}$ is obviously real. If we now analytically change the potential back to the desired form (Fig. 3), then certain contours of integration in function space will have to be rotated in order for the integral to remain convergent. Examining this in slightly greater detail allows one to show that half of the contour of integration away from the bounce in the direction of the zero mode must be rotated to the imaginary direction.[16,17] Therefore for $\beta<\beta_0$ one finds

$$\begin{aligned}\mathrm{Im}\mathscr{F}&=Z^{-1}(2\beta)^{-1}\,|\det{}_+(-\partial_t{}^2-\omega_0{}^2)\,|^{-1/2}\\&\quad\times\exp(-\beta V_0)\\&=[(\sinh\beta\hbar\omega/2)/(2\beta\sin\beta\hbar\omega_0/2)]\\&\quad\times\exp(-\beta V_0)\ .\end{aligned}\tag{2.9}$$

For $\beta>\beta_0$, one must also deal with the zero mode of $\hat{\mathscr{M}}$. However, this is a simple consequence of the fact that $\bar{x}(t)$ is not time translation invariant. To preserve the symmetry one must simultaneously expand about all time translates of the bounce, $\bar{x}(t+t_0)$, $0\le t_0<\beta h$. This is accomplished by standard collective-coordinate techniques, which automatically remove the zero mode of $\hat{\mathscr{M}}$.[17] One finds

$$\begin{aligned}\mathrm{Im}\mathscr{F}&=\tfrac{1}{2}\hbar Z^{-1}(W/2\pi\hbar)^{1/2}\,|\det{}'_+[-\partial_t{}^2+V''(\bar{x}(t))]\,|^{-1/2}\exp\{-S_E[\bar{x}(t)]/\hbar\}\\&=[(\sinh\beta h\omega/2)/(2\pi\tau'/\hbar)^{1/2}]\exp\{-S_E[\bar{x}(t)]/\hbar\}\ .\end{aligned}\tag{2.10}$$

Here $W\equiv\int_0^{\beta h}dt\,[\dot{\bar{x}}(t)]^2=2\int_{x_1}^{x_2}dx\{2[V(x)-E]\}^{1/2}$, and $\tau'=d\tau(E)/dE$. ($\det'_+$ indicates the determinant with zero modes removed. See Appendix A for its evaluation.)

This shows how the instability of the metastable state is reflected in the presence of nontrivial saddle points in the functional integral. Note that the energy of the bounce with the correct period $\beta\hbar$ is simply the energy for which the probability of escape is largest, i.e., it maximizes the product of the Boltzmann factor, $\exp(-\beta E)$, times the tunneling probability

$$\exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}dx\{2[V(x)-E]\}^{1/2}\right].$$

The critical temperature $T_0=\beta_0{}^{-1}=\hbar\omega_0/2\pi$ is the temperature above which thermal excitation over the top of the barrier is the dominant decay mechanism. The fact that the periodic bounce $\bar{x}(t)$ becomes static above this temperature reflects the fact that this is a classical process.

Finally, one may ask how to relate the imaginary part of the free energy to the actual decay rate Γ. This requires a further WKB calculation. For temperatures $T<T_0$, one finds[18]

$$\Gamma=(2/\hbar)\mathrm{Im}\mathscr{F}\ , \tag{2.11}$$

which reduces to the familiar relation $\Gamma=(2/\hbar)\times\mathrm{Im}E$, when $T\to0$. For $T>T_0$, one finds[18]

$$\Gamma=(\omega_0\beta/\pi)\mathrm{Im}\mathscr{F}\ , \tag{2.12}$$

which reduces to the classical result $\Gamma=(\omega/2\pi)\times\exp(-\beta V_0)$ as $T\to\infty$. [These formulas are correct outside a narrow transition region where $T-T_0=O(h^{3/2})$. They are also correct in multidimensional systems. See Ref. 18 for further details.]

Once a tunneling process has occurred to a new classically allowed region, the subsequent evolution will be governed by classical mechanics until a new equilibrium state is neared.[19] For further discussion of this point see Sec. VI.

III. PERTURBATION THEORY, ZERO TEMPERATURE, AND STABILITY

The action from which we begin our construction of quantum gravity is the Einstein-Hilbert action. This is defined for a Lorentzian metric g_{ab} on a manifold M as

$$I=\frac{1}{16\pi G}\int_M R(-g)^{1/2}d^4x\ . \tag{3.1}$$

R is the Ricci scalar of the metric g_{ab}. In cases of interest, M will have some boundary ∂M, typically a boundary at infinity. If we consider variations of the metric in M whose normal derivatives vanish on ∂M, then extrema of I yield the vacuum Einstein equations

$$R_{ab}=0\ . \tag{3.2}$$

However, in a functional integral, we wish to have an action which reproduces the vacuum Einstein equations under all variations of the metric that vanish on ∂M. To find an action with these properties, we add a surface term to the action,[20] giving

$$I=\frac{1}{16\pi G}\int_M R(-g)^{1/2}d^4x+\frac{1}{8\pi G}\int_{\partial M}K(\pm h)^{1/2}d^3x+C\ . \tag{3.3}$$

h_{ab} is the induced metric on the boundary, the $+$ $(-)$ being taken depending on whether the boundary is spacelike (timelike). K is the trace of the second fundamental form. C is a term which depends only on the metric h_{ab}. It could be absorbed into the measure in the functional integral. However, for spacetimes which admit a single asymptotically flat region, so that ∂M is a timelike tube at infinity, C can be written as

$$C=\frac{-1}{8\pi G}\int K^0(h)^{1/2}d^3x\ . \tag{3.4}$$

K^0 is the second fundamental form of ∂M embedded in flat space. With this choice, the action for Minkowski spacetime is zero.

In classical general relativity, one can ask about the stability of flat spacetime. One would expect that stability would be guaranteed by the positivity of the energy. The total energy of an asymptotically flat spacetime is the Arnowitt-Deser-Misner (ADM) mass.[21] This is

$$E_{\mathrm{ADM}}=\int_{\partial\Sigma}(\hat{h}^k_{l;k}-\hat{h}^k_{k;l})d^2S^l\ , \tag{3.5}$$

where $\hat{h}_{ab}$ is the metric induced on a spacelike hypersurface Σ which has a boundary at infinity which is a large S^2. In vacuum gravitation, $E_{\mathrm{ADM}}\geq0$.[11,12] This result holds even if Σ has a complicated topology resulting from the presence of black holes. If we couple matter with an energy-momentum tensor T_{ab} to the gravitational field, the Einstein equations become

$$R_{ab}-\tfrac{1}{2}Rg_{ab}=8\pi GT_{ab}\ . \tag{3.6}$$

If the energy-momentum density of matter is nonnegative in all frames, i.e., if $T^{00}\geq|T^{ab}|$ for any orthonormal tetrad, then $E_{\mathrm{ADM}}\geq0$.[11,12] Furthermore, if $E_{\mathrm{ADM}}=0$ then the spacetime is Minkowski space. This indicates that Minkowski space is classically stable in the presence of fixed matter sources, since there is nothing into which it could decay.

We wish to explore the structure of quantum gravity rather than classical gravity at both zero and nonzero temperature. In order to do this we follow the methods outlined in Sec. II.

We wish to construct a functional integral representation for the partition function of quantum gravity. A formal construction is not much more difficult here than in the better known case of an Abelian or non-Abelian gauge theory.[22,23] For zero temperature we recover the vacuum-to-vacuum amplitude discussed by Faddeev and Popov[24] and others.[25] We shall therefore give a brief outline of the construction.

Start with the canonical variables[21,26]: $\hat{h}_{ij}(x)$, the components of the three metric, and the canonically conjugate momenta $\pi_{ij}(x)$ $(i,j,k,\ldots=1,2,3)$. The canonical commutation relations are

$$[\pi_{ij}(x),\hat{h}_{kl}(x')]=-i\delta_{i(k}\delta_{l)j}\delta(x,x') .$$

We work within an unphysical Hilbert space spanned by all $h_{ij}(x)$. Physical states must be invariant under gauge (general coordinate) transformations. This requires that they be annihilated by four constraints

$$\mathscr{H}_i=-2\pi_i{}^j{}_{;j} , \tag{3.7}$$

$$\mathscr{H}_0=\mathscr{H}_\perp=\hat{h}^{-1/2}(\pi_{ij}\pi^{ij}-\tfrac{1}{2}\pi^2)-\hat{h}^{1/2}R .$$

(A semicolon denotes the covariant derivative with respect to the metric $\hat{h}_{ij}$.) One can construct a projection operator $\hat{\Lambda}$ that projects out the physical subspace as

$$\hat{\Lambda}\equiv\int\mathscr{D}N_a(\vec{x})\exp\left[i\int d\vec{x}N_b(\vec{x})\mathscr{H}^b(\vec{x})\right] . \tag{3.8}$$

The Hamiltonian is simply

$$\hat{H}=\int d^3x\,\mathscr{H}_\perp-P_\perp ,\quad P_\perp\equiv\oint_{r\to\infty}d^2S^l(\hat{h}_{lr,r}-\hat{h}_{rr,l}) . \tag{3.9}$$

Now we follow the steps of Sec. II, and derive a function integral for

$$Z=\mathrm{tr}(\hat{\Lambda}e^{-\beta H})=\lim_{N\to\infty}\mathrm{tr}(\hat{\Lambda}e^{-\epsilon\hat{H}})^N \quad (\epsilon=\beta/N)$$
$$=\int\mathscr{D}N_a(\vec{x},t)\mathscr{D}\pi_{ij}(\vec{x},t)\mathscr{D}\hat{h}_{ij}(\vec{x},t)\exp\left[\int_0^\beta dt\left[\int d^3x(N_a\mathscr{H}^a+i\pi_{ij}\dot{\hat{h}}^{ij})-P_\perp\right]\right] , \tag{3.10}$$

where the integration is over metrics $\hat{h}_{ij}(\vec{x},t)$ and $N_a(\vec{x},t)$ which are strictly periodic in Euclidean time t: $\hat{h}_{ij}(\vec{x},0)=h_{ij}(\vec{x},\beta)$. The integration over π_{ij}, which only appears quadratically, can now be done explicitly. ly. This can then be written in the standard, Euclidean form of the functional integral for gravity, first derived by Faddeev and Popov[24] (N_a has been rewritten in terms of g_{0a}):

$$Z=\int\mathscr{D}[g_{ab}(x)]\exp[-\hat{I}(g)+\text{gauge-fixing terms}] , \tag{3.11}$$

where the Euclidean action is given by

$$\hat{I}[g]=\frac{-1}{16\pi G}\int Rg^{1/2}d^4x-\frac{1}{8\pi G}\int K\hat{h}^{1/2}d^3x+C . \tag{3.12}$$

Below, we shall specify the form of the gauge-fixing terms.

The functional integral is evaluated by integrating over all metrics which are positive definite and obey appropriate boundary conditions. We are interested in two types of boundary condition which correspond to the zero-temperature vacuum and to the canonical ensemble at temperature $T=1/\beta$.

The boundary conditions appropriate to the vacuum are termed asymptotically Euclidean (AE).[27] An AE metric is one in which the metric approaches the flat metric on R^4 outside some compact set. For the action to be finite, the metric must then look like

$$ds^2=\left[1+\frac{\alpha}{r^2}\right]\delta_{ab}dx^adx^b+O(r^{-3}) , \tag{3.13}$$

where r is a four-dimensional radial coordinate and α a function of the coordinates, but independent of r. The boundary at infinity is topologically S^3.

The boundary conditions for the canonical ensemble at temperature $T=\beta^{-1}$ are termed asymptotically flat (AF).[27] An AF metric is one in which the metric approaches the flat metric on $R^3\times S^1$ outside some compact set. Finite action requires the metric to be asymptotically

$$ds^2=d\tau^2+\left[1+\frac{\alpha}{r^2}\right]\delta_{ij}dx^idx^j$$
$$+\text{terms which fall off faster}\quad (i,j=1,2,3) . \tag{3.14}$$

Here r is a three-dimensional radial coordinate. α can be a function of the coordinates, but is independent of r; τ is a coordinate which is periodic with period β. The boundary of infinity is topologically $S^2\times S^1$. This case will be discussed in detail in Secs. IV and V.

This functional integral construction of quantum gravity is poorly understood. Are we to integrate over all manifolds or perhaps only those topologically equivalent to flat space? How can one render the functional integral well defined when R can be arbitrarily large? These and other questions are the subject of much recent research and speculation.[28] At the moment the only feasible way to treat the functional integral is by saddle-point methods. This is adequate for a treatment of the small perturbations about Minkowski space and for a semiclassical analysis of vacuum stability.

The saddle-point evaluation starts by constructing stationary points of the action, namely, solutions of the Euclidean Einstein equations. Expansions about these saddle points are performed by writing

$$g_{ab}=g_{ab}^{(\text{saddle point})}+\phi_{ab}\ . \tag{3.15}$$

Treating ϕ_{ab} as a quantum field and $g_{ab}^{(\text{saddle point})}$ as a c-number background field will generate the usual perturbation expansion, which can be expressed in terms of Feynman diagrams. The saddle-point metric g_{ab} (normally assumed to be a nonsingular geodesically complete four-manifold) is colloquially termed a gravitational instanton. Different instantons will have various physical interpretations. But first we must find all gravitational instantons consistent with our AE boundary conditions. The positive-action theorem, first proved by Schoen and Yau,[10] states that for any AE metric with $R=0$, the action I is non-negative and $I=0$ if and only if g_{ab} is flat. However the action for any AE instanton must be zero. This follows from the fact that any AE instanton will be a solution of $R_{ab}=0$. Such a metric will always admit a uniform dilatation $g_{ab}\rightarrow\lambda g_{ab}$, which will map the old solution into a new solution. However such a dilatation will map the action $I\rightarrow\lambda I$. But such a dilatation could be produced by a coordinate transformation $x^a\rightarrow\lambda x^a$, which must leave the action invariant. Thus, the action for any AE instanton can only be zero. The positive-action theorem then guarantees that such a metric is flat. For zero temperature we need only examine the perturbations about flat space. Since the action is not positive definite for metrics that do not satisfy $R=0$, there is cause for concern that these perturbations might be unstable.

Let us, therefore, examine the pertubations about an arbitrary saddle-point metric, $\hat{g}_{ab}$. It is useful at this stage to specify the gauge that we will employ. It is simplest to consider the covariant Lorentz gauge

$$\nabla_a(\phi^{ab}-\tfrac{1}{2}\hat{g}^{ab}\phi)=X^b\ , \tag{3.16}$$

where X^b is some specified function of the coordinates. (∇_a is the covariant derivative with respect to the background metric g_{ab}.) We then employ the 't Hooft averaging procedure. The net result is an effective action that contains the field ϕ_{ab} and a set of anticommuting Faddeev-Popov vector fields η^a. This effective action is rather complicated. To simplify it, we introduce the following decompositions[29]:

$$\phi_{ab}=\phi_{ab}^{TT}+\tfrac{1}{4}\hat{g}_{ab}\phi+(\nabla_a\xi_b+\nabla_b\xi_a-\tfrac{1}{2}\hat{g}_{ab}\nabla_c\xi^c)\ . \tag{3.17}$$

ϕ_{ab}^{TT} is the transverse tracefree ($\nabla_a\phi^{abTT}=0$, $\hat{g}^{ab}\phi_{ab}^{TT}=0$) part of ϕ_{ab}; ϕ is the trace part; and ξ_b is the longitudinal tracefree part. The two vector fields ξ_b and η_b are then decomposed according to the Hodge-de Rham decomposition

$$\begin{aligned}\eta_b&=\nabla_b\chi+\eta_b^c+\eta_b^H\ ,\\ \xi_b&=\nabla_b\psi+\xi_b^c+\xi_b^H\ .\end{aligned} \tag{3.18}$$

η_b^c,ξ_b^c are the coexact parts of η and ξ and are hence divergence free. η_b^H,ξ_b^H are the harmonic parts of η and ξ. The number of square-integrable harmonic vectors is equal to the dimensionality of the first cohomology group $H^1(M,R)$ on the manifold associated with metric $\hat{g}_{ab}$. Since we are almost always dealing with simply connected manifolds, we will ignore the harmonic sector in what follows.

We now expand the effective action in terms of ϕ and η. There are no terms linear in ϕ since the vacuum Einstein equations are satisfied. The quadratic terms are

$$\frac{1}{16\pi G}[\tfrac{1}{4}\phi_{ab}^{TT}G^{abcd}\phi_{cd}^{TT}]\ , \tag{3.19a}$$

$$\frac{1}{16\pi G}[-\tfrac{1}{2}\phi F\phi]\ , \tag{3.19b}$$

$$\frac{1}{16\pi G}[2X_{ab}(\psi)X^{ab}(F\psi)]\ , \tag{3.19c}$$

$$\frac{1}{16\pi G}[Y_{ab}(\xi_e^c)Y^{ab}(C_f^e\xi^{cf})]\ , \tag{3.19d}$$

$$\frac{1}{16\pi G}[X_{ab}(\chi)X^{ab}(F\chi)]\ , \tag{3.19e}$$

$$\frac{1}{16\pi G}[Y_{ab}(\eta_e^c)Y^{ab}(C_f^e\eta^{cf})]\ , \tag{3.19f}$$

where

$$\begin{aligned}
&G_{abcd}=-g_{ac}g_{bd}\Box-2R_{acbd}\ ,\\
&X_{ab}(\psi)=(\nabla_a\nabla_b+\nabla_b\nabla_a-\tfrac{1}{2}\hat{g}_{ab}\Box)\psi\ ,\\
&F=-\tfrac{1}{8}\Box\ ,\\
&Y_{ab}(\eta_e)=\nabla_a(\eta_e\delta_b^e)+\nabla_e(\eta_e\delta_a^e)\ ,\\
&G_f^e=-\delta_f^e\Box\ .
\end{aligned} \tag{3.20}$$

These quadratic terms determine the propagators of the fields in the theory. Further terms can be obtained by expanding the effective action to higher order and would include the vertices of the theory. The metric perturbation ϕ_{ab} has ten degrees of freedom, of which five correspond to a spin-2 piece (ϕ_{ab}^{TT}), three to a spin-1 piece (ξ_b^c), and two to spin-0 pieces (ϕ,ψ). The Faddeev-Popov terms contain a spin-1 piece (η_b^c), and a spin-0 piece (χ). If we consider perturbations about flat space ($\hat{g}_{ab}=\eta_{ab}$) then the operators G, F, and C are all manifestly positive definite. However, the term $\phi F\phi$ in the effective action is negative definite. This is the perturbation-theory remnant of the fact that certain conformal transformations can be made on a given metric such as to make the Euclidean action arbitrarily negative.[9] Thus we might think that Z, which is proportional to $[\det(F)]^{-1/2}$, contains a factor of $i^{(\dim F)}$ and that the perturbations about flat space are unstable. However, this negative-metric piece is not significant. First, a detailed examination of the propagator shows that this spin-zero piece does not couple to the conserved energy-momentum tensor of other fields. Thus it cannot represent a physical instability of the system. Second, a more sophisticated analysis, which is performed in a family of covariant gauges, shows that the analog of operator F is a gauge-dependent operator, in contrast to the operators G and C which are explicitly gauge independent.[29] In fact, if we choose to work in the gauge

$$\nabla_a(\phi^{ab}-\hat{g}^{ab}\phi)=X^b \tag{3.21}$$

rather than the Lorentz gauge, we would discover that the operator F vanishes identically. Since all physical observables are gauge invariant, they must be independent of F and so we cannot be troubled by such terms. Third, even if we were to ignore the above arguments and proceed to calculate physical observables in perturbation theory we would find an explicit cancellation of $\det F$. This is because a factor of $\det F$ occurs three times in the evaluation of Z. Two factors of $(\det F)^{-1/2}$ arise from the integrations over ϕ and ψ, and a factor of $\det F$ comes from integrating the Faddeev-Popov ghosts.

In curved space, the situation is slightly different. If the operator F is positive definite then the terms (3.19c) and (3.19e) are also positive definite, when $R_{ab}=0$ as is the case for a classical solution. Suppose that a normalizable eigenfunction of F is ϕ_n with eigenvalue λ_n. Then

$$\begin{aligned}
\int_M \phi_n F\phi_n g^{1/2}d^4x &= \int_M(\nabla_a\phi_n)(\nabla^a\phi_n)g^{1/2}d^4x\\
&\quad+\int_{\partial M}\phi_n(\nabla_a\phi_n)d\Sigma^a\\
&=\lambda_n\int_M \phi_n{}^2 g^{1/2}d^4x\ .
\end{aligned} \tag{3.22}$$

Since ϕ_n must vanish on the boundary ∂M of the manifold M, it follows that $\lambda_n>0$ for all square-integrable eigenfunctions.

The operator C acts on divergence-free vectors. Suppose that it has an eigenfunction ζ_a with eigenvalue λ_n. Then an identity due to Yano and Nagano[30]

$$\begin{aligned}
\int_M [(\nabla_a\zeta^a)^2&-\tfrac{1}{2}(\nabla_a\zeta_b+\nabla_b\zeta_a)(\nabla^a\zeta^b+\nabla^b\zeta^a)\\
&+\zeta^a\Box\zeta_a-2\zeta^aR_{ab}\zeta^b]g^{1/2}d^4x=0
\end{aligned} \tag{3.23}$$

shows that C is positive semidefinite, the zeros being associated only with Killing vectors. However, deformations constructed from Killing vectors are not included in functional integrals as this would overcount field configurations. This is because a Killing vector refers to a continuous symmetry of a background metric and maps a space into the identical space. Therefore (3.19d) and (3.19f) are positive for $R_{ab}=0$.

The situation is rather different for the operator G. One cannot prove that it is positive definite, and indeed, in Sec. V, where we discuss finite-temperature instantons, we will encounter a space in which G has both zero and negative eigenvalues.[31] The zero modes are associated with transformations between distinct solutions of the Einstein equations with the same action. There is one such mode for each degree of freedom of the instanton, and these modes will be handled by the collective-coordinate method. The negative eigenvalues of G do have physical significance and will lead to the instability of hot flat space.

We therefore see that flat space at zero temperature is stable quantum mechanically as well as classically. Of course we can only verify this for small perturbations about flat space, but it seems unlikely, in view of the positive-action theorem, that nonperturbative instabilities could arise. However, one must note that quantum gravity is probably nonrenormalizable beyond the one-loop level, and that any statement regarding quantum gravity to all orders in perturbation theory is dangerous.

IV. FINITE-TEMPERATURE PERTURBATION THEORY

We shall now investigate quantum gravity at finite temperature. We wish to describe the properties of a system placed in an arbitrarily large spatial volume which is kept at some fixed temperature $T=1/\beta$. The equilibrium state of such a system will be described by the canonical partition function

$$Z \equiv e^{\beta\mathscr{F}} = \mathrm{Tr}\, e^{-\beta H}\Big|_{\text{physical degrees of freedom}} \tag{4.1}$$

for which we derived a functional integral representation in the previous section [Eq. (3.11)]. This system describes the purest vacuum of all; in the absence of matter only fluctuations of the metric field are present. It is these that we must sum over in the functional integral, integrating over all asymptotically flat, Euclidean four-metrics, which are periodic in Euclidean time with period β: $g_{ab}(\tau,x)=g_{ab}(\tau+\beta,x)$.

Once again we only know how to treat this problem in the semiclassical approximation. To this end we must first find all saddle points of the classical action, i.e., periodic Euclidean finite-action solutions of Einstein's equations, and expand about each one. There is one trivial periodic solution, namely, flat space. The metric $g_{ab}(x) =\delta_{ab}$ is clearly periodic and has zero action. The contribution of this saddle point to $\mathscr{F}$, to lowest order in perturbation theory, will simply be the free energy of an ideal gas of gravitons at temperature T. In higher orders the interaction free energy of the gravitons will appear and will produce an (Jeans) instability. In the following section we shall consider the contribution of other (instanton) saddle points.

Perturbation theory about flat space at finite temperature proceeds much the same way as at zero temperature, the only difference being that the fields $g_{ab}(x)$ are periodic in t with period β. Thus the Euclidean frequencies p_0 are quantized in units of $2\pi n/\beta=\omega_n$ and integrals over p_0 are replaced by discrete sums

$$\int dp_0 \rightarrow \frac{2\pi}{\beta}\sum_{\omega_n} .$$

Otherwise the zero-temperature Feynman rules are unchanged. Note that if fermionic matter fields are included in the theory, they must be antiperiodic in t, and therefore their frequencies are quantized in units of $\omega_n=(2n+1)\pi/\beta$. On the other hand Faddeev-Popov ghosts, although fermionic, serve to represent a bosonic determinant and thus must be periodic.[23,32]

The free energy is given by the sum of all vacuum graphs, evaluated with finite-temperature propagators. To lowest order

$$\begin{aligned} Z &= e^{\beta\mathscr{F}} \\ &= [\det(-\Box)]^{-1/2}_{T,\text{physical degrees of freedom}} . \end{aligned} \tag{4.2}$$

$\Box$ is the appropriate second-order differential operator. At zero temperature, $\mathscr{F}$ is (quartically) divergent; however the temperature-dependent part of $\mathscr{F}$ is ultraviolet finite. The calculation is straightforward, yielding the standard result for the free energy of a relativistic gas of massless particles with two (helicity) degrees of freedom

$$\frac{\mathscr{F}_0(T)}{V} = -\frac{\pi^2 T^4}{45} = \frac{2}{\beta}\int \frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta k}) . \tag{4.3}$$

One can also readily calculate higher-order [in $(16\pi G)^{1/2}$] corrections to the free energy, arising from the self-interactions of the gas of gravitons. These will be given by vacuum Feynman graphs that contain cubic (or higher-order) vertices. There are two sources of trouble that appear in this perturbative expansion. First, since pure gravity is unrenormalizable, the ultraviolet divergences will be uncontrollable. We have nothing to say about this problem. However, in addition we expect infrared instabilities to show up once we allow for graviton-graviton interactions. As discussed above, a homogeneous matter distribution develops a (Jeans) instability for time-dependent fluctuations of wavelength bigger than $1/k_J$, where $k_J =(4\pi G\rho)^{1/2}$ ($V_s=1$). We expect such an instability to appear in our gas of gravitons at finite temperature since it does contain "matter", namely, the thermally excited gravitation modes themselves. These carry energy and thus are a source for the gravitational field, leading to the Jeans instability.

How does this instability appear in the partition function? The standard analysis of the Jeans instability, as performed in the Introduction, considers *time-dependent* fluctuations of a homogeneous medium. The Jeans wave number k_J is the maximum wave number for which the frequency of these fluctuations is still real. One can interpret this result by saying that due to the antiscreening effects of gravity in a relativistic gas of density ρ (velocity of sound = 1), the graviton acquires an imaginary "mass," $m_J{}^2 = -4\pi G\rho$. This is the analog of the usual plasmon.[33] In our Euclidean calculation, however, we are interested in the static equilibrium properties of the system, in particular in the response of the gravitational field to spatial, time-independent fluctuations of the medium. Here too we expect gravitational antiscreening, which will produce an instability; however the static graviton "mass" need not have the Jeans value. This too is familiar from the analogous case of a plasma.[23] There one finds zero-momentum excitations of the photon have a mass equal to $m_{\rm el}/\sqrt{3}$. On the other hand, the inverse screening length of the plasma, which governs the long-range correlations of the charge density $\rho(x)$, is given by $m_{\rm el}{}^2 = \frac{1}{3}g^2T^2$, namely,

$$\langle \rho(\vec{x})\rho(0)\rangle \underset{|\vec{x}|\to\infty}{\sim} e^{-m_{\rm el}|\vec{x}|} .$$

If we calculate the electric screening of a plasma by evaluating Euclidean functional integrals for a charged gas at finite temperature, it is the latter mass that is generated. To evaluate the plasmon frequency one must analytically continue the resulting phonon propagator back to Minkowski space.

We shall now present a very simple argument that the graviton mass which appears in Euclidean propagators is in fact twice the Jeans value, namely,

$$m_g{}^2 = -16\pi G\rho(T) , \tag{4.4}$$

where ρ is the thermal density of gravitons ($\rho = \pi^2 T^4/15$). In fact the above result holds for any kind of massless "matter."

We wish to consider the static correlation of the gravitational field with itself in the presence of a thermal gas of massless particles. Imagine placing a very small test mass M into the system at the origin. At large distances we can use the weak-field (Newtonian) approximation, whence the gravitation potential ϕ, defined by $g_{00} = -1 + 2\phi$, satisfies

$$-\nabla^2\phi = 4\pi GM\delta^3(\vec{x}) + 4\pi G\delta\rho(\phi, T) . \tag{4.5}$$

$\delta\rho(\phi,T) = \rho(\phi,T) - \rho(0,T)$ is the change in the energy density of the thermal gas at temperature T due to the gravitation potential ϕ. We can evaluate the energy density far away from the origin, where ϕ can be regarded as small and uniform, by applying the equivalence principle. Thus the energy density of the gas at $\vec{x}$ is the same as if the temperature were $T[1+\phi(x)]$ and there was no gravitational field [since if T is the temperature of the heat bath at infinity, at $\vec{x}$ all energies are redshifted by an amount $1+\phi(x)$]. Therefore,

$$\rho(\phi,T) = N \int \frac{d^3k}{(2\pi)^3} \frac{k}{\exp[\beta k/(1+\phi)] - 1} = (1+\phi)^4 \rho(T) . \tag{4.6}$$

(N = number of degrees of freedom of the massless particles.) Therefore far away from the source, where $\phi << 1$, we have

$$[-\nabla^2 - 16\pi G\rho(T)]\phi(\vec{x}) = 4\pi GM\delta^3(\vec{x}) . \tag{4.7}$$

Thus for static weak fields the graviton acts as if it had an imaginary mass given by Eq. (4.4).

According to this argument the graviton should develop an imaginary mass when it couples to thermally excited matter of any kind. We shall illustrate how this emerges in perturbation theory in the simple case of gravitons coupled to massless fermions. We wish to calculate the propagator of the graviton field $h_{ab} = g_{ab} - \delta_{ab}$ to one-loop order. The full propagator may be expressed in terms of the one-particle irreducible self-energy, $\Pi_{ab,cd}(\omega_n, k)$,

$$G_{ab,cd}(\omega_n,\vec{k}) = [(\omega_n{}^2 + \vec{k}^2)\delta_{c(a}\delta_{b)d} + \Pi_{ab,cd}(\omega_n,\vec{k})]^{-1} + \text{gauge terms} , \tag{4.8}$$

which in turn is given by the one-loop Feynman diagram in Fig. 6. The vertex is given by the energy-momentum tensor of a massless fermion. At $T=0$ gauge invariance, plus Euclidean invariance, implies that $k^a\Pi_{ab,cd}(k) = 0$, and this in turn requires that at $T=0$, $\Pi_{ab,cd}(0) = 0$. However at finite temperature the energy k_0 is quantized, and there is no such constraint on the longitudinal self-energy $\Pi_{00,00}$. In fact, one can easily show that if we first set $k_0=0$, and then examine $\Pi_{ab,cd}(0,\vec{k})$ as $\vec{k}\to 0$ that only $\Pi_{00,00}(0,\vec{k})$ can be nonvanishing. To evaluate this term we simply evaluate the contribution of the diagram exhibited in Fig. 6 to $\Pi_{00,00}(0,0)$:

$$\Pi_{00,00}(0,0)=-16\pi G\int \frac{d^3p}{(2\pi)^3}\frac{1}{\beta}\sum_n \mathrm{Tr}\left[\gamma_0 p_0\frac{1}{p}\gamma_0 p_0\frac{1}{p}\right], \tag{4.9}$$

where we recall that the fermion energy p_0 take values of $(2n+1)\pi/\beta=\omega_n$. This can be calculated by standard contour techniques[23] and yields the result

$$\Pi_{ab,cd}(0)=\delta_{a0}\delta_{b0}\delta_{c0}\delta_{d0}(-\tfrac{14}{15}\pi^3GT^4)\ . \tag{4.10}$$

This shows indeed that the longitudinal graviton h_{00} develops a one-loop "mass" due to thermal fluctuations, of magnitude

$$m_g{}^2=-\tfrac{14}{15}\pi^3GT^4=-16\pi G\rho_f\ , \tag{4.11}$$

in accord with our expectation (recall that the density of a massless fermi gas is $\rho_f=\frac{7}{120}\pi^2T^4$).

The same effect will be produced even if there are no explicit matter fields present. According to our previous argument the one-loop contribution of thermal gravitons to $\Pi_{00,00}(0)$ will produce a "mass" of[34]

$$m_g{}^2=-16\pi G\rho_g=-\tfrac{16}{15}\pi^3GT^4\ . \tag{4.12}$$

The fact that the graviton acquires an imaginary "mass" at finite temperature means that flat space is unstable. Flat space, which was an absolute minimum of our classical action (with, of course, AE boundary conditions), becomes merely a saddle point of the effective action once we take into account the interactions of the thermal gravitons. (This is illustrated in Fig. 7.) The mechanism for instability is clear, large-scale density fluctuations of the gas of gravitons tend to grow owing to the attractive (antiscreening) gravitational forces. Presumably, these eventually collapse to form black holes. Indeed if we were to calculate the higher-order contributions to the free energy, we would encounter the increasing infrared-divergent "ring" diagrams of Fig. 8. The sum of these yields a contribution to the free energy of

$$\frac{1}{2\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}\ln[1+\Pi(k)/k^2]\ , \tag{4.13}$$

which is complex [since $\Pi_{00,00}(0)<0$].

This Jeans instability of hot flat space suffices to call into question the ability to treat hot gravity by semiclassical methods. In fact unless there is some stabilizing mechanism in the theory it is questionable whether there exist *any* fixed-temperature equilibrium states. If someone provides an indefinite amount of energy in order to keep the walls of our container at a finite temperature, it might be that gravitational collapse continues to occur until the resulting black hole engulfs the walls themselves.

Nevertheless we shall continue to employ the semiclassical approximation in the following section, where we investigate the contribution of instanton saddle points. This is not only because of the interest in elucidating the significance of the Euclidean Schwarzschild solution (the instanton), but also since the instability generated by this mechanism is totally different from the Jeans in-

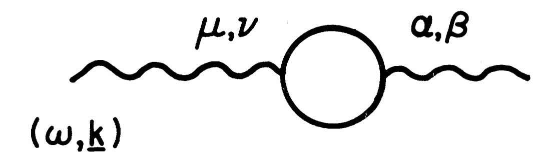

FIG. 6. The diagram which gives a contribution to the polarization tensor.

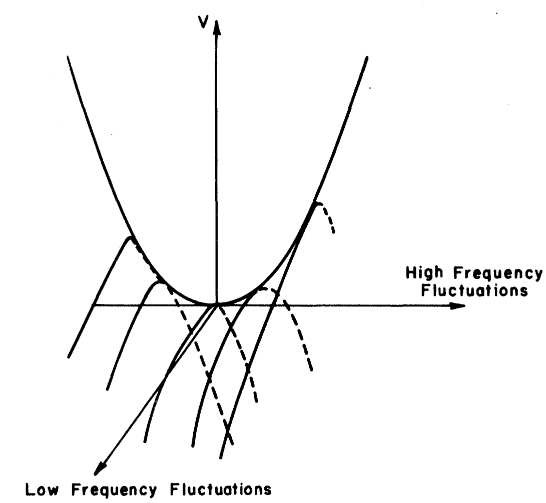

FIG. 7. The effective one-loop potential surface close to flat space-time at nonzero temperature. The potential increases for short-wavelength fluctuations but decreases for long-wavelength fluctuations.

stability. It corresponds, not to long-wavelength fluctuations about flat space, but to the spontaneous nucleation of black holes. Furthermore, the radius of these black holes is smaller than the Jeans length by a factor of T/m_P (m_P=Planck mass). Thus one could imagine enclosing the system in a finite volume of size less than the Jeans length, which would eliminate all but the black-hole nucleation instability.

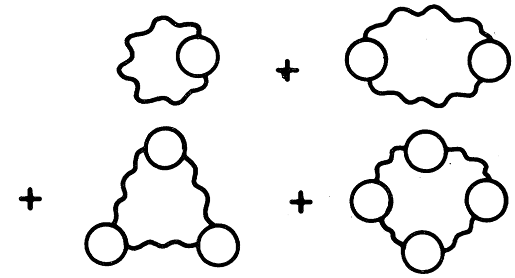

FIG. 8. The ring diagrams that contribute to the imaginary part of F.

V. GRAVITATIONAL INSTANTONS

In the previous section, we discussed a particular AF gravitational instanton, namely, flat space, with the topology of $R^3 \times S^1$. However, unlike the AE, zero-temperature case, flat space is not the unique instanton; there exist other periodic solutions of the Euclidean equations of motion. A familiar instanton is the Euclidean section of the Schwarzschild solution,[35] which is a special case of the Kerr instanton.[36] In Boyer-Lindquist coordinates (t,r,θ,ϕ) this metric takes the form

$$ds^2=(r^2-a^2\cos^2\theta)\left[\frac{dr^2}{\Delta}+d\theta^2\right]+\frac{1}{r^2-a^2\cos^2\theta}(\Delta(dt+a\sin^2\theta\,d\phi)^2+\sin^2\theta[(r^2-a^2)d\phi-a\,dt]^2)\ , \tag{5.1}$$

where

$$\Delta=r^2-2GMr-a^2\ . \tag{5.2}$$

This describes a two-parameter family of solutions. In addition to the translational degrees of freedom, there is a rotation parameter a and a mass parameter M. This metric is positive definite as long as the radial coordinate r is restricted to the region $GM+(G^2M^2+a^2)^{1/2}\le r<\infty$. The region $r<GM+(G^2M^2+a^2)^{1/2}$ is not part of the space that we are interested in. The locus $r=GM+(G^2M^2+a^2)^{1/2}$ will be a conical singularity unless we identify the point (t,r,θ,ϕ) with the point $(t+2\pi\kappa^{-1},\ r,\ \theta,\ \phi+2\pi\Omega\kappa^{-1})$, where

$$\kappa=\frac{(G^2M^2+a^2)^{1/2}}{2GM[GM+(G^2M^2+a^2)^{1/2}]} \tag{5.3}$$

and

$$\Omega=\frac{a}{2GM[GM+(G^2M^2+a^2)^{1/2}]}\ . \tag{5.4}$$

Furthermore, $\theta=0,\pi$ define symmetry axes with rotations about them generated by the Killing vector $\partial/\partial\phi$. For these axes to be nonsingular, ϕ must be an angular variable with period 2π.

Since our fields must be periodic in t, with period β, it is clear that κ must be fixed to equal

$$\kappa=2\pi T\ . \tag{5.5}$$

Also Ω must equal the chemical potential associated with the angular momentum. The instanton is also characterized by its topological properties. Consider the two-surface $r=GM+(G^2M^2+a^2)^{1/2}$, which is the fixed point of the orbit of the Killing vector $\partial/\partial t$. Such a fixed-point set is sometimes called a "bolt."[37] One can use the G-index theorems to find the Euler character and Hirzebruch signature of the manifold. This bolt has zero self-intersection number and an Euler number of two; thus they equal two and zero, respectively. Finally the action of the instanton is

$$I=\frac{\beta M}{2}\ . \tag{5.6}$$

It has been widely conjectured that the Kerr instanton is the unique AF instanton other than flat space.[38] In any case it is the only one we shall consider.

The above form for the action looks very much like the contribution of a particle, or soliton, of mass $M/2$. Does the instanton represent the contribution of black holes to the partition function?

We do not believe so. Recall that M is not a fixed mass, but rather determined in terms of Ω and T by Eqs. (5.3)–(5.5). If we consider the Schwarzschild case, where $a=\Omega=0$, and thus $\kappa=2\pi T=1/4GM$, the action is given by

$$I=\frac{\beta^2}{16\pi G}=\frac{m_P{}^2}{16\pi T^2}\,, \tag{5.7}$$

which is unlike the contribution of any fixed-mass particle to the partition function. In fact we shall argue that the instanton is not a soliton, but rather provides a mechanism for the nucleation of black holes.

There are also, however, other configurations which are arbitrarily close to solutions of the vacuum Einstein equations. These will be needed when we actually calculate the instanton contributions to Z. These configurations are in a distinct topological class to the Kerr solution. We will discuss in detail the sector which has Euler character $\chi=4$ and zero angular momentum. Part of the $\chi=2N$ sector is discussed by Gibbons and Perry,[39] and the region where the rotation is not zero can be obtained by analytic continuation of the solutions of Hauser and Ernst,[40] Kinnersley *et al.*,[41] and Kramer and Neugebauer.[42] These are all very complicated but qualitatively similar to the case discussed below. We discuss the metric

$$ds^2=Vdt^2+V^{-1}[e^{2k}(d\rho^2+dz^2)+\rho^2 d\phi^2]\,. \tag{5.8}$$

ρ,ϕ,z form a cylindrical polar coordinate system, ϕ being identified with period π. The solution is considered to be static and axially symmetric about the $\rho=0$ axis. Hence V and k are functions of ρ and z only:

$$V=\frac{r_1'+r_1'-2GM}{r_1'+r_1''+2GM}\,\frac{r_2'+r_2''-2GM}{r_2'+r_2''+2GM}\,, \tag{5.9}$$

$$k=\tfrac{1}{4}\sum_{n,m=1,2}\ln\frac{r_n'r_m''+(z-z_n-GM)(z-z_m+GM)+\rho^2}{r_n'r_m'+(z-z_n-GM)(z-z_m-GM)+\rho^2}\;\frac{r_n''r_m'+(z-z_n+GM)(z-z_m-GM)+\rho^2}{r_n''r_m''+(z-z_n+GM)(z-z_m-GM)+\rho^2}\,, \tag{5.10}$$

$$r_n'^2=\rho^2+(z-z_n-GM)^2\,,\qquad r_n''^2=\rho^2+(z-z_n+GM)^2\,. \tag{5.11}$$

The Killing vector $\partial/\partial t$ has fixed points at $\rho=0$, $z_1-GM<z<z_1+GM$; and $\rho=0$, $z_2-GM<z<z_2+GM$. These fixed points appear to be rods in the ρ-z plane. However, they are in fact two-surfaces in a four-manifold. For these surfaces to be free of conical singularities, t must be identified with period $8\pi GM$ exactly as in the Kerr case. As ρ, z tend to infinity, $V\to 1$ and $k\to 0$ and thus the metric becomes flat. The periodicity in t means that this metric has AF boundary conditions with a temperature of $T=(8\pi GM)^{-1}$. This metric now has $R_{ab}=0$ everywhere except where $\rho=0$, $z_1+GM<z<z_2-GM$, assuming that $\Delta z=z_2-z_1+2GM>0$. This is the locus corresponding to the gap between the rods in the ρ-z one. There is a conical singularity here which cannot be eliminated by any identification of coordinates compatible with these already performed. However, a conical singularity has the effect of introducing a δ function into the Ricci curvature scalar. Consequently, the contribution to the action from such a singularity is finite. In this case, the action turns out to be

$$I=8\pi GM^2-\beta\frac{GM^2}{(\Delta z+2GM)}\,. \tag{5.12}$$

This is the action we would expect for two "bolts," each of mass M together with a Coulomb interaction between the two masses. In this sense instantons behave as particles; they have normal long-range gravitational interactions. This indicates that we should not expect to be able to find any stationary axisymmetric solutions. But if the masses are separated by arbitrarily large distances, we get arbitrarily close to solutions of the vacuum Einstein equations. In the leading semiclassical approximation we must sum over all these configurations.

We shall consider only the Schwarzschild instanton, with $\Omega=a=0$. To evaluate its contribution to

Z we must integrate over the Gaussian fluctuations about the background Schwarzschild metric. In particular we shall study the stability of such fluctuations using the methods developed in Secs. II and III.

The operators F and C are positive definite. They can be treated by standard methods. The operator G is rather more complex. To determine its eigenvalues we study the solutions of

$$-\Box\phi^{ab}-2R^{acbd}\phi_{cd}=\lambda\phi^{ab}\,, \tag{5.13}$$

where ϕ^{ab} are transverse, tracefree, and normalizable. A variant of this problem has been treated by Regge and Wheeler.[43] They investigated the Lorentz version of this problem with $\lambda=0$. Their methods were refined by subsequent workers, Vishveshwara,[44] Zerilli,[45] Press and Teukolsky,[46] Stewart,[47] and Chandrasekhar.[48] $\lambda=0$ corresponds to a small perturbation of the black hole that remains a classical solution. These authors searched for runaway solutions of the form $\exp(-i\omega t)\times$(function of spatial variables with ω complex). They demonstrated that $\mathrm{Im}\omega=0$ for all solutions of (5.13) and concluded that black holes were classically stable objects.

We, on the other hand, are interested in solutions to (5.13), where g_{ab} is the Euclidean Schwarzschild solution and λ is not necessarily zero. Positive (negative) values of λ will correspond to stable (unstable) Gaussian fluctuations about our instanton.

This equation can be separated in (t,r,θ,ϕ) coordinates, and is exhibited in Appendix B. We then follow the approach of Regge and Wheeler[43] and divide the space of eigenfunctions into even and odd parity:

$$\phi_{ab}^{(\mathrm{even})}=\begin{bmatrix} H_0(r) & H_1(r) & K_0(r)\partial_\theta & K_0(r)\partial_\phi \\ (\mathrm{sym}) & H_2(r) & K_1(r)\partial_\theta & K_2(r)\partial_\phi \\ (\mathrm{sym}) & (\mathrm{sym}) & r^2[G_1(r)+G_2(r)]\partial_\theta{}^2 & r^2G_2(r)(\partial_\theta\partial_\phi-\cot\theta\partial_\phi) \\ (\mathrm{sym}) & (\mathrm{sym}) & (\mathrm{sym}) & r^2[G_1(r)\sin^2\theta+G_2(r)(\partial_\phi{}^2+\sin\theta\cos\theta\,\partial_\theta)] \end{bmatrix}\exp(i\omega t)Y_{lm}(\theta,\phi)\,, \tag{5.14}$$

$$\phi_{ab}^{(\mathrm{odd})}=\begin{bmatrix} 0 & 0 & \dfrac{-h_0(r)}{\sin\theta}\partial_\phi & h_0(r)\sin\theta\,\partial_\theta \\ 0 & 0 & \dfrac{-h_1(r)}{\sin\theta}\partial_\phi & h_1(r)\sin\theta\,\partial_\theta \\ (\mathrm{sym}) & (\mathrm{sym}) & h_2(r)\left[\dfrac{1}{\sin\theta}\partial_\theta\partial_\phi-\dfrac{\cos\theta}{\sin^2\theta}\partial_\phi\right] & \tfrac{1}{2}h_2(r)\left[\dfrac{1}{\sin\theta}\partial_\phi{}^2+\cos\theta\,\partial_\theta-\sin\theta\,\partial_\theta{}^2\right] \\ (\mathrm{sym}) & (\mathrm{sym}) & (\mathrm{sym}) & -h_2(r)(\sin\theta\partial_\theta\partial_\phi-\cos\theta\,\partial_\phi) \end{bmatrix}$$
$$\times\exp(i\omega t)Y_{lm}(\theta,\phi)\,. \tag{5.15}$$

Substitution of these forms into (5.13), together with the conditions of tracefree and transversality applied to ϕ_{ab}, lead to sets of coupled ordinary differential equations in r. By applying Sturm-Liouville techniques, it is possible to show that for even perturbations with $l\geq 2$, and for odd perturbations with $l\geq 1$, that any eigenvalue λ must be positive. The argument fails for $l=0$ or 1 perturbations. If $l=0$ and $\omega>0$, then $\lambda>0$. If $l=1$ and $\omega=0$, then we find three zero modes which are

$$\phi_{ab}^{(i)}=\nabla_a\nabla_b\phi^{(i)}\,, \tag{5.16}$$

$$\phi^{(i)}=(r-GM)\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}. \tag{5.17}$$

At first sight these zero modes look like gauge transformations. However, the vectors $\xi_a^{(i)}=\nabla_a\phi^{(i)}$ are nonnormalizable. Thus, these modes do not correspond to nonsingular gauge transformations and must be included in the functional integral. They simply represent translations in the x, y, and z directions of the origin of the Schwarzschild instanton. They are dealt with by the standard collective-coordinate method.[49]

For $l=0$, we are forced to resort to numerical methods. Write the perturbations as

$$\phi^{ab}_{(-)}=\operatorname{diag}\left[\left[1-\frac{2m}{r}\right]^{-1}h_0(r),\left[1-\frac{2m}{r}\right]h_1(r),k(r),k(r)\csc^2\theta\right]\exp(-i\omega t)\ . \tag{5.18}$$

Since ϕ^{ab} is tracefree,

$$h_0+h_1+2k=0\ . \tag{5.19}$$

Since it is transverse, it follows that

$$\frac{r-2GM}{r}h_1'+\frac{2r-3GM}{r^2}h_1+2(2GM-r)\frac{k}{r^2}-\frac{GM}{r^2}h_0=0\ , \tag{5.20}$$

where prime$\equiv d/dr$. The radial equations now become

$$\frac{r-2GM}{r}h_1''+\frac{4r^2-22GMr+24G^2M^2}{r^2(r-3GM)}h_1'-\frac{8GM}{r^2(r-3GM)}h_1=-\lambda h_1\ . \tag{5.21}$$

This equation has regular singular points at $r=0$, $r=2GM$, and $r=3GM$ and an irregular singular point at $r=\infty$. Its solutions are not explicitly known. Near $r=2GM$, $h\sim(r-2GM)^\sigma$, $\sigma=0,-1$. For large r, $h\sim\exp(\pm|\lambda|^{1/2}r)$. The solutions for large r are acceptable only for an appropriate choice of sign, and at $r=2GM$, only if $\sigma=0$. The technique for finding eigenfunctions of this type is to integrate out from $r=2GM$ for trial values of λ. Only one value of λ was found to be consistent by this procedure, namely, $\lambda\simeq-0.19\,(GM)^{-2}$. If we start with $h_0(2GM)=h_1(2GM)=-k(2GM)+1$, then we discover that h_0, h_1, and k all tend to zero at infinity monotonically. This is in agreement with the naive expectation that the lowest eigenvalue corresponds to the "smoothest" eigenfunction. Defining the normalization as

$$N^2=\int_M\phi^{ab}_{(-)}\phi_{(-)ab}g^{1/2}d^4x\ , \tag{5.22}$$

we discover that

$$N^2\simeq112\beta(GM)^{-4}\ . \tag{5.23}$$

Henceforth, we will deal with the normalized eigenfunction $\phi^{ab}_{(-)}N^{-1}=\tilde{\phi}^{ab}$.

We have therefore discovered an unstable mode for small fluctuations about the Schwarzschild instanton. This instanton is therefore not a strict minimum of the action, but rather a saddle point. Its role in the thermodynamics of hot gravity is similar to that of the configuration $x=a$ in our toy model (see Fig. 3). It behooves us to investigate what happens when we roll off the top of the barrier. We therefore consider the effect of an infinitesimal perturbation of this type. It generates the metric

$$g_{ab}=g_{ab}^{(\text{Schwarzschild})}+\epsilon\tilde{\phi}_{ab}\ . \tag{5.24}$$

This space is spherically symmetric, periodic, and static, but it is not a solution of the vacuum Einstein equations. To investigate further, we notice that since

$$g_{ab}=\operatorname{diag}\left[\left[1-\frac{2GM}{r}\right]\left[1+\frac{\epsilon h_0}{N}\right],\left[1-\frac{2GM}{r}\right]^{-1}\left[1+\frac{\epsilon h_1}{N}\right],r^2\left[1+\frac{\epsilon k}{N}\right],r\left[1+\frac{\epsilon k}{N}\right]\sin^2\theta\right]\ , \tag{5.25}$$

it is convenient to define a new circumferential radial coordinate ρ so that spheres $\rho=$const have area $A=4\pi\rho^2$:

$$\rho=r\left[1+\frac{\epsilon k}{N}\right]^{1/2}\ . \tag{5.26}$$

Thus, the area of the fixed point of $\partial/\partial t$, a two-sphere, which used to have area $A=16\pi G^2M^2$ now has area $A=16\pi G^2M^2+0.94\epsilon G$. Note that this surface remains nonsingular without any change in the periodicity of t. Since the "mass" of such a two-sphere is defined to be $m=(A/16\pi G^2)^{1/2}$, this mass of the "bolt" of the new configuration is

$$M+0.0094\epsilon(GM)^{-1} . \qquad (5.27)$$

Another measure of mass is the total mass at infinity. This is determined by the trace of the second fundamental form on the boundary at infinity. This is determined by the boundary contribution to the action, and this mass remains unchanged. Accordingly we may think of the transformation as producing a spherically symmetric cloud of material outside the "bolt." This matter will have positive or negative energy density depending on whether ϵ is positive or negative. The action however, will decrease because although the boundary term remains constant, there is a contribution from the volume term:

$$I=4\pi GM^2-0.00094/(GM)^2\epsilon^2 . \qquad (5.28)$$

Following the discussion of the toy model in Sec. II, we shall interpret the Schwarzschild instanton as indicating a finite probability for black holes, of mass $M=\beta/8\pi G$, to nucleate. The rate of nucleation will be calculated below. The unstable mode will correspond to the subsequent expansion (or collapse) of the black hole as it absorbs (or emits) thermal radiation.

One might be puzzled as to whether we should include such instantons at all in the functional integral. After all they are topologically distinct from flat space (Euler character 2 instead of zero), and perhaps there exists an infinite barrier that prevents such configurations from developing. This objection is clearly fallacious since the instanton action is finite; thus the nucleation rate is nonzero. However we can also show that by singular distortions of the metric (which however never cause the action to diverge), one can "continuously" deform the instanton into flat space. To do this we take the Schwarzschild metric and identify the Euclidean time with period β. However, we now take the mass at infinity m to be arbitrary rather than $M=\beta/8\pi G$. The resultant space will have a conical singularity at $r=2Gm$. The action is

$$I=\tfrac{1}{2}m\beta+2Gm^2\left[\frac{\beta}{4mG}-2\pi\right]$$

$$=m\beta-4\pi Gm^2 . \qquad (5.29)$$

The action is extremal when $m=\beta/8\pi G$, as expected. This configuration allows for a continuous variation from zero action at $m=0$, flat space, to a maximum at $m=\beta/8\pi G$.

As m increases further, the action decreases without bound. This illustrates the well-known fact that in relativity a topology change can be accomplished continuously without generating a configuration with infinite action. This deformation cannot be achieved in perturbation theory. To generate such a perturbation, we would need to consider the tensor

$$\phi_{ab}=\frac{\partial}{\partial m}(g_{ab})$$

$$=\text{diag}\left[-\frac{2}{r},\frac{2r}{(r-2GM)^2},0,0\right] . \qquad (5.30)$$

This is both transverse and tracefree. However, the norm of this mode N is given by

$$N^2=\int \phi^{ab}\phi_{ab}g^{1/2}d^4x \qquad (5.31)$$

and is divergent. It is therefore not included in the perturbation expansion about the Schwarzschild instanton.

The nonnormalizable mode may be regarded as a process whereby mass is directly moved in from infinity to the bolt. This is in contrast to the normalizable mode which corresponds to moving mass from a finite distance. An attempt to picture this is presented in Fig. 9.

Before proceeding to evaluate the contribution of these instantons to the partition functions we must

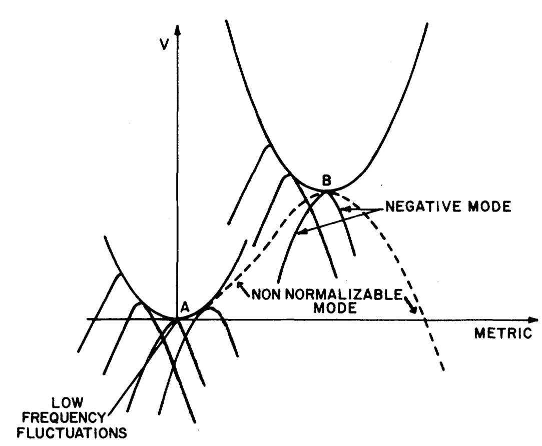

FIG. 9. The one-loop effective potential including the instanton contribution. (A) is flat spacetime. Decrease in the action from (A) represents the Jeans instability. The saddle point at (B) represents the Schwarzschild instanton. The continuous line representing a decrease of the action is the negative mode. The dashed line is the non-normalizable negative mode.

deal with a renormalization problem that does not occur in the flat-space calculation. It turns out that in curved space there is an ultraviolet divergence that occurs even at the one-loop level, which requires a separate counterterm. In a dimensional regularization scheme its contribution to the action is, on shell, (i.e., when g_{ab} is a solution of Einstein's equations),[50]

$$\Delta I = \frac{1}{16\pi^2}\frac{1}{n-4}\frac{53}{45}\int_M R_{abcd}R^{abcd}\sqrt{g}\,d^4\,. \tag{5.32}$$

This term is proportional to the Euler character of the manifold M on which g_{ab} is defined. For flat space it vanishes, but is nonzero for the instanton. To deal with this one must introduce a regulator mass μ (which we presume should be taken of order the Planck mass m_P). This μ dependence will appear in the contribution of the instanton to the partition function. These remarks apply to all theories of gravity, including supergravity, except for a particular version of $N=8$ supergravity where this counterterm vanishes on shell.[51]

We shall now evaluate (approximately) the contribution to the partition function from the Gaussian fluctuations about each topological sector. This can be done using the standard techniques of ζ-function regularization.[52] The flat space contribution is, as before,

$$Z^{(0)} = \exp\left[\frac{V\pi^2}{45\beta^3}\right], \tag{5.33}$$

i.e., the contribution from thermal gravitons in a box of volume V at temperature $T=1/\beta$. The contribution from the Schwarzschild instanton is

$$Z^{(1)} = \frac{i}{2}\exp(-4\pi GM^2)\left[\exp\left[\frac{V\pi^2}{45\beta^3}\right]\right] \times(\mu\beta)^{212/45}\left[\frac{M}{2\pi\beta}\right]^{3/2}V\,. \tag{5.34}$$

The factor of $i/2$ occurs from the one normalizable negative mode; the first exponential contains the classical action of the Schwarzschild instanton. The second exponential arises from the thermal (= quantum) fluctuations about the instanton. At first sight, it might seem surprising that this term is identical to the flat-space result. However we note that the ultraviolet-finite part of $(1/V)\ln Z^{(1)}$ in the infinite-volume limit can only depend on T, since the factor of GM that appears in the background metric is itself inversely proportional to T. Therefore, $(1/V)\ln Z^{(1)}$ must, on dimensional grounds, be proportional to T^3. The constant of proportionality can be determined to be $\pi^2/45$ in the high-temperature limit. The factor $(\mu\beta)^{212/45}$ arises from the anomalous scaling behavior associated with the counterterm (5.32). The final term, proportional to the spatial volume V, occurs from the integration over the translational degrees of freedom of the instanton. The uncalculated finite part of the determinant, which is a temperature-independent constant, has been absorbed into the definition of μ. Written this way it looks like the single-particle partition function for an object of mass M at a temperature β^{-1} in a box of volume V. However, since $\beta=8\pi GM$, the final factors can be rewritten as $(V/64\pi^3G^{3/2})$, which is independent of M. Similarly, we can estimate the contribution from the N-instanton sector, neglecting the classical interaction between the instantons, as

$$Z^{(N)} = \left[\frac{i}{2}\right]^N \exp(-4\pi GNM^2)\exp\left[\frac{V\pi^2}{45\beta^3}\right](\mu\beta)^{212N/25}\frac{1}{N!}\left[\frac{V}{64\pi^3G^{3/2}}\right]^N. \tag{5.35}$$

Here we have assumed that there exists a negative mode associated with each bolt, thus giving a factor of i^N. The first exponential is the classical action of the N-instanton configuration, and the second the thermal graviton term. The final term results from the collective-coordinate integration. Since each instanton is indistinguishable this produces the familiar factor of $(1/N!)(V/64\pi^3G^{3/2})^N$. The above treatment is the standard dilute-gas approximation, which should be valid for widely separated instantons. If we now sum over all sectors,

$$Z=\sum_{N=0}^{\infty} Z^{(N)}\,, \tag{5.36}$$

we obtain

$$Z \simeq \exp\left[\frac{V\pi^2}{45\beta^3}\right]\exp\left[\left[\frac{i}{2}\right]\left[\frac{V}{64\pi^3G^{3/2}}\right](\mu\beta)^{212/45}\exp(-4\pi GM^2)\right]. \tag{5.37}$$

This calculation is analogous to the toy model of Sec. II. There are some differences however. In the case of potential theory we saw that for large temperatures the decay of the metastable vacuum was given by the static saddle point at the top of the barrier, and at low temperatures quantum tunneling also occurs. In our case there is only the former mechanism at all temperatures, and the height of the barrier is proportional to $\beta=1/T$. Even with these differences in mind we see no alternative to the conclusion that the instanton represents thermal nucleation of Schwarzschild metrics—black holes, with a nucleation rate per unit volume given by Γ, where

$$\Gamma=\frac{\omega_0\beta}{\pi}\,\mathrm{Im}\left[\frac{F}{V}\right], \tag{5.38}$$

where

$$\omega_0{}^2=-\lambda=\frac{0.19}{(GM)^2}=\left[\frac{1.74}{\beta}\right]^2. \tag{5.39}$$

Therefore the vacuum decay rate (per unit volume) is

$$\Gamma=\frac{0.87}{\beta}(\mu\beta)^{212/45}\frac{m_P{}^3}{64\pi^3}e^{-m_P{}^2/16\pi T^2}. \tag{5.40}$$

VI. INTERPRETATION AND CONCLUSIONS

In a typical low-temperature quantum decay problem, one may regard the Euclidean bounce solution (instanton) as describing the quantum tunneling from one classically allowed region to another. After the tunneling process, the system is in a highly excited state above the true vacuum. Therefore its subsequent evolution is essentially classical. The appropriate initial data are determined by the field of the Euclidean instanton at the moment it reaches its classical turning point, that is, when the momenta conjugate to the field vanish. Alternatively, if the temperature is sufficiently high so that the relevant instanton describes a process of thermal activation to the top of the barrier (as opposed to tunneling), then the only difference is that the subsequent classical evolution may be either toward the true vacuum or back to the original metastable state.[16]

In the case of the black-hole nucleation, the situation is slightly different since the instability is absent classically, but only arises due to the presence of thermal (=quantum) fluctuation. Therefore the evolution following black-hole nucleation is not quite classical, but may be consistently computed semiclassically. The necessary ingredients are simply the two effects of absorption of thermal radiation into the hole, plus the emission of thermal radiation due to the Hawking process.[53] The Schwarzschild instanton describes the nucleation of a black hole of critical mass $M=1/8\pi GT$, for which these two effects are in unstable equilibrium. Subsequent fluctuations will, with equal probability, cause the black hole either to grow indefinitely or to evaporate. The reason for this instability is that the black hole has negative specific heat. Far from the unstable equilibrium one can estimate the rates at which the black hole grows or shrinks.

Suppose that the black hole is much hotter than the surrounding matter. Then, if we were only to consider gravitons, the hole will lose mass at a rate of

$$\dot{M}\sim-\frac{1}{G^2M^2}. \tag{6.1}$$

On the other hand, if it were much cooler than the surrounding medium, it would accrete at a rate[54] of

$$\dot{M}\sim T_0{}^4(GM)^2, \tag{6.2}$$

T_0 being the temperature at the walls of the box.

It is amusing to apply our results on finite-temperature instabilities to the standard model of the early universe.[55]

The global expansion in a standard Friedmann-Robertson-Walker model is known to affect the Jeans instability drastically. Instead of exponentially growing large-scale fluctuations, linear perturbations can grow at most as a power of t. The actual spectrum of irregularities one finds at late times is extremely sensitive to assumed initial conditions. In addition to these linearized perturbations, one may try to estimate the probability of direct nucleation of black holes. The nucleation rate for pure gravity is given by Eq. (5.40). It may easily be extended to a general theory containing any number of matter fields. One finds for massless fields

$$\Gamma(T)=0.87T(\mu/T)^{\theta}\frac{m_P{}^3}{64\pi^3}\exp\left[-\frac{m_P{}^2}{16\pi T^2}\right], \tag{6.3}$$

where[56]

$$\theta=\tfrac{1}{45}(212N_2-\tfrac{233}{4}N_{3/2}-13N_1+\tfrac{7}{4}N_{1/2}+N_0). \tag{6.4}$$

N_s is the number of spin-s fields.

Suppose that the universe is given by a Robertson-Walker line element with $k=0$,

$$d^2s=-dt^2+R^2(t)(dx^2+dy^2+dz^2)\ , \tag{6.5}$$

then the number of black holes that have been nucleated in the period from time t_1 to time t_2 per unit comoving volume at time t will be

$$N\sim\frac{1}{R^3(t_2)}\int_{t_1}^{t_2}R^3(t)\Gamma(T(t))dt\ . \tag{6.6}$$

The temperature of the universe $T(t)$ is related to the scale factor $R(t)$ by

$$T(t)R(t)=T(t_2)R(t_2)\ . \tag{6.7}$$

Assuming the universe to be radiation dominated (as it surely must be when any significant nucleation takes place), then

$$R(t)\sim G^{1/4}t^{1/2}\ . \tag{6.8}$$

Hence

$$T(t)\sim G^{-1/4}t^{-1/2}\ . \tag{6.9}$$

Thus

$$N\sim\frac{1}{{t_2}^{3/2}}\int_{t_1}^{t_2}dt\,t^{1+\theta/2}\exp\left[-\frac{{m_P}^2t}{16\pi}\right]G^{(\theta-1)/4}\times\frac{0.87\mu^\theta {m_P}^3}{64\pi^3}\ . \tag{6.10}$$

As $t_1\rightarrow 0$, this integral will only converge if $\theta>-4$. This estimate is presumably invalid for temperatures greater than the Planck temperature, and so for times t_1 less than the Planck time. However, it illustrates that the rate of black-hole production is sensitive to θ. It is impossible to estimate absolute rates without some knowledge of μ, although it is widely supposed that $\mu\sim m_P$. Presumably, this indicates that this process is quite important in the very early universe.

We can now estimate the probability of finding a black hole nucleated by this process. Any black hole will in fact evaporate in this scenario. Suppose a black hole of mass M is nucleated at $t=t_0$ at a temperature of $T_0\sim G^{-1/3}\,t_0^{-1/2}$. A black hole, at best, can accrete at a rate of

$$\dot M(t)\sim M^2T^4G^2\sim GM^2t^{-2}\ . \tag{6.11}$$

Thus, its mass $M(t)$ is given by

$$M(t)\sim\frac{t}{1+(t_0G^{-1}-M_0)t/M_0t_0}G^{-1}\ . \tag{6.12}$$

Its temperature $T(t)$ is, as $t\rightarrow\infty$,

$$T(t)\sim\frac{t_0G^{-1}-M_0}{M_0t_0}\ , \tag{6.13}$$

which is constant. Since the blackbody background temperature is cooling like $t^{-1/2}$, there will necessarily come a time when the hole will be hotter than its background, and it will evaporate. (This conclusion also holds in a matter-dominated era.)

From this, we see that any black hole we might observe must have an evaporation time scale longer than 1 Hubble time. This corresponds to an initial mass of around 10^{15} g, which would be nucleated at around $t\sim10^{-6}$ sec. The number of such black holes per unit comoving volume now is

$$N\sim10^{-143+37\theta/2}\exp(-10^{36})\ \text{cm}^{-3} \tag{6.14}$$

assuming $\mu\sim m_P$.

This illustrates the incredible improbability of this phenomenon at any reasonable temperature.

Note added in proof. We understand that B. Allen[57] has confirmed some of the numerical work quoted in Sec. V.

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APPENDIX A: ONE-DIMENSIONAL DETERMINANTS

Consider the equation

$$-\ddot u(t)+V(t)u(t)=\lambda u(t) \tag{A1}$$

on the interval $0\le t\le\tau$. Let $u_\lambda^1(t)$ $[{u_\lambda}^2(t)]$ be the solution with initial data $u_\lambda^1(0)=1$, $\dot u_\lambda^1(0)=0$ $[u_\lambda^2(0)=0,\ \dot u_\lambda^2(0)=1]$. Form the matrix

$$M_\lambda(\tau)\equiv\begin{bmatrix}u_\lambda^1(\tau) & u_\lambda^2(\tau)\\ \dot u_\lambda^1(\tau) & \dot u_\lambda^2(\tau)\end{bmatrix}\ . \tag{A2}$$

Note that if $u(t)$ is any solution of (A1), then

$$\begin{bmatrix}u_\lambda(\tau)\\ \dot u_\lambda(\tau)\end{bmatrix}=M_\lambda(\tau)\begin{bmatrix}u_\lambda(0)\\ \dot u_\lambda(0)\end{bmatrix}\ . \tag{A3}$$

Furthermore, $\det M_\lambda(\tau)=u_\lambda^1(\tau)\dot u_\lambda^2(\tau)-u_\lambda^2(\tau)\times\dot u_\lambda^1(\tau)=1$ since the Wronskian is constant in time. Consequently, $-{\partial_t}^2+V(t)$ on the interval $[0,\tau]$ with periodic boundary conditions has an eigenvector with eigenvalue λ if and only if $\text{tr}(M_\lambda(\tau)\ -1)=0$. This implies that

$$\det{}_{+}[(-\partial_t{}^2+V(t)-\lambda)/(-\partial_t{}^2-\lambda)]$$
$$=\mathrm{tr}(M_\lambda(\tau)-1)/\mathrm{tr}(M_\lambda^0(\tau)-1)\ , \qquad (A4)$$

where

$$M_\lambda^0(\tau)=\begin{bmatrix}\cos\sqrt{\lambda}\,\tau & (1/\sqrt{\lambda})\sin\sqrt{\lambda}\,\tau\\ -\sqrt{\lambda}\sin\sqrt{\lambda}\,\tau & \cos\sqrt{\lambda}\,\tau\end{bmatrix},$$

since both sides of the equation are meromorphic functions of λ with identical poles and zeros (and both go to one as $\lambda\to\infty$ with $\arg\lambda\neq 0$). Finally, we may define the overall normalization on functional determinants so that

$$\det{}_{+}[-\partial_t{}^2+V(t)-\lambda]=\mathrm{tr}(M_\lambda(\tau)-1)\ . \qquad (A5)$$

This agrees with the standard result for a harmonic oscillator,

$$(\det{}_{+}[-\partial_t{}^2+\omega^2])^{-1/2}$$
$$=\mathrm{Tr}\{\exp[-\beta(\tfrac{1}{2}\hat{p}^2+\tfrac{1}{2}\hat{\omega}^2\hat{x}^2)]\}$$
$$=e^{-\beta\hbar\omega/2}/(1-e^{-\beta\hbar\omega})\ .$$

since $\mathrm{tr}(M_0(\beta\hbar)-1)=2(\cos\beta\hbar\omega-1)$.

Functional determinants with any other choice of boundary conditions may also be computed using the matrix $M_\lambda(\tau)$. For example, determinants with antiperiodic and Dirichlet boundary conditions, respectively, are given by

$$\det{}_{-}(-\partial_t+V(t))=\mathrm{tr}(M_\lambda(\tau)+1)$$
and $\qquad (A6)$
$$\det{}_D(-\partial_t+V(t)-\lambda)=\pi\hbar[M_\lambda(\tau)]_{12}\ .$$

Finally, we must compute $\det'_{+}[-\partial_t+V''(\bar{x}(t))]$. Since

$$\det{}'_{+}(-\partial_t+V'')=-(\partial/\partial\lambda)\det(-\partial_t+V''-\lambda)\,|_{\lambda=0}$$
$$=-(\partial/\partial\lambda)\mathrm{tr}(M_\lambda(\tau)-1)\,|_{\lambda=0}$$

we need $M_\lambda(\tau)$ correct to $O(\lambda)$. This may be easily computed using the known zero-mode $\dot{\bar{x}}(t)$, and perturbing in λ. If the bounce is translated in time so that $\ddot{\bar{x}}(0)=0$, then

$$u_0^1(t)=\dot{\bar{x}}(t)/\dot{\bar{x}}(0),\quad u_0^2(t)=\dot{\bar{x}}(t)\dot{\bar{x}}(0)\int_0^t dt'/(\dot{\bar{x}}(t'))^2$$

and

$$u_\lambda^{1,2}(t)=u_0^{1,2}(t)+\lambda\int_0^t dt'[u_0^2(t)u_0^1(t')-u_0^1(t)u_0^2(t')]\times u_0^{1,2}(t')+O(\lambda^2).$$

This yields

$$\mathrm{tr}(M_\lambda(\tau)-1)=\lambda\int_0^\tau dt\int_0^\tau dt'(\dot{\bar{x}}(t)/\dot{\bar{x}}(t'))^2+O(\lambda^2)\ ,$$

so that

$$\det{}'_{+}[-\partial_t^2+V''(\bar{x}(t))]=-\left[\int_0^\tau dt(\dot{\bar{x}}(t))^2\right]\times\left[\int_0^\tau dt/(\dot{\bar{x}}(t))^2\right]$$
$$=W(E)\tau'(E)\ , \qquad (A7)$$

where

$$E=V(\bar{x}(t))-\tfrac{1}{2}(\dot{\bar{x}}(t))^2\ ,$$

$$W(E)=2\int_{x_1}^{x_2}dx[2(V(x)-E)]^{1/2}$$

and

$$\tau'(E)=(\partial/\partial E)\,2\int_{x_1}^{x_2}dx/[2(V(x)-E)]^{1/2}\ .$$

This verifies equation (2.10).

APPENDIX B: THE COMPONENTS OF EQ. (5.13) IN THE SCHWARZSCHILD INSTANTON

We display the components of $-\Box\phi^{ab}-2R^{abcd}\phi_{cd}=\lambda\phi^{ab}$ in Schwarzschild coordinates [we would like to thank Roberta Young for checking Eqs. (B1)–(B10) using CAMAL, an algebra handling program ($G=1$)]:

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{00}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{00}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{00}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{00}+\frac{2(m+r)}{r^2}\frac{\partial}{\partial r}\phi^{00}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{00}$$
$$+\frac{2m^2}{r^3(r-2m)}\phi^{00}+\frac{4m}{(r-2m)^2}\frac{\partial}{\partial t}\phi^{01}+\frac{2m(2r-3m)}{r(r-2m)^3}\phi^{11}-\frac{2m}{(r-2m)}\phi^{22}-\frac{2m\sin^2\theta}{(r-2m)}\phi^{33}=-\lambda\phi^{00}\ , \qquad (B1)$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{01}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{01}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{01}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{01}-\frac{2(m-r)}{r^2}\frac{\partial}{\partial r}\phi^{01}$$
$$+\frac{2m}{(r-2m)}\frac{\partial}{\partial t}\phi^{11}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{01}-\frac{2m}{r^2}\frac{\partial}{\partial t}\phi^{00}+\frac{2(2m-r)}{r^2}\frac{\partial}{\partial\theta}\phi^{02}+\frac{2(2m-r)}{r^2}\frac{\partial}{\partial\theta}\phi^{03}$$
$$-\frac{2r^2-4mr+4m^2}{r^3(r-2m)}\phi^{01}+\frac{2(2m-r)}{r^2}\cot\phi^{02}=-\lambda\phi^{01}\ , \qquad (B2)$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{02}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{02}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{02}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2}\phi^{02}+\frac{4(r-m)}{r^2}\frac{\partial}{\partial r}\phi^{02}$$
$$+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{02}+\frac{2m}{(r-2m)^2}\frac{\partial}{\partial t}\phi^{12}+\frac{2}{r^3}\frac{\partial}{\partial\theta}\phi^{01}-\frac{2\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{03}+\left[\frac{4m}{r^3}+\frac{1}{r^2}(1-\cot^2\theta)\right]\phi^{02}=-\lambda\phi^{02}\,, \tag{B3}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{03}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{03}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{03}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{03}+\frac{4(r-m)}{r^2}\frac{\partial}{\partial r}\phi^{03}+\frac{3\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{03}$$
$$+\frac{2m}{(r-2m)^2}\frac{\partial}{\partial t}\phi^{13}+\frac{2}{r^3\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{01}+\frac{2\cot\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{02}+\frac{2m}{r^3}\phi^{03}=-\lambda\phi^{03}\,, \tag{B4}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{11}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{11}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{11}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{11}-\frac{2(3m-r)}{r^2}\frac{\partial}{\partial r}\phi^{11}$$
$$+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{11}-\frac{4m}{r^2}\frac{\partial}{\partial t}\phi^{01}+\frac{4}{r^2}(2m-r)\frac{\partial}{\partial\theta}\phi^{12}+\frac{4}{r^2}(2m-r)\frac{\partial}{\partial\theta}\phi^{12}+\frac{4}{r^2}(2m-r)\frac{\partial}{\partial\phi}\phi^{13}$$
$$+\frac{2m(2r-3m)(r-2m)}{r^5}\phi^{00}+\frac{16mr-14m^2-4r^2}{r^3(r-2m)}\phi^{11}+\frac{2(r-2m)(r-3m)}{r^2}\phi^{22}$$
$$+\frac{2(r-2m)(r-3m)}{r^2}\phi^{33}\sin^2\theta+\frac{4(2m-r)}{r^2}\cot\theta\phi^{12}=-\lambda\phi^{11}\,, \tag{B5}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{12}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{12}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{12}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{12}+\frac{4(r-2m)}{r^2}\frac{\partial}{\partial r}\phi^{12}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{12}-\frac{2m}{r^2}\frac{\partial}{\partial t}\phi^{02}$$
$$+\frac{2}{r^2}(2m-r)\frac{\partial}{\partial\theta}\phi^{22}+\frac{2}{r^2}(2m-r)\frac{\partial}{\partial\phi}\phi^{23}+\frac{2}{r^3}\frac{\partial}{\partial\theta}\phi^{11}-\frac{2\cot\theta}{r^3}\frac{\partial}{\partial\theta}\phi^{13}+\frac{1}{r^2}(1-\cot^2\theta)\phi^{12}$$
$$+\frac{2(2m-r)}{r^2}\cot\theta\phi^{23}+\frac{2(r-2m)}{r^2}\sin\theta\cos\theta\phi^{33}+\frac{4(2m-r)}{r^3}\phi^{12}=-\lambda\phi^{12}\,, \tag{B6}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{13}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{13}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{13}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{13}+\frac{4(r-2m)}{r^2}\frac{\partial}{\partial r}\phi^{13}$$
$$+\frac{3\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{13}-\frac{2m}{r^2}\frac{\partial}{r^2}\phi^{03}+\frac{2(2m-r)}{r^2}\frac{\partial}{\partial\theta}\phi^{23}+\frac{2}{r^2}(2m-r)\frac{\partial}{\partial\phi}\phi^{33}+\frac{2}{r^3\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{11}$$
$$+\frac{2\cot\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{12}+\frac{6(2m-r)\cot\theta}{r^2}\phi^{23}-\frac{4(r-2m)}{r^3}\phi^{13}=-\lambda\phi^{13}\,, \tag{B7}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{22}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{22}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\phi^{22}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2}\phi^{22}+\frac{6r-10m}{r^2}\frac{\partial}{\partial r}\phi^{22}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{22}$$
$$+\frac{4}{r^3}\frac{\partial}{\partial\theta}\phi^{12}-\frac{4\cot\theta}{r^2}\frac{\partial}{\partial\phi}\phi^{23}+\frac{2m(2m-r)}{r^6}\phi^{00}+\frac{2(r-3m)}{r^4(r-2m)}\phi^{11}+\frac{4m\sin^2\theta}{r^3}\phi^{33}+\frac{2\cos^2\theta}{r^2}\phi^{33}$$
$$+\frac{2(r-2m)}{r^3}\phi^{22}+\frac{2}{r^2}(1-\cot^2\theta)\phi^{22}=-\lambda\phi^{22}\,, \tag{B8}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{23}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{23}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{23}+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\phi^2}\phi^{23}+\frac{6r-10m}{r^2}\frac{\partial}{\partial r}\phi^{23}+\frac{3\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{23}$$
$$+\frac{2}{r^3}\frac{\partial}{\partial\theta}\phi^{13}-\frac{2\cot\theta}{r}\frac{\partial}{\partial\phi}\phi^{33}+\frac{2}{r^3\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{12}+\frac{2\cot\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{22}+\left[\frac{3}{r^2}(1-\cot^2\theta)-\frac{8m}{r^3}\right]\phi^{23}=-\lambda\phi^{23}\,, \tag{B9}$$

$$\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}\phi^{33}+\frac{r-2m}{r}\frac{\partial^2}{\partial r^2}\phi^{33}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi^{33}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\phi^{33}+\frac{6r-10m}{r_2}\frac{\partial}{\partial r}\phi^{33}$$
$$+\frac{5\cot\theta}{r^2}\frac{\partial}{\partial\theta}\phi^{33}+\frac{4}{r^3\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{13}+\frac{4\cot\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\phi^{23}+\frac{2m(2m-r)}{r^6\sin^2\theta}\phi^{00}+\frac{2(r-3m)}{r^4\sin^2\theta(r-2m)}\phi^{11}$$
$$+\frac{4\cot\theta}{r^3\sin^3\theta}\phi^{12}+\frac{(4m+2r\cot^2\theta)}{r^3\sin^2\theta}\phi^{22}+\frac{2}{r^3}[(r-2m)+r\cot^2\theta]\phi^{33}=-\lambda\phi^{33}\,. \quad \text{(B10)}$$

These equations are closely related to those of Edelstein and Vishveshwara,[58] who considered perturbations to the Lorentzian Schwarzschild solution, but in a gauge different from ours.

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