PHYSICAL REVIEW D VOLUME 25, NUMBER 12 15 JUNE 1982

Topological obstructions to gauge-independent descriptions of broken symmetry

Joe Kiskis
Department of Physics, University of California at Davis, Davis, California 95616
(Received 26 February 1982)

When a gauge group G is broken to a subgroup $H(\phi)$ by a Higgs scalar ϕ, a fermion field ψ will split into irreducible representations of $H(\phi)$. The construction of fields $\psi^i(\phi,\psi)$ that are linear in ψ, express the splitting gauge independently, and transform irreducibly with respect to $H(\phi)$ is considered. It is found that in some cases the construction is possible while in others there are topological obstructions. Connections with the notion of complementarity are discussed.

I. INTRODUCTION

Included in the theorist's creed is the belief that the physical content of gauge theories can be expressed gauge invariantly. Gauge-dependent statements are suspect. For example, the vacuum expectation value of the Higgs scalar field is assumed to be nonzero in some gauges, but it can be shown to vanish in others.[1–3] As a result, its usefulness as an order parameter distinguishing the Higgs and confining regions has been questioned.[3] The physical picture associated with the Higgs mechanism is certainly gauge dependent. Thus a gauge-independent or "symmetric" description of the Higgs mechanism that does not rely on a particular gauge fixing has been advocated and discussed.[3–5] In addition, there is a notion of complementary.[6–8] This states that in certain circumstances the Higgs and confining regions are actually smoothly connected.

For a simple example, consider a theory with gauge group $G=\mathrm{SU}(2)$ broken to $H=1$ by a doublet Higgs scalar. Let there also be a fermion doublet ψ. After symmetry breaking, radiative corrections give the two components of ψ different mass. On the other hand, one might argue that gauge invariance is never really broken,[1,2] the components of ψ are related by gauge transformations, and thus they cannot have different masses. This apparent contradiction is resolved with the observation that the separation of ψ into two components is gauge independent. In this case, it is actually gauge invariant. The fields $\phi^\dagger\psi$ and $i\phi^T\sigma_2\psi$ are gauge invariant and linear in ψ. For the usual choice of the vacuum expectation value of ϕ, they reduce to the upper and lower components of ψ. The question is: Can these constructions be generalized to more complicated models including cases with $H\neq 1$?

Fröhlich, Morchio, and Strocchi[3] have discussed this and related issues. They have shown that it is always possible to give a gauge-independent decomposition of the fermion representation space into subspaces associated with irreducible representations of H. The decomposition is effected by certain projection operators. However, the fields constructed with the projection operators have fewer physical degrees of freedom than is indicated by the number of components, and they do not transform irreducibly with respect to H. Thus it is reasonable to attempt a slightly more ambitious program.

Let ψ_j $(j=1,\ldots,d)$ be the components of a fermion field that carries a d-dimensional representation of G. We will attempt to associate with ψ sets of fields $\psi^i(\phi,\psi)$ that depend on the scalar field and are linear in ψ. The fields are labeled by i and the components by j,

$$\psi^i_j(\phi,\psi)\ \text{ with } j=1,\ldots,d_i \text{ and } \sum_i d_i=d\ . \tag{1.1}$$

These fields should decompose ψ in a gauge-independent way: (a) ψ^i and ψ^j should not mix under any gauge transformation if $i\neq j$; (b) each ψ^i should transform irreducibly with respect to $H(\phi)$, the stability group of ϕ.

Fields with these properties can be used in constructing an explicit gauge-independent link between different gauge-dependent descriptions of the theory. They have an important role in describing the Higgs mechanism "symmetrically" and in establishing a smooth connection between the Higgs and confinement regions (when such exists).

While the construction of Ref. 3 is always possible, this more ambitious one is not. Thus, it may differentiate models on the basis of some physical property. We have not been able to clearly identify this property. But the preceding paragraph suggests that complementary is involved. (Other indications to this effect will be noted as they arise). An imperfect understanding of spontaneous symmetry breaking and confinement has prevented us from giving the correspondence between physical and mathematical results that would be ideal.

Let us state the problem again. Consider as given a gauge group G, a fermion field ψ, a Higgs scalar ϕ, an orbit Φ of ϕ by G and thus an unbroken group $H(\phi)$ for $\phi\in\Phi$. Problem: Can fields ψ^i with properties (a) and (b) be constructed? Answer: Sometimes yes, and sometimes no.

The problem involves an intricate interplay of algebraic topology, representation theory, and the global structures of G, Φ and space-time. Since a general solution does not seem to be possible, several examples will be discussed. Simple examples that illustrate the ranges of issues, methods, and results have been selected. The methods are topological. No doubt, many of the results can be obtained using algebraic methods. This has not been done here. Such a study is likely to give additional insights.

Section II gives some preliminary discussion and a complete statement of the problem. Section III discusses general methods of attack. Section IV contains the examples.

In a particular way, this paper addresses the general question: Can all structures in a given gauge be expressed gauge independently? We will see that there are difficulties.

II. PRELIMINARY DISCUSSION

Begin with a connected gauge group G. The fermions are assumed to carry an irreducible representation of dimension d. The field ψ is a vector in a d-dimensional complex vector space Ψ. The representation g_F is a homomorphism from G into the unitary transformations of Ψ:

$$g_F\colon G\to U(d)\ . \tag{2.1}$$

Thus G acts by

$$(g,v)\in G\times\Psi\to g_F{}^{-1}(g)v\ . \tag{2.2}$$

In Ψ, there is the usual inner product written $u^\dagger v$ for $u,v\in\Psi$. The action of U preserves this. In physics problems, it is convenient and customary to introduce a basis of orthornomal vectors in Ψ:

$$\{w_i, i=1,\ldots,d\}\ ,\quad w_i^\dagger w_j=\delta_{ij}\ . \tag{2.3}$$

(The subscripts label the vectors in the basis, not the components thereof.) A vector is then referred to this basis. For instance,

$$\psi_i=w_i^\dagger\psi\ . \tag{2.4}$$

When ψ is rotated to

$$\psi'=g_F{}^{-1}\psi\ , \tag{2.5}$$

then

$$\psi_i'=\sum_j w_i^\dagger g_F{}^{-1}w_j\psi_j\ . \tag{2.6}$$

There are many orthornormal bases and no natural way to choose among them. An arbitrary choice is made. It should be noted that while U(d) acts transitively on the orthonormal bases, G does not in general.

The theory also contains a scalar field ϕ. The representation g_s of G carried by the scalar field may be reducible. The action of G on ϕ determines the orbits of ϕ by this action of G. At any point in the discussion, it will be assumed that ϕ is restricted to a particular specified orbit Φ. This is a gauge-invariant statement. It is roughly analogous to the assertion that ϕ gets a certain vacuum expectation value. In either case, it is a mathematical statement. Only a study of the dynamics can determine whether or not it is a reasonable approximation to the physical situation.

The stability group $H(\phi)$ of a vector ϕ will be defined in the usual way as the subgroup of G that fixes ϕ:

$$H(\phi)=\{g\in G\mid g_s{}^{-1}(g)\phi=\phi\}\ . \tag{2.7}$$

If two vectors lie on the same orbit, then they have conjugate stability groups. Introduce the set of subgroups conjugate to H:

$$\mathscr{H}=\{g^{-1}Hg, g\in G\}\ . \tag{2.8}$$

Now consider the action of $H(\phi)$ on Ψ. In general, the representation of $H(\phi)$ carried by Ψ will be reducible. Thus, Ψ can be decomposed into mutually orthogonal subspaces,

$$\Psi=V_1(\phi)+V_2(\phi)+\cdots \tag{2.9}$$

with dimensions that add up to d,

$$d=d_1+d_2+\cdots\ . \tag{2.10}$$

Each ordered decomposition is an element of the complex flag manifold[9] $\mathrm{Fl}(d;d_1,d_2,\ldots)\equiv D$.

Each $V_i(\phi)$ carries an irreducible representation of $H(\phi)$. If all of these representations are different, then the decomposition of (2.9) is unique. However, if V_i and V_j carry the same representation of H, then there is no unique way to decompose the space V_i+V_j into V_i and V_j. An arbitrary choice must be made.

As ϕ moves about in Φ, $H(\phi)$ moves about in $\mathscr{H}$, and the $V_i(\phi)$ move about in Ψ. Specifically, if

$$\phi=g_s{}^{-1}(g)\phi_0 \, , \tag{2.11}$$

then

$$H(\phi)=g^{-1}H(\phi_0)g \tag{2.12}$$

and

$$V_i(\phi)=g_F{}^{-1}(g)V_i(\phi_0) \, . \tag{2.13}$$

In fact, let us use these expressions to define $H(\phi)$ and $V_i(\phi)$. Choose a point $\phi_0\in\Phi$. Determine its stability group $H(\phi_0)$. Determine the decomposition

$$V=V_1(\phi_0)+V_2(\phi_0)+\cdots \tag{2.14}$$

(including any arbitrary choices that must be made). Then, to obtain $H(\phi)$, and $V_i(\phi)$, find a g that satisfies (2.11) and use (2.12) and (2.13). There is no ambiguity here even though there are many g's that solve (2.11). All g's that satisfy (2.11) "differ" by an element of $H(\phi_0)$ that fixes ϕ_0 in (2.11) *and* $H(\phi_0)$ in (2.12) and $V_i(\phi_0)$ in (2.13). Thus, there are continuous maps,

$$H:\Phi\to\mathscr{H}, \text{ i.e., } \phi\to H(\phi) \, , \tag{2.15}$$

$$V:\Phi\to D, \text{ i.e., } \phi\to\{V_1(\phi),V_2(\phi),\ldots\}. \tag{2.16}$$

(For comparison, the projectors of Ref. 3 are projectors onto the V_i.)

The case of no "broken symmetry" corresponds to

$$H(\phi)=G \, , \tag{2.17}$$

and the possibility of completely "broken symmetry" is associated with

$$H(\phi)=1 \, . \tag{2.18}$$

As has already been mentioned, it is useful to introduce a basis in Ψ. (Basis will always mean orthonormal basis). When $H=G$, this is done in the manner already discussed. In particular, there is no natural basis since all subspaces are mixed up by the action of H. An arbitrary choice is made.

When $H(\phi)=1$, each $V_i(\phi)$ is one dimensional. In this case, it is natural to choose basis vectors that depend upon ϕ in such a way that $w^i(\phi)$ is a basis vector for $V_i(\phi)$. To do this, start again at ϕ_0. Select (in an arbitrary way) from each $V_i(\phi_0)$ a normalized vector $w^i(\phi_0)$. Then using (2.11) again, define

$$w^i(\phi)=g_F{}^{-1}(g)w^i(\phi_0) \, . \tag{2.19}$$

Since $H=1$, this is unambiguous. It follows that

$$w^i(g_s{}^{-1}(g)\phi)=g_F{}^{-1}(g)w^i(\phi) \, . \tag{2.20}$$

Now refer ψ to the basis $w^i(\phi)$:

$$\psi^i(\phi,\psi)=w^{i\dagger}(\phi)\psi \, . \tag{2.21}$$

Under a gauge transformation by $g\in G$

$$\phi'=g_s{}^{-1}(g)\phi \text{ and } \psi'=g_F{}^{-1}(g)\psi \tag{2.22}$$

so

$$\psi^i(\phi',\psi')=w^{i\dagger}(\phi')\psi'=w^{i\dagger}(\phi)\psi=\psi^i(\phi,\psi) \, . \tag{2.23}$$

This should be contrasted with (2.4)–(2.6) which refer to the case $H=G$ and are now rewritten:

$$w_i(\phi)=w_i \, , \tag{2.24}$$

$$\psi_i(\phi,\psi)=w_i^\dagger\psi \, , \tag{2.25}$$

$$\psi_i(\phi',\psi')=\sum_j w_i^\dagger g_F{}^{-1}(g)w_j\psi_j(\phi,\psi) \, . \tag{2.26}$$

In summary, when $H(\phi)=G$, the decomposition of Ψ is

$$\Psi=V_1(\phi)=V_1 \text{ with } d_1=d \tag{2.27}$$

and all the ψ_i mix. They carry the d-dimensional representation of $H(\phi)$ and G. When $H(\phi)=1$, the decomposition of Ψ is

$$\Psi=V_1(\phi)+\cdots+V_d(\phi) \text{ with } d_1=d_2=\cdots=1 \tag{2.28}$$

and the ψ^i do not mix. They each carry the one-dimensional trivial representation of $H(\phi)$ and G.

The task is now clear: Search for a generalization of these trivial results to cases where H is a proper subgroup of G. The natural thing to do is to search for a ϕ-dependent basis in Ψ such that the first d_1 vectors are a basis for $V_1(\phi)$, the next d_2 are a basis for $V_2(\phi)$, etc. Label such a basis with two indices $w_j^i(\phi)$. The superscript indicates the subspace, and the subscript runs 1 to d_i. So

$$w_j^i(\phi), \; j=1,\ldots,d_i \tag{2.29}$$

is a basis for $V_i(\phi)$.

Examine the transformation properties of the w's. Elementary orthogonality and completeness arguments show that

$$g_F^{-1}(g)w_j^i(\phi)=\sum_k w_k^i(g_s^{-1}(g)\phi)T_{kl}^{i\,-1} \tag{2.30}$$

and

$$w_j^i(g_s^{-1}(g)\phi)=\sum_k g_F^{-1}(g)w_k^i(\phi)T_{kj}^i \tag{2.31}$$

with

$$T_{kj}^i=w_k^{i\dagger}(\phi)g_F(g)w_j^i(g_s^{-1}(g)\phi) \tag{2.32}$$

and

$$T_{kj}^{i\,-1}=w_k^{i\dagger}(g_s^{-1}(g)\phi)g_F^{-1}(g)w_j^i(\phi)\ . \tag{2.33}$$

This shows that the selection of w's is gauge invariant in the sense that vectors with different values of i do not mix under arbitrary gauge transformations.

In the case that $h\in H(\phi)$, these specialize to

$$g_F^{-1}(h)w_j^i(\phi)=\sum_k w_k^i(\phi)T_{0kj}^{i\,-1} \tag{2.34}$$

and

$$w_j^i(g_s^{-1}(h)\phi)=w_j^i(\phi)=\sum_k g_F^{-1}(h)w_k^i(\phi)T_{0kj}^i \tag{2.35}$$

with

$$T_{0kj}^i=w_k^{i\dagger}(\phi)g_F(h)w_j^i(\phi) \tag{2.36}$$

and

$$T_{0kj}^{i\,-1}=w_k^{i\dagger}(\phi)g_F^{-1}(h)w_j^i(\phi)\ . \tag{2.37}$$

This shows explicitly that for each i the T_{0kj}^i give a representation of $H(\phi)$ carried by the $w_j^i(\phi)$.

For the case $H=G$, there is only one d-dimensional subspace. So the basis appearing in (2.24) should now be written w_j^1. For the case $H=1$, there are d one-dimensional subspaces. The basis vectors of (2.19) should now be written $w_1^i(\phi)$. Specialization of (2.30)–(2.37) to these extreme cases clarifies the earlier discussion.

The fermion field ψ can be refered to the basis w_j^i of Ψ:

$$\psi_j^i(\phi,\psi)=w_j^{i\dagger}(\phi)\psi\ . \tag{2.38}$$

A gauge transformation gives

$$\psi_j^i(\phi',\psi')=w_j^{i\dagger}(g_s^{-1}(g)\phi)g_F^{-1}(g)\psi \tag{2.39}$$

$$=\sum_k w_k^{i\dagger}(\phi)\psi T_{kj}^{i*} \tag{2.40}$$

$$=\sum_k T_{jk}^{i\,-1}\psi_k^i(\phi,\psi)\ . \tag{2.41}$$

It is important to note that under arbitrary gauge transformations, fields with different values of i do not mix. Thus, the separation of fields into these groups is gauge independent.

In the case that $h\in H(\phi)$,

$$\psi_j^i(\phi',\psi')=\sum_k T_{0jk}^{i\,-1}\psi_k^i(\phi,\psi)\ . \tag{2.42}$$

This shows that for each i the $\psi_j^i(\phi,\psi)$ carry a d_i-dimensional irreducible representation of $H(\phi)$.

For comparison, note that the projection operators of Ref. 3 can be constructed from the w's:

$$P^i(\phi)=\sum_j w_j^i(\phi)w_j^{i\dagger}(\phi)\ .$$

Objects such as $P^i\psi$ carry a d-dimensional representation of G and thus a d-dimensional reducible representation of $H(\phi)$. Although this may be all that is really needed, the ψ_j^i are in some ways a more convenient and natural construction.

The remainder of this paper discusses when the w's can and cannot be constructed. As mentioned, this may be related to complementarity. When the w's can be constructed, the ψ_j^i establish a direct, gauge-independent connection between the fields usually used in the regions of broken and unbroken symmetry. This suggests complementarity. When the w's cannot be constructed, the connection between the descriptions is less direct (such as through the $P^i\psi$ which can always[3] be constructed). A more pronounced physical difference between the two regions is indicated.

III. GENERAL CONSIDERATIONS

In this section, the construction of the w's will be discussed. General techniques will be presented. These will be applied to the examples in the next section.

Henceforth, a d_i-dimensional orthonormal basis will be refered to as a d_i frame. The subspace it spans is a d_i plane. A d_i frame determines a unique d_i plane in Ψ. It is central to this work that a d_i plane does not determine a unique d_i frame. There are many d_i frames associated with a given d_i plane. The frames are related to each other by $U(d_i)$ transformations.

Also, recall the structure of an orbit Φ. The group G acts transitively on Φ with stability subgroup H. Thus

$$\Phi\approx G/H\ . \tag{3.1}$$

This coset space is topologically nontrivial in general. Our problem is not really to find $w(\phi)$'s (which can always be done), but rather to find $w(\phi)$'s that are continuous in ϕ as it ranges over Φ. While in principle it may be possible to deal with operators that are not continuous in ϕ, it would be awkward. A point on the orbit at which to place the discontinuity would have to be chosen. This is certainly not gauge invariant and thus violates the spirit of the approach.

The simplest possibility is method I. To obtain the w's, choose a point ϕ_0 and choose the frames $w_j^i(\phi_0)$. Frames at ϕ can then be obtained by an action of G that carries ϕ_0 to ϕ. However, there are in general many elements of G that carry ϕ_0 to ϕ. They differ be elements of H. The vectors they give at ϕ will agree if any only if

$$w_j^i(\phi_0)=g_F{}^{-1}(h)w_j^i(\phi_0) \quad \text{for } h\in H(\phi_0)\ . \tag{3.2}$$

Since for each i, the $w_j^i(\phi_0)$ carry an irreducible representation $H(\phi_0)$, (3.2) is satisfied only for the one-dimensional trivial representations of $H(\phi_0)$. Thus, with this method, one can generate

$$w_1^i(\phi)=g_F{}^{-1}(g)w_1^i(\phi_0) \quad (\phi=g_S{}^{-1}(g)\phi_0) \tag{3.3}$$

for the subspaces V_i that carry a trivial representation of H. The resulting ψ_1^i are gauge-invariant objects. When $H=1$, all V_i carry a trivial representation of H. The method completely solves this problem. (This is just a rewording of the discussion in Sec. II.) When $H\neq 1$, it gives only the gauge-invariant combinations. Finally, note that $H=1$ is a case where complimentary applies and also all the w_j^i exist.

Method II begins again by choosing frames at ϕ_0. Let W be the set of frames obtained from those by any action of G. let $I(\phi_0)$ be the stability group of the frames chosen at ϕ_0. Then I is invariant in G, and the coset space is a group

$$W\approx G_F=G/I\ . \tag{3.4}$$

For the sake of this problem, it can be assumed, with no loss of generality, that

$$I\cap H=1\ . \tag{3.5}$$

[If the right-hand side of (3.5) were not 1, everything could be divided by it. This would give a new and equivalent problem for which (3.5) would be true.] With (3.5) the inverse image in G of a point in Φ can intersect with the inverse image in G of a point in G_F at most one point.

Associated with a given $\phi\in\Phi$ there are many $g\in G$ and many frames in W. The object is to choose one of these frames. But due to (3.5), this is equivalent to the choice of a g. Now from (3.1), a projection map

$$p: G\rightarrow\Phi \tag{3.6}$$

is obtained that associates to each element of G its coset in Φ. Thus the problem is to find a continuous map

$$r: \Phi\rightarrow G \tag{3.7}$$

such that

$$p(r(\phi))=\phi\ . \tag{3.8}$$

This is the problem of finding a section of the principal bundle (3.6). The section exists if and only if the bundel is trivial. In general it is not, and this method will fail. However, there are cases where it works. It can be checked before more complicated methods are applied.

Two very simple cases where it must work are $H=1$ or $H=G$. When $H=1$, both p and r are the identity map on G. Equivalently, (2.19) gives the solution. When $H=G$, Φ is a single point. The original arbitrary choice of frame at ϕ_0 gives r.

The previous method requires that the G that carries the frame for V_1 from ϕ_0 to ϕ is the same as the g that does the job for V_2, etc. In method III, there is a different g_i for each V_i. Let $H_i(\phi_0)$ be the subgroup of $H(\phi_0)$ that fixes $V_i(\phi_0)$ pointwise. A separate problem is obtained for each subspace. Equation (3.6) becomes

$$p_i: G/H_i\rightarrow\phi\ , \tag{3.9}$$

(3.7) becomes

$$r_i: \phi\rightarrow G/H_i\ , \tag{3.10}$$

and r_i must satisfy

$$p_i(r_i(\phi))=\phi\ . \tag{3.11}$$

If any V_i carries a faithful representation of H, then $H_i=1$, and this is the same as the previous method. Also for a trivial subspace $H_i=1$, p_i becomes the identity map on Φ, so r_i is easily constructed. This is equivalent to (3.3).

Since H_i acts as the identity on V_i, it is invariant in H. Thus, (3.9) is a principal bundle. Again this bundle is not always trivial; so this more general will not always work.

Now consider the most general attack, method IV. Drop the requirement that the frames at ϕ come from those at ϕ_0 by any action of G. The set F of all d frames in Ψ can be identified with $U(d)$. H determines a partition of d according to (2.10).

Each frame determines a decomposition of Ψ: the first d_1 vectors span V_1, the next d_2 span V_2, etc. Given a decomposition of Ψ according to (2.9), one finds a d frame by assembling a d_1 frame in V_1, a d_2 frame in V_2, etc. All other d frames that will give this decomposition can be obtained by an action of $U(d_1)\times U(d_2)\times \cdots$. $U(d_i)$ acts on V_i. A principal bundle with projection P and fiber $U(d_1)\times U(d_2)\times \cdots$ is obtained,

$$P: F\rightarrow D \ . \tag{3.12}$$

According to (2.16) there is a map V associating with each ϕ a decomposition. The goal is to associate with each ϕ an appropriate d frame. The d frame is appropriate if it projects in (3.12) to the decomposition associated with ϕ. Thus, the mathematical problem is to lift the map V of (2.16) to a map

$$R: \Phi\rightarrow F \tag{3.13}$$

such that

$$P(R(\phi))=V(\phi) \ . \tag{3.14}$$

In some cases this is possible and in some it is not. The examples will illustrate this in more detail.

If the w's exist, then local fields can be constructed,

$$\psi_j^i(\phi,\psi,x)=\psi_j^i(\phi(x),\psi(x))=w_j^{i\dagger}(\phi(x))\psi(x) \ . \tag{3.15}$$

The only further generalization that is possible is to relax the condition that w be local. At the classical level, this can be treated, and an interesting mathematical structure results. Monopoles and instantons enter the picture. However, quantization will be very difficult. Thus we have chosen not to discuss this possibility further here.

This section has been an overview of the formal aspects of the problem. It is difficult to state (much less prove) general theorems due to the large number of cases that must be considered. Instead the structure of the problem was given. In the next section, some examples will be analyzed. These will illustrate a range of techinques and results.

IV. EXAMPLES

A. SU(2) with doublet Higgs bosons and doublet fermions[3–5,7,8,10]

The gauge group is $G=\mathrm{SU}(2)$. Take for ϕ_0 any nonzero vector. The only SU(2) transformation that fixes ϕ_0 is the identity. Thus $H=1$. The two-dimensional fermion representation then splits into two trivial one-dimensional representations. The general discussion of Sec. III shows that $w_1^i(\phi)$ and $w_1^2(\phi)$ can always be found.

$$w_1^1(\phi)=\phi \ ,w_1^2(\phi)=-i\sigma_2\phi^* \ . \tag{4.1}$$

Then

$$\psi_1^1=\phi^\dagger\psi \text{ and } \psi_1^2=i\phi^T\sigma_2\psi \tag{4.2}$$

are each G and H invariants.

B. U(2) with doublet Higgs barons and doublet fermions

This is essentially "SU(2)×U(1)" Weinberg-Salam theory.[3,8] Although very much like A, it is included to show that the existence of the w's is not restricted to cases wherein the resulting ψ's are invariants as in (4.2).

Now $H=\mathrm{U}(1)$, and for

$$\phi_0\propto\binom{0}{1} \ , \tag{4.3}$$

$$H(\phi_0)=\begin{bmatrix} e^{i\theta}0 \\ 0\ 1\end{bmatrix} \ , \quad 0\leq\theta<2\pi \tag{4.4}$$

Ψ splits into two one-dimensional subspaces. V_1 carries a trivial representation of H, and V_2 carries a charge-one representation. Since $H\neq 1$, a choice for the w's at ϕ_0 cannot be transported to ϕ unambiguously. Following Sec. III, the next thing to do is check the structure of the bundle (3.6). In this case

$$\Phi\approx\mathrm{U}(2)/\mathrm{U}(1)\approx S^3 \tag{4.5}$$

and (3.6) becomes

$$\mathrm{U}(2)\rightarrow S^3 \ . \tag{4.6}$$

A section of this bundle can be obtained by selecting from each coset the element with determinant one. The bundle is trivial. This section suggests a choice for the $w(\phi)$'s. They should be obtained by transporting with an SU(2) action. Thus (4.1) and (4.2) are obtained again. But there is a difference. While ψ_1^1 is still invariant, $\psi_1^2(\phi,\psi)$ carries a charge-one representation of $H(\phi)$.

C. SU(2) with three triplets of Higgs bosons and a doublet fermion[7]

Assume that the orbit Φ is characterized by the scalars being linearly independent. Then

$$H=Z_2=\{1,-1\}\ . \tag{4.7}$$

Also

$$G=\mathrm{SU}(2)\approx S^3 \tag{4.8}$$

and

$$\Phi=G/H=\mathrm{SO}(3)\approx \mathrm{RP}^3\ . \tag{4.9}$$

The bundle becomes

$$\mathrm{SU}(2)\to \mathrm{RP}^3 \tag{4.10}$$

with fiber Z_2. If it were trivial, then it would be true that

$$SU(2)\approx \mathrm{RP}^3\times Z_2 \tag{4.11}$$

topologically. In particular SU(2) would be disconnected. Since SU(2) is connected, (4.9) is nontrivial and there is no section. Method III does not improve the situation since the fermion subspaces carry faithful representations of H.

Consider method IV. For this case,[11]

$$F\approx \mathrm{U}(2)\approx \mathrm{U}(1)\times \mathrm{SU}(2)\approx S^1\times S^3\ . \tag{4.12}$$

(The second homeomorphism is not a group isomorphism.) The fiber is

$$\mathrm{U}(1)\times \mathrm{U}(1)\approx S^1\times S^1\ , \tag{4.13}$$

and the base is

$$D\approx \mathrm{U}(2)/[\mathrm{U}(1)\times \mathrm{U}(1)]\approx S^2\ . \tag{4.14}$$

Equation (4.14) follows from the fact that in this simple case a decomposition is determined by a single direction in C^2. Thus

$$D\approx \mathrm{CP}^2\ . \tag{4.15}$$

Then a look at CP^2 reveals that it is homeomorphic to S^2. Thus the bundle (3.12) is

$$P: S^1\times S^3\to S^2 \tag{4.16}$$

with fiber (4.13). The object is to lift the map

$$V: \mathrm{RP}^3\to S^2 \tag{4.17}$$

to

$$R: \mathrm{RP}^3\to S^1\times S^3 \tag{4.18}$$

so that (3.14) is satisfied.

It will now be shown that this cannot be done. The map P of (4.16) can be considered in two stages,

$$S^1\times S^3\to S^3\to S^2\ . \tag{4.19}$$

Thus if V lifts to $S^1\times S^3$, then it lifts to S^3. Now there is a cellular decomposition of RP^3 in which the two-cell is RP^2.[12] Thus if V lifts to $S^1\times S^3$, then this RP^2 lifts to S^3. Since RP^2 is of lower dimension than S^3, a map $\mathrm{RP}^2\to S^3$ will miss some point on S^3. Since S^3 with one point removed is contractible, any map $\mathrm{RP}^2\to S^3$ is homotopically trivial. So the map $\mathrm{RP}^2\to S^3\to S^2$ will also be homotopically trivial. Thus if V lifts to $S^1\times S^3$, then V restricted to the two-cell of RP^3 must be homotopically trivial. However, a look at V reveals that this is nontrivial.

So for this case, w's and ψ's cannot be constructed. It is also true that, in this case, complementarity does not hold.[7]

D. SU(2) with three triplets of Higgs bosons and a triplet of fermions

Both the scalars and the fermions are invariant under the Z_2 center of SU(2). Thus, the gauge group can be taken to be SO(3). Assume again that Φ is characterized by the linear independence of the scalars. Then, with no further loss of generality, assume that the scalars are orthogonal.

Now SO(3) is completely broken, so method I will produce three gauge-invariant ψ's. Clearly, they are the projections of ψ onto the three scalar directions.

E. SU(2) with a triplet of Higgs bosons and a doublet of fermions.[8]

Assume that ϕ_0 is in the "3" direction. Then

$$H=\mathrm{U}(1)=e^{i\sigma_3\theta},\ \ 0\le\theta<2\pi\ . \tag{4.20}$$

The two-dimensional fermion representation splits into two nontrivial one-dimensional representatiuons of H with opposite charges.

The bundle (3.6) is

$$\mathrm{SU}(2)\to S^2 \tag{4.21}$$

with fiber U(1). This is the Hopf fibration and is a nontrivial principal bundle.[13] There is no section. Since each fermion subspace carries a faithful representation of H, method III will not work either.

Consider again method IV. The structure is the same as in C except that now

$$\Phi\approx S^2\ . \tag{4.22}$$

Since

$$\pi(S^2,\mathrm{U}(2))=\pi_2(\mathrm{U}(2))=\pi_2(S^1\times S^3)$$
$$=\pi_2(S^1)+\pi_2(S^3)=0 \qquad (4.23)$$

all maps $S^2\rightarrow\mathrm{U}(2)$ are homotopic to the constant map. Thus if the map

$$V:S^2\rightarrow S^2 \qquad (4.24)$$

is homotopic to the identity map, it will not lift. A direct examination of V shows that it is the identity map. The construction of the ψ_j^i cannot be carried out. This is also a case where complementarity is not expected to hold.[8]

F. SU(2) with a triplet of Higgs bosons and a triplet of fermions

As in D, the gauge group can be taken to be SO(3). In this case, it is convenient to consider structures over the reals rather than the complexes. ϕ_0 and Φ are as in E and

$$H=\mathrm{SO}(2)\ . \qquad (4.25)$$

Ψ splits to V_1 and V_2. V_1 carries a trivial representation of H. V_2 carries the defining two-dimensional representation. Thus it is easy to obtain

$$\psi_1^1=\hat{\phi}\cdot\psi\ . \qquad (4.26)$$

The others cannot be constructed. To show this, go directly to method IV. For this case, F is O(3). D is RP^2. Φ is S^2. Since

$$\pi_2(\mathrm{SO}(3))=0\ , \qquad (4.27)$$

V will lift only if it is homotopically trivial. V assigns to each $\hat{\phi}\in S^2$ a decomposition in D. With RP^2 viewed as lines through the origin in R^3, V assigns to $\hat{\phi}$ the line along which it lies. Thus V is the double cover

$$S^2\rightarrow\mathrm{RP}^2\ . \qquad (4.28)$$

A look at the homotopy sequence confirms intuition. This is homotopically nontrivial.

Thus the ψ_i^2 cannot be found. This example shows that the problem with the previous examples was not that the fermions were in the doublet representation.

G. SO(4) with vector Higgs bosons and vector fermions

This example is included to show that there are nontrivial examples wherein the w's can be constructed. $G=\mathrm{SO}(4)$ is broken to

$$H=\mathrm{SO}(3)\ . \qquad (4.29)$$

Ψ splits into V_1 and V_2. V_1 carried the one-dimensional trivial representation. V_2 carries the three-dimensional vector representation of H.

For this case,

$$\Phi=\mathrm{SO}(4)/\mathrm{SO}(3)=S^3\ . \qquad (4.30)$$

It is a famous result that S^1, S^3, and S^7 are parallelizable.[13] Thus the bundle

$$\mathrm{SO}(4)\rightarrow S^3 \qquad (4.31)$$

is trivial. The w's can all be constructed. One way to do it is to use the fact that S^3 can be identified with the group of unit quaternions. Then

$$w_1^1=\phi=(\phi_1,\phi_2,\phi_3,\phi_4)\ ,w_2^2=(-\phi_3,\phi_4,\phi_1,-\phi_2)\ ,$$
$$w_1^2=(\phi_4,\phi_3,-\phi_2,-\phi_1)\ ,w_3^2=(\phi_2,-\phi_1,\phi_4,-\phi_3)\ . \qquad (4.32)$$

Note that there is an analogous SO(8) example since $\mathrm{SO}(8)\rightarrow\mathrm{SO}(8)/\mathrm{SO}(7)\approx S^7$ is also trivial. The Cayley numbers are used in the construction in this case. It seems to be the lowest-dimensional example of a simple, compact group with a "proper" trivial bundle.

We have also studied some other cases. These do not involve any concepts beyond those already illustrated. Just the results will be given.

The first is a piece of the SU(5) model, $G=\mathrm{SU}(5)$. An adjoint Higgs representation gives[14]

$$H=[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)]/Z_6\ . \qquad (4.33)$$

A fundamental fermion splits

$$\underline{5}\rightarrow(3,1)+(1,2)\ . \qquad (4.34)$$

The w's cannot be found.

Let $G=\mathrm{SU}(3)$, and break it with a triplet Higgs boson to $H=\mathrm{SU}(2)$. Then triplet fermions split $\underline{3}\rightarrow\underline{1}+\underline{2}$. w_1^1 can be found, but w_i^2 cannot be.

When an adjoint Higgs representation breaks SU(3) to U(2), triplet fermions split $\underline{3}\rightarrow\underline{1}+\underline{2}$. The "1" carries a nontrivial representation of U(2). Not even the associated w_1^1 can be constructed.

Finally,[15] when a $\underline{27}$ breaks SU(3) to $\mathrm{SO}(3)\times Z_3$, the triplet fermion does not split: $3\rightarrow3$. Thus, w's can be constructed.

Note added. Connections among topological charges, spacetime topology, and spontaneous symmetry breaking have been discussed by C. J. Isham, Phys. Lett. 102B, 251 (1981); J. Phys. A 14, 2943 (1981).

ACKNOWLEDGMENTS

This work was stimulated by discussions with N. Snyderman. It is a pleasure to acknowledge other conversations with A. Edelson, R. Kirby, A. Krener, W. Pfeffer, I. Singer, E. Spanier, and J. Wagoner. We would like to thank J. Gunion for reading the manuscript. The research was supported by the Alfred P. Sloan Foundation and the United States Department of Energy.

---

[1]S. Elitzur, Phys. Rev. D 12, 3978 (1975).

[2]G. De Angelis, D. De Falco, and F. Guerra, Phys. Rev. D 17, 1624 (1978).

[3]J. Fröhlich, G. Morchio, and F. Strocchi, Phys. Lett. 97B, 249 (1980); Nucl. Phys. B190 [FS3], 553 (1981).

[4]T. Banks and E. Rabinovici, Nucl. Phys. B160, 349 (1978).

[5]G. 't Hooft, in *Recent Developments in Gauge Theories,* edited by G. 't Hooft *et al.* (Plenum, New York, 1980), p. 117.

[6]K. Osterwalder and E. Seiler, Ann. Phys. (N. Y.) 110, 440 (1978).

[7]E. Fradkin and S. Shenker, Phys. Rev. D 19, 3682 (1979).

[8]S. Dimopoulos, S. Raby, and L. Susskind, Nucl. Phys. B173, 208 (1980).

[9]R. Bishop and R. Crittenden, *Geometry of Manifolds* (Academic, New York, 1964).

[10]K. Macrae, Phys. Rev. D 22, 1966 (1980).

[11]C. Chevalley, *Theory of Groups I* (Princeton Univ. Press, Princeton, N. J., 1946).

[12]J. Vick, *Homology Theory* (Academic, New York, 1973).

[13]N. Steenrod, *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton, N. J.,1951).

[14]D. Scott, Nucl. Phys. B171, 95 (1980); M. Daniel, G. Lazarides, and Q. Shafi, Nucl. Phys. B170 [FS1], 156 (1980).

[15]R. Slansky, T. Goldman, and G. Shaw, Phys. Rev. Lett. 47, 887 (1981); A. Balachandran, V. Nair, and C. Trahern, Nucl. Phys. B196, 413 (1982).