Harrison-Neugebauer-type transformations for instantons: Multicharged monopoles as limits

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A new formulation, leading to agreeable simplifications, is given, in Sec. II, for constructing axially symmetric, multicharged monopoles through nonlinear superpositions. The ansatz introduced for this end is related to a modification of Yang's R gauge, which we call the "spherical" R gauge. This aspect of Sec. II is taken up in the Appendix. In Sec. III, we generalize our formalism to give a parallel construction for a particular hierarchy of instanton configurations which have the above-mentioned monopoles as static limits obtained through rescaling. Harrison-Neugebauer-type transformations are adapted to the case of finite action through a technique (conveniently termed the "de Sitter trick," though we are concerned here with flat Euclidean space) found useful previously. This is recapitulated in Sec. III. The action and a crucial regularity constraint are studied in Sec. IV. Possible further developments are indicated in the concluding remarks.

I. INTRODUCTION

The title of this paper may appear somewhat surprising. The transformations of Harrison[1] and Neugebauer[2] are associated with *static* axial symmetry, whereas instantons must be time dependent to avoid a trivial divergence of the action. *In flat space,* such transformations have so far been applied to construct axially symmetric, static monopole solutions.[3-7] They may have finite energy and magnetic charge, but the infinite domain $[-\infty,\infty]$ of the Minkowski or the Euclidean time t must trivially lead to infinite action in both contexts. We solve this problem by formulating our ansatz (see Sec. III) in terms of coordinates such that the Euclidean coordinates (t,r,θ,φ) are replaced by $(\tau,\rho,\theta,\varphi)$, where τ is *periodic,* with a period 2π for our normalization. We then adapt the above-mentioned transformations to an ansatz where only ρ and θ appear. For this "τ-static" case one can of course have finite actions if one can avoid other types of singularities. This is achieved. One can then transform back to the coordinates (t,r,θ,φ). In terms of the Euclidean time t our self-dual, finite-action solutions satisfying all the required regularity constraints will be neither static nor periodic. Considering always flat space, the coordinates (τ,ρ) serve as a technical device to incorporate in the intermediate steps some of the simplicity of static solutions.

The next question one might ask is, assuming that such a construction is possible, what can be its interest, particularly in view of the fact that we have already the general Atiyah-Drinfeld-Hitchin-Manin[8,9] construction for all instanton solutions. The answer lies in the many interesting special properties of the restricted classes of instanton solutions selected out by our technique. The simplest class thus obtained has already been explored in some detail in our previous papers.[10,11] It leads to what we have called a "summable chain" of instantons. Such a chain in a very simple scaling limit gives the singly charged-Prasad-Sommerfield (PS) monopole, the t dependence disappearing in this limit. [Throughout this paper by "monopole" we will mean static, self-dual Euclidean SU(2) gauge fields, the gauge potential component A_t replacing the Higgs scalar Φ.] It was shown that one can construct for this class totally explicitly compact Green's functions (even including the explicit inversion of a notorious matrix arising for a scalar field in the adjoint representation[10]) and quantum fluctuation determinants where the final space-time integral can be evaluated analytically.[11] This last fact permitted us to evaluate explicitly, and for arbitrary index, the corrections to the so-called "dilute-gas approximation." We have also shown recently how the same technique can furnish complex self-dual solutions with real, finite action which give, in an analogous limit, Manton's complex monopoles.[12] In this paper we extend our technique to construct higher-order "chains" which have as static, infinite-action limits multicharged monopoles obtained through nonlinear superpositions.[3-7] In particular we follow throughout very closely the formalism of Forgacs, Horvath, and Palla,[3-6] referred to as FHP hereafter. Our study remains incomplete in several respects. We hope to obtain more results in subsequent papers.

Thus we find a particular hierarchy in the space of instanton solutions, the "n-chain" having a monopole of magnetic charge n as the static limit. This is of interest in both ways—for studying instantons and also for studying monopoles. On one hand, it selects out special classes of instantons with, it is hoped, eventually exploitable special properties. On the other hand, taking a very simple limit in known results for instantons, one obtains immediately the corresponding results for monopoles without practically any labor at all. This was illustrated for the "1-chain"[10,11] elsewhere.
steps, though it involves some repetitions. In Sec. II (and the Appendix) we reconstruct the static FHP monopoles in a somewhat different gauge. This serves a double purpose. Firstly it introduces some very agreeable simplifications in the construction of the monopoles. *The simplification at the level of monopole of charge 1 is indeed spectacular.* The new features of our gauge are thus first displayed in an already familiar context. The most important property is exhibited in Sec. III, where our gauge is shown to permit a generalization for an analogous treatment of instantons. Other points of interest will be found in the final remarks.

II. NONLINEAR SUPERPOSITION OF MONOPOLES IN THE "SPHERICAL" R GAUGE

For the flat Euclidean metric

$$ds^2=dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)$$

and the SU(2) gauge group, we take the ansatz, implying axial symmetry,

$$\begin{aligned}
A_t&=\frac{f_r}{f}\frac{\sigma_3}{2}+\frac{\psi_r}{f}\frac{\sigma_1}{2}\,,\\
A_r&=\frac{\psi_r}{f}\frac{\sigma_2}{2}\,,\\
A_\theta&=\frac{\psi_\theta}{f}\frac{\sigma_2}{2}\,,\\
(\sin\theta)^{-1}A_\varphi&=\frac{f_\theta}{f}\frac{\sigma_3}{2}+\frac{\psi_\theta}{f}\frac{\sigma_1}{2}\,,
\end{aligned}\tag{2.1}$$

where $f_r=\partial_r f$, $f_\theta=\partial_\theta f$, and so on and $f(r,\theta)$ and $\psi(r,\theta)$ do not depend on the azimuthal angle φ. (One can replace A_t by a Higgs scalar along with $t\to -it$. We use the term *monopole* in this sense.)

The self-duality constraints

$$\begin{aligned}
F_{tr}&=(r^2\sin\theta)^{-1}F_{\theta\varphi}\,,\\
F_{t\theta}&=(\sin\theta)^{-1}F_{\varphi r}\,,\\
(\sin\theta)^{-1}F_{t\varphi}&=F_{r\theta}\,,
\end{aligned}\tag{2.2}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu]$$

lead to

$$f\tilde{\Delta}f-(\vec{\nabla}f)^2+(\vec{\nabla}\psi)^2=0\,,\tag{2.3}$$

$$f\tilde{\Delta}\psi-2(\vec{\nabla}\psi)\cdot(\vec{\nabla}f)=0\,,\tag{2.4}$$

where (since $\partial\varphi\approx 0$ for our space of solutions)

$$\vec{\nabla}\equiv\left[\partial_r,\frac{1}{r}\partial_\theta\right]\tag{2.5}$$

and

$$\begin{aligned}
\tilde{\Delta}&\equiv\partial_r{}^2+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)\\
&=\Delta-\frac{2}{r}\partial_r
\end{aligned}\tag{2.6}$$

Δ being the complete Laplacian. Setting

$$\epsilon=f+i\psi\,,\quad \bar{\epsilon}=f-i\psi\tag{2.7}$$

the Eq. (2.3) and (2.4) can be condensed into

$$\tfrac{1}{2}(\epsilon+\bar{\epsilon})\tilde{\Delta}\epsilon-(\vec{\nabla}\epsilon)^2=0\,.\tag{2.8}$$

Thus in the Ernst equation the Δ is replaced by $\tilde{\Delta}$ in our case. The effect of this will soon become evident. (The consequences of axial-symmetry restrictions in the standard and our R gauge are briefly recapitulated in the Appendix. One may compare the two formulations at different stages.)

We have a so-called "Higgs vacuum" solution

$$f=e^r\,,\quad \psi=0\tag{2.9}$$

when

$$A_t=\frac{\sigma_3}{2}\,,\quad A_r=A_\theta=A_\varphi=0\,.$$

Thus spherical symmetry is not broken in this seed solution from which the iteration will start. This should be compared to the solution (the simplest nontrivial one possible for Ernst equation)

$$f=e^{z}\,,\quad \psi=0 \tag{2.10}$$

used as a starting point by FHP,[3-6] which leads to a complicated form already for the monopole of charge 1. For charge 1 we will obtain a remarkable simplification, as will be seen.

One may note that the inversion

$$r\rightarrow\frac{1}{r} \tag{2.11}$$

in (2.8) reduces it to the standard Ernst equation. But we will *not* start from (2.10) modified in this way ($z=r\cos\theta\rightarrow\cos\theta/r$). Nor does $f=e^{1/r}$, $\psi=0$ give a vacuum in the standard formalism. Thus though there is a correspondence [through (2.11)] between the solutions of the two formalisms, the properties of the solutions thus related can be very different. In practice one chooses directly the most convenient one for each case, as is evident on comparing (2.9) and (2.10).

We now state briefly the symmetries and transformations of (2.8) to be used in our construction of monopoles from (2.9). They should always be compared to the parallel formalism of FHP.[5,6] Ehler's transformation remains intact,

$$\epsilon'=\frac{a\epsilon+ib}{d-ic\epsilon}\,, \tag{2.12}$$

where one may set $(ad-bc)=1$. The Nangebauer-Kramer (NK) mapping I is now modified to

$$\begin{aligned}
&If=\frac{\sin\theta}{rf}\,,\\
&I\left[\frac{\psi_r}{f}\right]=\frac{i\psi_\theta}{rf}\,,\\
&I\left[\frac{1}{r}\frac{\psi_\theta}{f}\right]=-i\frac{\psi_r}{f}\,.
\end{aligned} \tag{2.13}$$

We define

$$\partial_\pm\equiv\tfrac{1}{2}\left[\partial_r\pm\frac{i}{r}\partial_\theta\right] \tag{2.14}$$

and

$$M_1=\frac{1}{2f}\partial_+\epsilon\,,\quad M_2=\frac{1}{2f}\partial_+\bar{\epsilon}\,, \tag{2.15}$$

$$N_1=\frac{1}{2f}\partial_-\bar{\epsilon}\,,\quad N_2=\frac{1}{2f}\partial_-\epsilon\,. \tag{2.16}$$

Equation (2.8) now leads to

$$\begin{aligned}
&\partial_-M_1=-M_1(N_1-N_2)-\frac{ie^{i\theta}}{4r\sin\theta}(N_2-M_1)\,,\\
&\partial_-M_2=-M_2(N_2-N_1)-\frac{ie^{i\theta}}{4r\sin\theta}(N_1-M_2)\,,\\
&\partial_+N_1=-N_1(M_1-M_2)+\frac{ie^{-i\theta}}{4r\sin\theta}(M_2-N_1)\,,\\
&\partial_+N_2=-N_2(M_2-M_1)+\frac{ie^{-i\theta}}{4r\sin\theta}(M_1-N_2)\,.
\end{aligned} \tag{2.17}$$

Using (2.13) one obtains

$$\begin{aligned}
&IM_1=-M_2+\frac{ie^{i\theta}}{4r\sin\theta}\,,\\
&IM_2=-M_1+\frac{ie^{i\theta}}{4r\sin\theta}\,,\\
&IN_1=-N_1-\frac{ie^{-i\theta}}{4r\sin\theta}\,,\\
&IN_2=-N_2-\frac{ie^{-i\theta}}{4r\sin\theta}\,.
\end{aligned} \tag{2.18}$$

We define (see Fig. 1)

$$\begin{aligned}
&R^2(c)=r^2+c^2-2rc\cos\theta\,,\\
&R(c)\sin\omega=r\sin\theta\,,
\end{aligned} \tag{2.19}$$

and

$$R(c)\cos\omega=r\cos\theta-c\,.$$

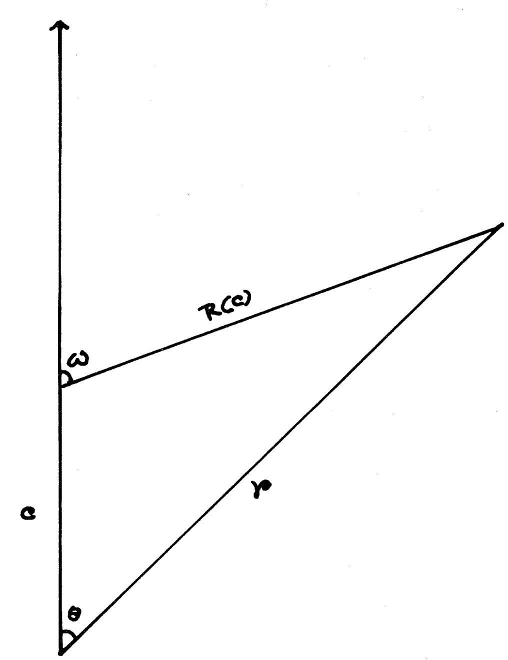

FIG. 1. Geometrical relationships of Eq. (2.19).

[More precisely $\omega=\omega(c)$.]

For a real c, ω has an evident geometrical meaning but we will include the possibility of complexification. Harrison's transformation[1,5,6] is modified for our case as follows. The pseudopotential satisfies

$$\partial_+ q=(M_2-M_1)q+e^{i(\omega-\theta)}(M_2-M_1q^2)\ ,$$
$$\partial_- q=(N_1-N_2)q+e^{-i(\omega-\theta)}(N_1-N_2q^2)\ . \tag{2.20}$$

Thus instead of just ω (Ref. 13), we have the difference $(\omega-\theta)$.

We define

$$p\equiv e^{i(\omega-\theta)} \tag{2.21}$$

and

$$\bar{q}=-\frac{p+q}{1+pq}\ . \tag{2.22}$$

Thus $\bar{q}$ is formally the same as for FHP[14] but a *difference is implicit in the definition of p.* For example, for $c=0$, our $p=1$, and hence

$$\bar{q}=-1\ ,\quad \text{for any } q\neq -1\ . \tag{2.23}$$

(For $q\to -1$, limiting forms are to be used.) Thus the appearance of the *difference* $(\omega-\theta)$ can simplify certain situations. This will indeed be the case.

It is instructive to write down at once the effect of the transformation[15]

$$B=IH\ ,$$

where H is the Harrison transformation. We obtain, finally,

$$BM_1=-\bar{q}\left[\frac{1}{q}M_2+\frac{ie^{i\omega}}{4r\sin\theta}\right],$$
$$BM_2=-\frac{1}{\bar{q}}\left[qM_1+\frac{ie^{i\omega}}{4r\sin\theta}\right],$$
$$BN_1=-\frac{1}{\bar{q}}\left[\frac{1}{q}N_1-\frac{ie^{-i\omega}}{4r\sin\theta}\right],$$
$$BN_2=-\bar{q}\left[qN_2-\frac{ie^{-i\omega}}{4r\sin\theta}\right]. \tag{2.24}$$

One can verify directly that the set (BM_1, BM_2, BN_1, BN_2) also satisfies the Eq. (2.17). (The set HM_1 etc. can immediately be obtained since $H=IB$.) Thus apart from certain constant relative phase factors we have formally essentially the same transformation as FHP.[15] Let us however explore their content by starting from (2.9).

For

$$f=e^r\ ,\quad \psi=0\ ,$$
$$M_1=M_2=N_1=N_2=\tfrac{1}{4}\ . \tag{2.25}$$

Note that they are all real for our choice of conventions. Also that now

$$2\,\mathrm{Tr}(A_t{}^2)=4(M_1+N_2)(M_2+N_1)=1\ .$$

Injecting (2.25) in (2.20) one obtains using (2.19)

$$q=\tanh\{\tfrac{1}{2}(R(c)-K)\}\ . \tag{2.26}$$

Thus again the initial differences compensate to give the same solution (with a change of sign) as for the FHP formalism.[16] Let us however continue. To generate the monopole of unit charge we choose

$$K=0 \text{ and } c=0\ ,$$

so that

$$q=\tanh\left[\frac{r}{2}\right]. \tag{2.27}$$

Now for our case $\bar{q}=-1$, simply [see (2.23)]. Combining (2.27), (2.24), (2.15), and (2.7) one obtains finally

$$f=\frac{\sinh r}{r}\sin\theta\ ,$$
$$\psi=\cos\theta\ . \tag{2.28}$$

This should be compared to the FHP result[17]

$$\epsilon=F^{-1}(r\sin\theta+iP)\ ,$$

where

$$F=\frac{r}{\sinh r}+r\cosh z\coth r-z\sinh z\ ,$$
$$P=z\cosh z$$
$$\quad -r\sinh z\coth z\quad (z=r\cos\theta)\ . \tag{2.29}$$

Before passing on to multicharged monopoles let us briefly note the following points. Injecting (2.28) into (2.1) one obtains, defining χ through

$$e^{\chi}=\frac{r}{\sinh r}\ , \tag{2.30}$$

$$A_t=-\chi_r\frac{\sigma_3}{2}\ ,\quad A_r=0\ ,$$
$$A_\theta=-e^{\chi}\frac{\sigma_2}{2}\ ,\quad A_\varphi=-e^{\chi}\sin\theta\frac{\sigma_1}{2}+\cos\theta\frac{\sigma_3}{2} \tag{2.31}$$

where

$$\chi_r = \frac{d\chi}{dr} = \left[\frac{1}{r} - \coth r\right] .$$

A gauge transformation

$$A_\mu \to U A_\mu U^{-1} + i(\partial_\mu U) U^{-1} , \tag{2.32}$$

where

$$U = e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{-i\pi\sigma_2/2} \tag{2.33}$$

leads to the familiar form

$$\begin{aligned} A_t &= \chi_r \frac{\vec{\sigma}\cdot\hat{r}}{2} \equiv \chi_r \frac{\hat{\sigma}_r}{2} , \\ \vec{\mathrm{A}} &= (e^{\chi} - 1)\frac{1}{r}\left[\hat{r} \times \frac{\vec{\sigma}}{2}\right] , \end{aligned} \tag{2.34}$$

i.e.,

$$A_r = 0 , \quad A_\theta = -(e^{\chi} - 1)\frac{\hat{\sigma}_\varphi}{2} ,$$

$$\frac{A_\varphi}{\sin\theta} = (e^{\chi} - 1)\frac{\hat{\sigma}_\theta}{2} ,$$

$\hat{\sigma}_r, \hat{\sigma}_\theta, \hat{\sigma}_\varphi$ being spherical projections of $\vec{\sigma}$. The *Cartesian* components are denoted by $\vec{\mathrm{A}}$ and

$$\hat{r} = (\sin\theta\cos\varphi , \sin\theta\sin\varphi , \cos\theta) .$$

We have found that one can generate this "1-monopole" from the same vacuum (2.9) in three ways.

(1) The first is the $B = IH$ transformation which we have just used, modified for our chosen gauge.

(2) A *complex gauge transformation* of (2.9) by

$$U = e^{-i\lambda\sigma_2/2} ,$$

where

$$\begin{aligned} \cos\lambda &= -\coth r , \\ \sin\lambda &= (i \sinh r)^{-1} \end{aligned} \tag{2.35}$$

leads to

$$f' = (\sinh r)^{-1} , \quad \psi' = i \coth r , \tag{2.36}$$

i.e.,

$$\begin{aligned} A_t &= -\coth r \frac{\sigma_3}{2} + (i \sinh r)^{-1}\frac{\sigma_1}{2} , \\ A_r &= (i \sinh r)^{-1}\frac{\sigma_2}{2} , \quad A_\theta = 0 , \quad A_\varphi = 0 . \end{aligned} \tag{2.37}$$

This is a complex vacuum and is again related to one used by FHP[18] in a similar context through the interchange $r \leftrightarrow z$. *But now instead of an H transformation, the simpler I transformation (2.13) leads to (2.31) from (2.36).*

(3) A complex Ehler's transformation, where in (2.12) along with (2.9) we set

$$-ia = -b = c = id = \frac{1}{\sqrt{2}} , \tag{2.38}$$

is another way of generating the complex vacuum (2.36). Then of course we use I again. Thus (2) and (3) are not really distinct here. But if such techniques can be made to work for more interesting cases a generalization of (2) could be particularly interesting since neither of its two steps involves any integration. That is why we have displayed it separately. In this paper we will however continue with the H (or B) transformations which avoids integration, after the first easy step, through a composition theorem.

The composition theorem for H transformations can be shown to be preserved, formally intact, for our case. Namely, when applying two successive H-type transformations one does not have to integrate (2.20) twice but for the second stage one has as for FHP[19]

$$q' = \frac{(\bar{q}_1 p_1 - \bar{q}_2 p_1)}{q_1(\bar{q}_1 p_1 - \bar{q}_2 p_2)} , \tag{2.39}$$

where

$$\bar{q}_i = -\frac{(p_i + q_i)}{(1 + p_i q_i)} \quad (i = 1,2)$$

and (p_1, q_2) and (p_2, q_2) are solutions for the same M's and N's of (2.20) but differing in the choice of parameters such as c and K in (2.26). Hence proceeding in a fashion strictly parallel to that of FHP we obtain thus composing two successive H transformations

$$(M_i^{(0)}, N_i^{(0)}) \xrightarrow{H_1} (M_i' N_i') \xrightarrow{H_2} (M_i, N_i) \tag{2.40}$$

the following results.

We define

$$\begin{aligned} Q_1 &\equiv (q_1 p_1 - q_2 p_2) , \quad Q_2 \equiv (q_1 p_2 - q_2 p_1) , \\ \bar{Q}_1 &\equiv (\bar{q}_1 p_1 - \bar{q}_2 p_2) , \quad \bar{Q}_2 \equiv (\bar{q}_1 p_2 - \bar{q}_2 p_1) . \end{aligned} \tag{2.41}$$

Then,

$$M_1=\frac{\bar{Q}_2}{\bar{Q}_1}\left[\frac{Q_1}{Q_2}M_1^{(0)}+\frac{(p_1{}^2-p_2{}^2)}{Q_2}\left[\frac{ie^{i\theta}}{4r\sin\theta}\right]\right],$$

$$M_2=\frac{\bar{Q}_1}{\bar{Q}_2}\left[\frac{Q_2}{Q_1}M_2^{(0)}-\frac{q_1q_2(p_1{}^2-p_2{}^2)}{Q_1}\left[\frac{ie^{i\theta}}{4r\sin\theta}\right]\right],$$

$$(2.42)$$

$$N_1=\frac{\bar{Q}_1}{\bar{Q}_2}\left[\frac{Q_1}{Q_2}N_1^{(0)}+\frac{q_1q_2(p_1{}^2-p_2{}^2)}{p_1p_2Q_2}\left[\frac{-ie^{-i\theta}}{4r\sin\theta}\right]\right],$$

$$N_2=\frac{\bar{Q}_2}{\bar{Q}_1}\left[\frac{Q_2}{Q_1}N_2^{(0)}-\frac{(p_1{}^2-p_2{}^2)}{p_1p_2Q_1}\left[\frac{-ie^{-i\theta}}{4r\sin\theta}\right]\right].$$

One can go on to the generalization to n-step using the same determinantal structure as FHP[20] and study the regularity constraints by constructing

$$2\,\mathrm{Tr}(A_t{}^2)=4(M_1+N_2)(M_2+N_1)\ . \qquad (2.43)$$

Since there are now no essentially new features we prefer to pass on directly to the generalization of the next section, where the transformations are adapted to the construction of *finite action* self-dual solutions. *The results for the monopoles can then anyhow be recovered through a very simple scaling limit.*

A final comment should however be added. From charge 2 onward there is no spectacular simplification comparable to (2.28) in our gauge. The simplification there is an agreeable but not an essential feature. The real utility of our gauge will be evident in Sec. III.

Something, however, of the simplicity of the initial stages (due to a spherically symmetric seed solution and $\bar{q}=-1$ at the origin) does filter through to higher orders. Thus for charge 3 (indeed for all odd charges) there is some simplification due to one "center" at the origin. In fact it might be interesting to try, in our gauge, to use the charge-1 solution with generalized parameters directly as a building block. We will not attempt such variations in this paper.

III. HARRISON-TYPE TRANSFORMATIONS FOR INSTANTONS: MULTICHARGED MONOPOLES AS SCALING LIMITS

We will use a technique which led us elsewhere to "summable chains" of instantons which have the singly charged Prasad-Sommerfield monopole as a limit[10,11] and to complex, self-dual solutions with finite, real actions which have Manton's complex monopoles as limiting configurations.[12] Indeed it has also been shown useful in constructing merons,[21] non-self-dual complex solutions with finite complex actions[22] and a special class of SU(3) instantons.[23] It will work again in the context of nonlinear superposition of monopoles. We recapitulate the basic points once again for the reader's convenience.

We start with the flat Euclidean metric

$$ds^2=dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\ . \qquad (3.1)$$

The coordinate transformation

$$(t+ir)=\tan\left[\frac{\tau+i\rho}{2}\right] \qquad (3.2)$$

gives

$$ds^2=(\cosh\rho+\cos\tau)^{-2}[d\tau^2+d\rho^2+\sinh^2\rho(d\theta^2+\sin\theta\,d\varphi^2)]$$

and the domain

$$-\infty<t<\infty\ ,\quad 0\le r<\infty \qquad (3.3)$$

now corresponds to

$$-\pi\le\tau\le\pi\ ,\quad 0\le\rho<\infty\ . \qquad (3.4)$$

The conformal factor in (3.2) has singularities. But in the construction of solutions of the gauge fields and the calculation of actions an overall conformal factor can be ignored. The regularity properties of the solutions can be checked directly in the spherical coordinates (3.1) (or, if necessary, finally in terms of the Cartesian ones). The interesting point is that since τ is periodic one can now construct "τ-static" (τ-independent) solutions of finite action. In general such solutions will neither be static nor periodic in terms of the Euclidean time t, which will provide us with the correct interpretation. [Changing the conformal factors in (3.3) to simply $(\cosh\rho)^{-2}$ one gets a De Sitter space of constant curvature. In this paper we will be concerned with the flat space (3.1) and use (3.2) as a technical device for constructing solutions.]

Another property will be of central interest for us. Rescaling,

$$\tau'=\lambda\tau\ ,\quad \rho'=\lambda\rho \qquad (3.5)$$

and making $\lambda\to\infty$,

$$\frac{4ds^2}{\lambda^2}\equiv ds'^2=d\tau'^2+d\rho'^2+\rho'^2(d\theta^2+\sin^2\theta\,d\varphi^2)\ , \qquad (3.6)$$

where now

$$-\infty<\tau'<\infty\ ,\quad 0\le\rho'<\infty\ .$$

(One can now substitute $\tau'=t',\rho'=r'$, since we have usual spherical coordinates.) Rescaling simultaneously the gauge potentials as

$$\begin{aligned}&A'_{\tau'}=\frac{1}{\lambda}A_\tau\ ,\quad A'_{\rho'}=\frac{1}{\lambda}A_\rho\ ,\\&A'_\theta=A_\theta\ ,\quad A'_\varphi=A_\varphi\end{aligned}\tag{3.7}$$

in the limit, the τ-independent solutions reduce to really static flat space solutions (t' now being the Euclidean time).

In this paper we will construct self-dual finite action, axially symmetric solutions $A_\mu(\rho,\theta)$. Through (3.2) they will be interpreted as time-dependent (t-dependent) instantons. Through (3.5) and (3.7) they will lead us to static monopoles of Sec. II.

At each step the results to follow should be compared with the corresponding ones of Sec. II. There will be great formal similarity. But the significance of the content will be quite different. *(In many cases we will use the same symbol as in the previous section for simplicity.* This should not cause any confusion.)

Now the ansatz is [with $f=f(\rho,\theta)$, $\psi=\psi(\rho,\theta)$, $f_\rho=\partial_\rho f$, etc.]

$$\begin{aligned}&A_\tau=\frac{f_\rho}{f}\frac{\sigma_3}{3}+\frac{\psi_\rho}{f}\frac{\sigma_1}{2}\ ,\\&A_\rho=\frac{\psi_\rho}{f}\frac{\sigma_2}{2}\ ,\\&A_\theta=\frac{\psi_\theta}{f}\frac{\sigma_2}{2}\ ,\\&(\sin\theta)^{-1}A_\varphi=\frac{f_\theta}{f}\frac{\sigma_3}{2}+\frac{\psi_\theta}{f}\frac{\sigma_1}{2}\ .\end{aligned}\tag{3.8}$$

The self-duality constraints for

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu,A_\nu]\ ,$$

namely,

$$\begin{aligned}&F_{\tau\rho}=(\sinh^2\!\rho\,\sin\theta)^{-1}F_{\theta\varphi}\ ,\\&F_{\tau\theta}=(\sin\theta)^{-1}F_{\varphi\rho}\ ,\\&(\sin\theta)^{-1}F_{\tau\varphi}=F_{\rho\theta}\ ,\end{aligned}\tag{3.9}$$

reduce to

$$f\tilde\Delta f-(\vec\nabla f)^2+(\vec\nabla\psi)^2=0\ ,\tag{3.10}$$

$$f\tilde\Delta\psi-2(\vec\nabla\psi\cdot\vec\nabla f)=0\ ,\tag{3.11}$$

where now

$$\vec\nabla\equiv\left[\partial_\rho,\frac{1}{\sinh\rho}\partial_\theta\right]\ ,\quad\text{for }\partial\varphi\approx0\tag{3.12}$$

and

$$\tilde\Delta\equiv\partial_\rho{}^2+\frac{1}{\sinh^2\!\rho\,\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\tag{3.13}$$

$$=\Delta-2\coth\rho\partial_\rho\ ,\tag{3.14}$$

Δ being the complete "Laplacian" used extensively for complex solutions,[12] where we have set $\partial_\varphi\approx0$ on the space or our solutions. Defining

$$\epsilon=f+i\psi\ ,\quad\bar\epsilon=f-i\psi\ ,\tag{3.15}$$

one has

$$\tfrac12(\epsilon+\bar\epsilon)\tilde\Delta\epsilon=(\vec\nabla\epsilon)^2\ .\tag{3.16}$$

Ehler's transformation remains formally as before; i.e.,

$$\epsilon'=\frac{a\epsilon+ib}{d-ic\epsilon}\tag{3.17}$$

or, setting $(ad-bc)=1$,

$$f'\pm i\left[\psi'-\frac{a}{c}\right]=c^{-2}\left[f\pm i\left[\psi+\frac{d}{c}\right]\right]^{-1}\ .\tag{3.18}$$

The mapping I of (2.13) is now

$$\begin{aligned}&If=\frac{\sin\theta}{(\sinh\rho)f}\ ,\\&I\left[\frac{\psi_\rho}{f}\right]=\frac{i\psi_\theta}{(\sinh\rho)f}\ ,\\&I\left[\frac{\psi_\theta}{(\sinh\rho)f}\right]=-i\frac{\psi_\rho}{f}\ .\end{aligned}\tag{3.19}$$

We define

$$\partial_\pm\equiv\frac12\left[\partial_\rho\pm\frac{i}{\sinh\rho}\partial_\theta\right]\tag{3.20}$$

and

$$\begin{aligned}&M_1=\frac{1}{2f}\partial_+\epsilon\ ,\quad M_2=\frac{1}{2f}\partial_+\bar\epsilon\ ,\\&N_1=\frac{1}{2f}\partial_-\bar\epsilon\ ,\quad N_2=\frac{1}{2f}\partial_-\epsilon\ .\end{aligned}\tag{3.21}$$

We define also

$$X_\pm \equiv \tfrac{1}{2}\partial_\pm\left\{\ln\left[\frac{\sin\theta}{\sinh\rho}\right]\right\}$$
$$= -\frac{1}{4}\left[\frac{\cosh\rho \mp i\cot\theta}{\sinh\rho}\right] . \tag{3.22}$$

This turns out to be the correct generalization of the factors $[\pm i e^{\pm i\theta}/(4r\sin\theta)]$ of Sec. II. [Note that

$$\left[\partial_r \pm \frac{i}{r}\partial_\theta\right]\left[\ln\frac{\sin\theta}{r}\right] = \pm i\frac{e^{\pm i\theta}}{r\sin\theta} . \Bigg]$$

The content of (3.16) can now be expressed as the following four first-order equations.

$$\begin{aligned} \partial_- M_1 &= -M_1(N_1-N_2)+(M_1-N_2)X_+ ,\\ \partial_- M_2 &= -M_2(N_2-N_1)+(M_2-N_1)X_+ ,\\ \partial_+ N_1 &= -N_1(M_1-M_2)+(N_1-M_2)X_- ,\\ \partial_+ N_2 &= -N_2(M_2-M_1)+(N_2-M_1)X_- . \end{aligned} \tag{3.23}$$

The transformation (3.19) gives

$$\begin{aligned} IM_1 &= -M_2+X_+ ,\\ IM_2 &= -M_1+X_+ ,\\ IN_1 &= -N_1+X_- ,\\ IN_2 &= -N_2+X_- . \end{aligned} \tag{3.24}$$

We have next to suitably generalize $R(c)$ and $e^{i(\omega-\theta)}$ appearing in (2.19) and (2.20), respectively. The necessary results have already been obtained elsewhere.[24] We recapitulate the results essential for our present purpose.

We define η through

$$\cosh\eta = \cosh c \cosh\rho - \sinh c \sinh\rho \cos\theta , \tag{3.25}$$

where c is a constant parameter which may be complex. This gives [with $\eta\equiv\eta(c)$]

$$\sinh^2\eta = (\sinh c \cosh\rho - \cosh c \sinh\rho\cos\theta)^2 + \sinh^2\rho\sin^2\theta \tag{3.26}$$
$$= (\cosh c\sinh\rho - \sinh c\cosh\rho\cos\theta)^2 + \sinh^2 c\sin^2\theta . \tag{3.27}$$

In the limit $\lambda\to 0$ after rescaling

$$(\rho,c,\eta)\to(\lambda\rho,\lambda c,\lambda\eta) , \tag{3.28}$$

one obtains (2.19) with $\eta=R(c)$ and $\rho=r$ after rescaling.

We define

$$\cos\gamma \equiv \frac{\partial\eta}{\partial\rho} = (\cosh c\sinh\rho - \sinh c\cosh\rho\cos\theta)/(\sinh\eta) \tag{3.29}$$

giving

$$\sin\gamma \equiv \frac{1}{\sinh\rho}\frac{\partial\eta}{\partial\theta} = \sinh c\sin\theta/\sinh\eta . \tag{3.30}$$

It can be verified that in the above-mentioned limit (3.28),

$$\gamma\to(\omega-\theta) . \tag{3.31}$$

The equations for the pseudopotential q now becomes

$$\begin{aligned} \partial_+ q &= (M_2-M_1)q + e^{i\gamma}(M_2-M_1q^2) ,\\ \partial_- q &= (N_1-N_2)q + e^{i\gamma}(N_1-N_2q^2) , \end{aligned} \tag{3.32}$$

where $\partial_\pm$ are given now by (3.20). Setting again

$$p\equiv e^{i\gamma} , \tag{3.33}$$

let

$$\bar{q} = -\frac{p+q}{1+pq} . \tag{3.34}$$

Note that again for

$$c=0 ,\quad p=1 \tag{3.35}$$

and (for $q\neq -1$)

$$\bar{q}=-1 .$$

Corresponding to (2.24) one has now

$$\begin{aligned} BM_1 &= -\bar{q}\left[\frac{1}{q}M_2+e^{i\gamma}X_+\right] ,\\ BM_2 &= -\frac{1}{\bar{q}}\left[qM_1+e^{i\gamma}X_+\right] ,\\ BN_1 &= -\frac{1}{\bar{q}}\left[\frac{1}{q}N_1+e^{-i\gamma}X_-\right] ,\\ BN_2 &= -\bar{q}\left[qN_2+e^{-i\gamma}X_-\right] . \end{aligned} \tag{3.36}$$

Applying the I of (3.19) one obtains

$$HM_1 = IBM_1 = \frac{q}{\bar{q}} M_1 + \frac{p+\bar{q}}{\bar{q}} X_+ = -\frac{q(1+pq)}{(p+q)} M_1 - \frac{q(p^2-1)}{p+q} X_+ \ . \tag{3.37}$$

Similarly HM_2, HN_1, and HN_2 are easily written down. For the vacuum to be used as the starting point we take

$$f = e^{\alpha\rho}\ , \quad \psi = 0\ , \tag{3.38}$$

where α is a constant. In (2.5) we normalized a scaling factor of r in f. *Here the constant α will play a crucial role.* This will soon become evident. From (3.38) one obtains

$$M_1 = M_2 = N_1 = N_2 = \frac{\alpha}{4}\ . \tag{3.39}$$

Injecting this in (3.32) one obtains using (3.30)

$$q = \tanh\{\tfrac{1}{2}[\alpha\eta(c) - \beta]\}\ , \tag{3.40}$$

where β is a constant of integration.

Now choosing $\beta = 0$,$c = 0$,

$$q = \tanh\tfrac{1}{2}\alpha\rho \tag{3.41}$$

and $\bar{q} = -1$, since $\gamma = 0$. Applying the B of (3.36) to (3.39), using (3.41) one obtains finally

$$f = \frac{\sinh\alpha\rho}{\alpha\sinh\rho}\sin\theta\ , \qquad \psi = \cos\theta\ . \tag{3.42}$$

The same gauge transformations as (2.33) leads to

$$\begin{aligned} A_\tau &= \chi_\rho \frac{\hat{\sigma}_\rho}{2}\ , \\ A_\rho &= 0\ , \\ A_\theta &= -(e^\chi - 1)\frac{\hat{\sigma}_\varphi}{2}\ , \\ (\sin\theta)^{-1}A_\varphi &= (e^\chi - 1)\frac{\hat{\sigma}_\theta}{2}\ , \end{aligned} \tag{3.43}$$

where

$$\hat{\sigma}_\rho = \vec{\sigma}\cdot\hat{\rho}\ ,$$

$$\hat{\rho} \equiv (\sin\theta\cos\varphi\ ,\sin\theta\sin\varphi,\ \cos\theta),\quad \text{etc}\ .$$

and now

$$e^\chi = \frac{\alpha\sinh\rho}{\sinh\alpha\rho}\ ,$$

$$\chi_\rho = (\coth\rho - \alpha\coth\alpha\rho)\ .$$

It has been shown elsewhere[10] that using (3.2) and a gauge transformation (3.43) can be given the standard 't Hooft form *for integer α,* namely,

$$A_{\mu a} = \eta^{(+)}_{\mu\nu a}\partial_\nu(\ln\Lambda)\ , \tag{3.44}$$

where

$$\Lambda = 1 + \sum_{k=1}^{\alpha-1} \frac{\csc^2 k\pi/\alpha}{(t - \cot k\pi/\alpha)^2 + r^2}\ . \tag{3.45}$$

A direct correspondence to the conformally extended form can also be achieved by choosing a gauge where

$$\Lambda = \sum_{k=0}^{\alpha-1} \frac{\sec^2 k\pi/\alpha}{(t - \tan k\pi/\alpha)^2 + r^2}\ . \tag{3.46}$$

This is our "summable chain" of instantons whose many attractive properties have been studied.[10,11] Evidently, from (3.45) or (3.46), one has an action

$$S = 8\pi^2(\alpha - 1)\ , \tag{3.47}$$

which can very easily be obtained directly from (3.43). It has also been shown that regularity conditions as $\rho \to \infty$ selects out the integer values of α (which can be taken to the positive without loss of generality) though the action density is nonsingular and integrable for arbitrary α.[25] We will come back to this aspect later on.

The following remarkable point must be noted. *We have generalized the Harrison-Neugebauer transformation in such* a *way that in one single step we have generated from the vacuum (3.38) an instanton configuration of arbitrary index.* We pay the price of having no free parameters. But on the other hand there is no restriction on the index whatsoever. This situation should be compared to stepwise increase of index associated with certain other classes of Bäcklund-type transformations.[26,27] In our formalism the stepwise aspect will be present but it will have a very different significance. The scaling limit (3.5), (3.6), and (3.7) with $\lambda = \alpha$ and $\alpha \to \infty$ leads from (3.42) and (3.43) to (2.28) and (2.34), respectively. Thus we obtain the charge-1 monopole in the infinite-action static limit. Iteration of our transformations will be shown to enable us to move up the sequence of these limiting configurations, namely, those of nonlinearly superposed monopoles. Before coming to iteration let us note one more point. Corresponding to the complex vacuum (2.36) we have here

$$f' = \frac{\alpha}{\sinh\alpha\rho}\ , \quad \psi' = i\alpha\coth\alpha\rho\ . \tag{3.48}$$

This can again be generated either by a complex gauge transformation by

$$U=e^{-i\zeta\sigma_2/2} \tag{3.49}$$

where

$$\cos\zeta=-\coth\alpha\rho\ ,\quad \sin\zeta=(i\sinh\alpha\rho)^{-1}$$

or by a complex Ehler's transformation with an evident choice of parameters providing

$$f'+i\psi'=-\alpha\left[\frac{e^{\alpha\rho}-1}{e^{\alpha\rho}+1}\right]. \tag{3.50}$$

We now come to the problem of iteration. The key property to be used here involving

$$p=e^{i\gamma}\ ,\quad X_{\pm}=\frac{1}{4}\left[\partial_\rho\pm\frac{i}{\sinh\rho}\partial_\theta\right]\left[\ln\frac{\sin\theta}{\sinh\rho}\right]$$
$$=-\frac{1}{4}\left[\frac{\cosh\rho\mp i\cot\theta}{\sinh\rho}\right],$$

where γ is defined by (3.29) and (3.30), is

$$\partial_{\pm}p=p^{\pm1}(p^2-1)X_{\pm}\ . \tag{3.51}$$

Using this one can derive, following the procedure of FHP,[5] their version of Cosgrove's composition theorem[28] for our case. In fact the equations to be solved and the resulting solution are found to be formally identical, though the significances of the symbols involved are different for our case. Let us recapitulate the content of this theorem.

Let (p_1,q_1) and (p_2,q_2) be two solutions of (3.32) with the same M_i,N_i $(i=1,2)$ but with *different* choices of parameters (for example two distinct sets c_1,β_1 and c_2,β_2 in (3.40). Corresponding to (p_1,q_1) one has a H transformation [defined by (3.36) and (3.37)], H_1 say, which leads to (M_i',N_i') starting from (M_i,N_i). Let q' be solution of (3.32) with (M_i',N_i') and $e^{i\gamma}=e^{i\gamma_2}=p_2$. The composition theorem states that one does not have to integrate directly (3.32) with (M_i',N_i') but simply write

$$q'=\frac{(\bar q_1p_2-\bar q_2p_1)}{q_1(\bar q_1p_1-\bar q_2p_2)}\ , \tag{3.52}$$

where

$$\bar q_i=-\frac{p_i+q_i}{1+p_iq_i}\ .$$

One also has

$$\bar q'=\frac{(q_1p_2-q_2p_1)}{\bar q_1(q_1p_1-q_2p_2)}=-\frac{p_2+q'}{1+p_2q'}\ . \tag{3.53}$$

Now one can compose two successive H transformations (and hence finally any number) algebraically after one first integration such as (3.40). This is the central result which makes the entire machinary work in practice. Our, by now familiar, scaling limit leads to (2.39), formally identical.

We now consider in some detail the effect of two successive H transformations on the vacuum (3.38). As in (2.41) we define

$$Q_1\equiv(q_1p_1-q_2p_2)\ ,\quad Q_2\equiv(q_1p_2-q_2p_1)\ , \tag{3.54}$$

where now

$$p_1=e^{i\gamma_1}\ ,\quad p_2=e^{i\gamma_2} \tag{3.55}$$

corresponding to parameters c_1 and c_2, respectively in (3.29), (3.30) and, *for the same* α,

$$q_1=\tanh\{\tfrac{1}{2}[\alpha\eta(c_1)-\beta_1]\}\ ,$$
$$q_2=\tanh\{\tfrac{1}{2}[\alpha\eta(c_2)-\beta_2]\}\ , \tag{3.56}$$

where η is given by (3.25). We define similarly

$$\bar Q_1\equiv(\bar q_1p_1-\bar q_2p_2)\ ,\quad \bar Q_2\equiv(\bar q_1p_2-\bar q_2p_1)\ . \tag{3.57}$$

It can be shown that the general result upon composing two successive H transformations starting from the set $M_i^{(0)}$, $N_i^{(0)}$ is

$$M_1=\frac{\bar Q_2}{\bar Q_1}\left[\frac{Q_1}{Q_2}M_1^{(0)}+\frac{(p_1^2-p_2^2)}{Q_2}X_+\right],$$
$$M_2=\frac{\bar Q_1}{\bar Q_2}\left[\frac{Q_2}{Q_1}M_2^{(0)}-\frac{q_1q_2(p_1{}^2-p_2{}^2)}{Q_1}X_+\right],$$
$$N_1=\frac{\bar Q_1}{\bar Q_2}\left[\frac{Q_1}{Q_2}N_1^{(0)}+\frac{q_1q_2(p_1{}^2-p_2{}^2)}{p_1p_2Q_2}X_-\right],$$
$$N_2=\frac{\bar Q_2}{\bar Q_1}\left[\frac{Q_2}{Q_1}N_2^{(0)}-\frac{(p_1{}^2-p_2{}^2)}{p_1p_2Q_1}X_-\right]. \tag{3.58}$$

In our case

$$M_1^{(0)}=M_2^{(0)}=N_1^{(0)}=N_1^{(0)}=\tfrac{1}{4} \tag{3.59}$$

and p_1, p_2, q_1, q_2 are given by (3.55) and (3.56). Instead of systematically describing all possibilities of the choice of the parameters and their consequences we state directly the results that leads in the scaling limit to the charge-2 monopole of FHP[1,6] adapted, of course, to the gauge of Sec. II.

The reality conditions are assured by choosing (p^* denoting the complex conjugate of p)

$$p_1^*=p_2{}^{-1}\ ,\quad q_1^*=q_2{}^{-1}\ . \tag{3.60}$$

In particular we set *(note the role of α)*

$$c_1=\frac{i\pi}{2\alpha} \text{ and } \beta_1=0\,, \tag{3.61}$$

$$c_2=-\frac{i\pi}{2\alpha} \text{ and } \beta_2=i\pi\,. \tag{3.62}$$

They assure (3.60). To study the regularity properties of the solutions thus obtained we have first to look at

$$\begin{aligned}2\,\mathrm{Tr}A_\tau{}^2&=4(M_1+N_2)(M_2+N_1)\\&=4\,|M_1+N_2|^2\equiv h^2\,,\end{aligned} \tag{3.63}$$

say, as one reaches typical danger zones such as $\theta\to 0$ and $\theta\to\pi/2$. [The total action will eventually be obtained from the asymptotic form of (3.63) as $\rho\to\infty$.] As $\theta\to 0$, carefully developing every thing up to order θ^2 and taking ratios of limiting forms one obtains

$$\lim_{\theta\to 0} h^2=\alpha^2\left[\tanh\alpha\rho-\frac{2\sinh 2\rho}{\alpha(\cosh 2\rho-\cos\pi/\alpha)}\right]^2. \tag{3.64}$$

Anticipating results to follow we state here that α will eventually be restricted to *integer* values ≥ 2 and that for $\alpha=2$ one has a pure gauge solution. We verify at this stage that there is no singularity on the z axis ($\theta=0$) and for $\alpha=2$ the right hand side is zero. Also rescaling

$$\rho=\frac{r}{\alpha}\,,\quad \tau=\frac{t}{\alpha} \tag{3.65}$$

when

$$A_\mu\to\alpha A_{\mu'}$$

$$(\mu=\tau\,,\ \mu'=t;\ \ \mu=\rho\,,\ \mu'=r)\,,$$

one should obtain as $\alpha\to\infty$ the static monopole case. Indeed $\rho=r/\alpha$ and $\alpha\to\infty$ gives for the right-hand side of (3.64) (since $r=z$ for $\theta=0$)

$$\alpha^2\left[\tanh z-\frac{2z}{z^2+(\pi/2)^2}\right]^2. \tag{3.66}$$

The factor α^2 is absorbed by the rescaling of the left-hand side. Thus we obtain the corresponding gauge-invariant expression of FHP[29] with $\pi/2$ for their constant *(their α)*.

Let us now examine the situation for $\cos\theta=0$. In particular we select the *interior* of the ring given by

$$\sinh\rho=\tan\frac{\pi}{2\alpha}\quad\left[\theta=\frac{\pi}{2}\right]. \tag{3.67}$$

This will permit a study of the behavior at the origin and it turns out that the ring (3.67) is the correct generalization of FHP's crucial domain $s\equiv(x^2+y^2)^{1/2}=\pi/2$. (We change *their* ρ to s for evident reasons.) Going in the other sense one of course finds immediately on rescaling as in (3.65) and making $\alpha\to\infty$ that (3.67) reduces to

$$s=\frac{\pi}{2}\quad\left[r=s \text{ for } \theta=\frac{\pi}{2}\right]. \tag{3.68}$$

For our case, after careful calculations one obtains defining (for

$$\sinh\rho\leq\tan(\pi/2\alpha)$$

or

$$\cosh\rho\leq[\cos(\pi/2\alpha)]^{-1})$$

$$\cos\eta_0=\cos\frac{\pi}{2\alpha}\cosh\rho \tag{3.69}$$

when

$$\begin{aligned}\sin^2\eta_0&=\sin^2\frac{\pi}{2\alpha}-\cos^2\frac{\pi}{2\alpha}\sinh^2\rho\\&=-\sinh^2\rho+\sin^2\frac{\pi}{2\alpha}\cosh^2\rho\end{aligned} \tag{3.70}$$

that for $\theta=\pi/2$

$$h^2=\alpha^2\left[\frac{2\sin^2(2\pi/2\alpha)\cos\alpha\eta_0[\sin\eta_0\cos\alpha\eta_0-(1/\alpha)\cos\eta_0\sin\alpha\eta_0]}{\sin\eta_0[\sin^2\eta_0-\sin^2(\pi/2\alpha)\sin^2\alpha\eta_0]}-1\right]^2. \tag{3.71}$$

For $\alpha=2$ the right-hand size again vanishes. Again, setting

$$\rho=\frac{r}{\alpha}=\frac{s}{\alpha} \text{ for } \theta=\frac{\pi}{2} \tag{3.72}$$

and letting $\alpha\to\infty$,

$$\sin\eta_0\to\frac{[(\pi/2)^2-s^2]^{1/2}}{\alpha}\equiv\frac{\delta}{\alpha} \tag{3.73}$$

say. Hence

$$\frac{h^2}{\alpha^2}\to\left[\frac{2(\pi/2)^2\cos\delta(\delta\cos\delta-\sin\delta)}{\delta[\delta^2-(\pi/2)^2\sin^2\delta]}-1\right]^2. \tag{3.74}$$

We thus again recover the gauge-invariant expression of FHP.[30] As in (3.74), singularities can indeed be verified to be absent in (3.71) both as

$$\rho\to 0 \text{ and as } \sinh\rho\to\tan\frac{\pi}{2\alpha}\ .$$

As $\eta_0\to 0$, developing in powers of η_0 the factors $\eta_0{}^3$ are seen to cancel out in the ratio given by the first term in the brackets of (3.71). This corresponds to the ring $\sinh\rho=\tan\pi/2\alpha$. As $\rho\to 0$, $\eta_0\to\pi/2\alpha$ and $\cos\alpha\eta_0$ in the numerator of (3.71) cancels the zero of the denominator.

More generally one has in an arbitrary region

$$h^2=|4Q_1Q_2|^{-2}\left|\alpha(Q_1{}^2+Q_2{}^2)+4(e^{i\gamma}-e^{i\gamma^*})[Q_1(e^{i\gamma}+e^{i\gamma^*})X_+-Q_2(e^{-i\gamma^*}+e^{-i\gamma})X_-]\right|^2, \tag{3.75}$$

where

$$Q_1=e^{i\gamma}\tanh\tfrac{1}{2}\alpha\eta^*-e^{i\gamma^*}\coth\tfrac{1}{2}\alpha\eta^*$$

and (3.76)

$$Q_2=e^{i\gamma^*}\tanh\tfrac{1}{2}\alpha\eta-e^{i\gamma}\coth\tfrac{1}{2}\alpha\eta^*$$

and η and γ involve

$$c=\frac{i\pi}{2\alpha} \tag{3.77}$$

in (3.25) and (3.29), respectively. This choice of c has already been shown to eliminate singularities for $\theta=0$ (or similarly for $\theta=\pi$) and $\theta=\pi/2$. In this paper we will not attempt to recast h^2 of (3.75) into alternative more explicit forms. We just state that having assured the situation for $\theta=0$ and $\theta=\pi/2$, one can verify that singularities do arise elsewhere. The simple properties of the hyperbolic functions of ρ and the nonzero values of $\sin\theta$ and $\cos\theta$ elsewhere assure this. Another interesting region for us is that of $\rho\to\infty$. In the following section we will study this and the resulting form will permit us to calculate the action.

The general case of composing n successive transformations will be considered in a following paper. We state here that the procedure remains closely parallel to that of FHP.[5] The determinantal structure remains formally identical. The only apparent difference, as for the 2-step case considered, consists in the replacement of a factor $(4s)^{-1}$ [their $(4\rho)^{-1}$] by X_+ for M_1, M_2 and by for N_1, N_2. There will be no difficulty in writing down, formally, the general case. To appreciate the difference of content hidden by the formal analogy one has first to calculate the action. This we proceed to do in the next section for the 2-step case. For the 1-step case we already have the result (3.47).

IV. THE ACTION AND REGULARITY AS $\rho\to\infty$

The total action is

$$S=\int_{-\infty}^{\infty}dt\int_0^{\infty}dr \times\int_0^{\pi}d\theta\int_0^{2\pi}d\varphi r^2\sin\theta(\tfrac{1}{2}\operatorname{Tr}F_{\mu\nu}F^{\mu\nu})$$

$$(\mu,\nu=t,r,\theta,\varphi)\ . \tag{4.1}$$

In terms of the coordinates $(\tau,\rho,\theta,\varphi)$; for our self-dual solutions depending on (ρ,θ), S can be shown to reduce to

$$S=2\pi^2\int_0^{\pi}\sin\theta\, d\theta\left[\sinh^2\rho\,\partial_\rho\left[\frac{f_\rho{}^2+\psi_\rho{}^2}{f^2}\right]\right]_{\rho\to\infty}, \tag{4.2}$$

where

$$\frac{f_\rho{}^2+\psi_\rho{}^2}{f^2}=2\operatorname{Tr}A_\tau{}^2 =4(M_1+N_2)(M_2+N_1) \equiv h^2\ , \tag{4.3}$$

provided there is no singularity inside the asymptotic surface of integration. [In deducing (4.2) and in the asymptotic developments to follow we use techniques introduced for the case of complex solutions.[12]] Let us consider the case (3.75), for which we have shown that the finite region contains no singularities. We now have to develop h^2 up to order $e^{-2\rho}$ as $\rho\to\infty$. We note that, using (3.25), as $\rho\to\infty$

$$e^{\eta}=e^{\rho+\xi_1}(1+\xi_2{}^2e^{-2\rho})+O(e^{-3\rho}) \tag{4.4}$$

where

$$e^{\xi_1}=\cosh c-\sinh c\cos\theta \tag{4.5}$$

and

$$\xi_2=\frac{\sinh c\sin\theta}{\cosh c-\sinh c\cos\theta}=\frac{\partial\xi_1}{\partial\theta} . \tag{4.6}$$

($c=i\pi/2\alpha$, but we do not have to use this value explicitly here.) Hence *for* $\alpha>2$, if we keep terms only up to order $e^{-2\rho}$, one can set

$$\tanh(\tfrac{1}{2}\alpha\eta)\approx 1 \quad \text{as } \rho\to\infty . \tag{4.7}$$

One also obtains, writing $p_1=(p_2^*)^{-1}=p$ say,

$$p^{\pm 1}=e^{\pm i\gamma}=1\pm i2\xi_2e^{-\rho}-2\xi_2{}^2e^{-2\rho}+O(e^{-3\rho}) \tag{4.8}$$

and

$$X_{\pm}=-\frac{1}{4}\left[\frac{\coth\rho\pm i\cot\theta}{\sinh\rho}\right]$$
$$=-\tfrac{1}{4}(1\mp i2\cot\theta\, e^{-\rho}+2e^{-2\rho})+O(e^{-3\rho}) . \tag{4.9}$$

Hence

$$p_2=(p^*)^{-1}=1+i2\xi_2^*e^{-\rho}-2\xi_2^{*2}e^{-2\rho}+O(e^{-3\rho}) \tag{4.10}$$

Putting all these together one obtains (for $\alpha>2$)

$$\lim_{\rho\to\infty} h^2=4\{(\alpha/2-1)-e^{-2\rho}[2+2\cot\theta(\xi_2+\xi_2{}^*)-(\xi_2{}^2+\xi_2^{*2})]\}^2 . \tag{4.11}$$

Hence

$$S=2\pi^2\int_0^{\pi}\sin\theta\, d\theta\{4(\alpha/2-1)[2+2\cot\theta(\xi_2+\xi_2{}^*)-(\xi_2{}^2+\xi_2^{*2})]\} . \tag{4.12}$$

Now one can verify that

$$\int_0^{\pi}(2\cot\theta\,\xi_2-\xi_2{}^2)\sin\theta\, d\theta=\frac{2}{\sinh c}[\cosh\xi_1]_0^{\pi}=0 . \tag{4.13}$$

Hence

$$S=8\pi^2 2(\alpha-2) . \tag{4.14}$$

For $\alpha=2$ one has to keep further terms in (4.7). But this case turns out to be a pure gauge one, as was already indicated by $h=0$ for $\alpha=2$ in (3.64) and (3.71). *For* $0\le\alpha<2$ *the action can be shown to be divergent.* In fact if we consider negative values of α one has just to write $|\alpha|$ for α in (4.14). We have still a continuous spectrum. We now show how integer values of α are selected out by regularity constraints at $\rho\to\infty$. The technique used to extract this constraint has already been used with variations in our previous paper.[12,23,25] We briefly indicate the essential points. As $\rho\to\infty$, keeping only finite terms, one has the simple result, using the foregoing development

$$M_1,N_1,M_2,N_2 \text{ all}\to\tfrac{1}{4}(\alpha-2)+O(e^{-\rho}) \tag{4.15}$$

and hence

$$A_\tau\to(\alpha-2)\sigma_3/2 ,$$
$$A_\rho\to 0 ,\ A_\theta\to 0 ,\ A_\varphi\to 0 . \tag{4.16}$$

This looks quite all right, but this is the situation in terms of coordinates which makes the metric itself singular in the region in question [see (3.3)]. (For a De Sitter world this would be the cosmological horizon.) To study properly the situation we can go over to the coordinates (t,r,θ,φ). As $\rho\to\infty$, i.e., as $t\to 0$, $r\to 1$ there is no problem in (3.1). From (3.2) and (4.16) one can show that as $\rho\to\infty$,

$$A_t\to(\alpha-2)\cos\tau\frac{e^{\rho}}{2}\frac{\sigma_3}{2} ,$$
$$A_r\to(\alpha-2)\sin\tau\frac{e^{\rho}}{2}\frac{\sigma_3}{2} , \tag{4.17}$$

and A_θ, A_φ as before $\to 0$. Hence in order to have *finite* A_t and A_r one must have *vanishing* A_τ and A_ρ as $\rho\to\infty$. A finite A_τ as in (4.16) is not sufficient. The finite part in (4.16) can however be eliminated by gauge transforming with

$$U=e^{i(\alpha-2)\tau\sigma_3/2} \tag{4.18}$$

when

$$(i\partial_\tau U)U^{-1}=-(\alpha-2)\frac{\sigma_3}{2} . \tag{4.19}$$

So to get finite values for A_t and A_r as $\rho\to\infty$ one has consistently to start with the gauge transform of (3.8) by U. But then another problem arises.

$$U\sigma_2 U^{-1} = \cos(\alpha-2)\tau\sigma_2 + \sin(\alpha-2)\tau\sigma_1 \tag{4.20}$$

with a similar result for σ_1. Since τ has a period 2π, we see that after the gauge transformation of (3.8) by U one has *single-values potentials only if* α *is an integer.* [Note that U itself is single valued if α is an *even* integer. But this feature arises only because we are using a 2×2 matrix representation (instead of a 3-component column one) of the isovector A. For a column representation U would involve 3×3 SO(3) matrices and would be single valued for all integer α. Hence the real constraint is already obtained from (4.20).]

Thus finally we obtain a class of regular, finite-action, self-dual solutions of index

$$P=\frac{S}{8\pi^2}=2(\alpha-2), \quad \alpha=3,4,\ldots . \tag{4.21}$$

Of crucial importance for us is the factor 2 before $(\alpha-2)$. It should be compared to (3.47) where the index was

$$P=\frac{S}{8\pi^2}=(\alpha-1), \quad \alpha=2,3,\ldots . \tag{4.22}$$

The rule for obtaining the magnetic charge for the static limit after rescaling turns out to be

$$\text{magnetic charge} = \lim_{\alpha\to\infty} \frac{P}{\alpha} . \tag{4.23}$$

For (4.22) we thus had the charge-1 monopole. Similarly for (4.21) we obtain the charge-2 monopole. With this idea behind, we will call (3.43) or (3.44) the "1-chain" and that corresponding to (3.58) [with (3.61) and (3.62) the "2-chain"].

For the general n-chain, the generalization of (4.12) leading to

$$P=n(\alpha-n) \quad (\alpha=n+1,n+2\cdots) \tag{4.24}$$

turns out to be quite easily obtainable. A thorough study of the determinantal structure will be presented elsewhere.

V. REMARKS

In this paper we have stopped short at a certain stage. Further study in several directions is needed. The study of the general case ("n-chain") with calculation of the action should be straightforward up to a certain stage. But further exploration of the "2-chain" of Secs. III and IV can already present intriguing features.

For the "1-chain" we could show that the "τ-static" form (3.43) is gauge equivalent to the standard form given by Eqs. (3.44) to (3.46). What would be the ADHM parametrization of the "2-chain"? It may not be a simple task to find it out explicitly. But it would be a quite interesting result to have. The known results for Green's functions and fluctuation determinants for the ADHM case can then immediately be utilized with the additional welcome possibility of carrying through further the calculations explicitly and analytically as for the "1-chain." The corresponding results for the monopole limit will then again follow without effort. (Though for convenience we are using the terminology "2-chain," "n-chain," and so on, they may turn out in the standard gauge not to have, unlike the 1-chain, any chainlike aspect at all. This is however not a serious problem.) The standard representation, once found will also possibly lead to periodic forms of multicharged monopoles. This was the case for the 1-chain.

Another important aspect is the possibility of introducing parameters. The solutions we present are parameterless (the action or index fixing our α) as are the monopoles in the static limit. Can this constraint be relaxed? One possibility is that of extending our class conformally by generalizing (3.2) by including parameters. Even if it works, the scope here is limited. One can make a conjecture that what has been shown here to be true for parameterless nonlinear superpositions is also true for the more general monopole solutions.[31-34] That is, all static, self-dual monopoles should be obtained as limits of certain instanton configurations after a suitable rescaling.

The result should indeed hold. We will attempt elsewhere to generalize our technique to include separated monopoles as limits.

Even for the axially symmetric case, at each iteration we are presumably generating a very particular sequence of instantons in the successive Atiyah-Ward classes (see sources quoted in Ref. 31). It would be desirable to display this more explicitly. [The simplest case (3.45) of course coincides with 't Hooft's class.] In this context a generalization, parallel to that given here, of the Prasad-Rossi formalism[35,36] might be of interest. Here one has to generalize, in our fashion, the Helmholtz rather than the Ernst equation. The required generalization is already in our Ref. 12. This would provide a suitable point of departure.

Here we have chosen the FHP formalism since this gives directly the gauge potentials with the reality constraints incorporated in a transparent fashion.

Let us note one final point. The Ernst equation is of interest both in general relativity (static, axially symmetric metrics) and for Yang-Mills fields (axially symmetric monopoles). Our generalization (3.16) [reducing to the Ernst equation through (3.5) and (2.11)] gives instantons for the Yang-Mills fields. The content of such an equation should also be examined in the general-relativistic context.

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APPENDIX: AXIAL SYMMETRY IN THE STANDARD AND THE "SPHERICAL" R GAUGES

In the R gauge the potentials have the form (with $\lambda_\mu \equiv \partial_\mu \lambda$ and so on)

$$\lambda A_\mu = i\begin{bmatrix} \frac{1}{2}\lambda_\mu & 0 \\ \zeta_\mu & -\frac{1}{2}\lambda_\mu \end{bmatrix} = i\left[\lambda_\mu \frac{\sigma_3}{2} + \zeta_\mu \left[\frac{\sigma_1}{2} - \frac{i\sigma_2}{2}\right]\right], \tag{A1}$$

$$\lambda A_{\bar\mu} = -i\begin{bmatrix} \frac{1}{2}\lambda_{\bar\mu} & \bar\zeta_{\bar\mu} \\ 0 & -\frac{1}{2}\lambda_{\bar\mu} \end{bmatrix} = -i\left[\lambda_{\bar\mu} \frac{\sigma_3}{2} + \bar\zeta_{\bar\mu} \left[\frac{\sigma_1}{2} + \frac{i\sigma_2}{2}\right]\right], \tag{A2}$$

where λ is real and ζ, in general, complex. In Yang's formulation[37] which we call the standard one, μ and $\bar\mu$ are defined directly in terms of the Cartesian coordinates of the flat Euclidean space. One can choose

$$\mu = y,z, \quad \bar\mu = \bar y, \bar z ,$$

where

$$\begin{aligned} y &= \frac{1}{\sqrt{2}}(x_1 + ix_2), \quad z = \frac{1}{\sqrt{2}}(x_3 - ix_4) , \\ \bar y &= \frac{1}{\sqrt{2}}(x_1 - ix_2), \quad \bar z = \frac{1}{\sqrt{2}}(x_3 + ix_4) . \end{aligned} \tag{A3}$$

We introduced elsewhere[21,27] a different choice suitable for generalization to curved spaces with static spherical symmetry. For flat space, with which we are concerned here, it reduces to, in terms of the spherical coordinates (t,r,θ,φ),

$$\begin{aligned} y &= \tan\frac{\theta}{2} e^{i\varphi} , \quad z = \tfrac{1}{2}(r+it) , \\ \bar y &= \tan\frac{\theta}{2} e^{-i\varphi} , \quad \bar z = \tfrac{1}{2}(r-it) . \end{aligned} \tag{A4}$$

(We consider the base space to be always real.) We will call this the "spherical" choice. We now introduce the constraints of *reality and static axial symmetry* in this formulation. We set

$$\lambda = f\cot\frac{\theta}{2} , \quad \zeta = \psi\cot\frac{\theta}{2} e^{i\varphi} ,$$

where f and ψ are two real functions of r and θ $[f=f(r,\theta) , \psi=\psi(r,\theta)]$. Next we introduce a gauge transformation

$$A_\mu \to UA_\mu U^{-1} + i(\partial_\mu U)U^{-1} ,$$

where

$$U = e^{-i\pi\sigma_2/2} e^{i\varphi\sigma_3/2} . \tag{A6}$$

One obtains for the spherical components of the transformed field (which we continue to denote by A_μ)

$$\begin{aligned} A_t &= \frac{f_r}{f}\frac{\sigma_3}{2} + \frac{\psi_r}{f}\frac{\sigma_1}{2} , \\ A_r &= \frac{\psi_r}{f}\frac{\sigma_2}{2} , \\ A_\theta &= \frac{\psi_\theta}{f}\frac{\sigma_2}{2} , \\ (\sin\theta)^{-1}A_\varphi &= \frac{f_\theta}{f}\frac{\sigma_3}{2} + \frac{\psi_\theta}{f}\frac{\sigma_1}{2} . \end{aligned} \tag{A7}$$

This is our ansatz (2.1).

It is known that Manton's static axially symmetric ansatz[38] utilized by FHP[5] is related in an analogous fashion to the choice (A3). It is interesting to compare this ansatz with (A7) using another approach. To do this we start with the ansatz,

$$\begin{aligned} A_t &= g_1 \frac{\sigma_3}{2} + g_2 \frac{\sigma_1}{2} , \\ A_r &= \frac{\psi_r}{f} \frac{\sigma_2}{2} , \\ A_\theta &= \frac{\psi_\theta}{f} \frac{\sigma_2}{2} , \\ A_\varphi &= h_1 \frac{\sigma_3}{2} + h_2 \frac{\sigma_1}{2} . \end{aligned} \tag{A8}$$

Here $g_i(r,\theta)$ and $h_i(r,\theta)$ are also real axially symmetric, static functions. We want to solve for them in terms of ψ, f, and their derivatives using the self-duality constraints. These constraints are, with

$$\begin{aligned} &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] , \\ &F_{tr} = (r^2 \sin\theta)^{-1} F_{\theta\varphi} , \\ &F_{t\theta} = (\sin\theta)^{-1} F_{\varphi r} , \\ &(\sin\theta)^{-1} F_{\tau\varphi} = F_{r\theta} . \end{aligned} \tag{A9}$$

One set of solutions is

$$\begin{aligned} &g_1 = \frac{f_r}{f} , \quad g_2 = \frac{\psi_r}{f} , \\ &(\sin\theta)^{-1} h_1 = \frac{f_\theta}{f} , \quad (\sin\theta)^{-1} h_2 = \frac{\psi_\theta}{f} , \end{aligned} \tag{A10}$$

where f and ψ must satisfy

$$\begin{aligned} &f \tilde{\Delta} f - (\vec{\nabla} f)^2 + (\vec{\nabla} \psi)^2 = 0 , \\ &f \tilde{\Delta} \psi - 2(\vec{\nabla} \psi) \cdot (\vec{\nabla} f) = 0 , \end{aligned} \tag{A11}$$

where

$$\vec{\nabla} \equiv \left[\partial_r , \frac{1}{r} \partial\theta \right] \quad (\partial_\varphi \approx 0)$$

and

$$\begin{aligned} \tilde{\Delta} &\equiv \partial_r^2 + (r^2 \sin\theta)^{-1} \partial\theta (\sin\theta \, \partial_\theta) \\ &= \Delta - \frac{2}{r} \partial_r . \end{aligned} \tag{A12}$$

This is the solution given by our Eqs. (2.1) to (2.6). Another set of solutions is

$$\begin{aligned} &g_1 = \frac{1}{f} \left[\cos\theta \, f_r - \frac{1}{r} \sin\theta \, f_\theta \right] , \\ &g_2 = \frac{1}{f} \left[\cos\theta \psi_r - \frac{1}{r} \sin\theta \, \psi_\theta \right] , \\ &(r \sin\theta)^{-1} h_1 = \frac{1}{f} \left[\sin\theta \, f_r + \frac{1}{r} \cos\theta \, f_\theta \right] , \\ &(r \sin\theta)^{-1} h_2 = \frac{1}{f} \left[\sin\theta \, \psi_r + \frac{1}{r} \cos\theta \, \psi_\theta \right] , \end{aligned} \tag{A13}$$

where now Δ replaces $\tilde{\Delta}$ in (A11) giving

$$\begin{aligned} &f \Delta f - (\vec{\nabla} f)^2 + (\vec{\nabla} \psi)^2 = 0 , \\ &f \Delta \psi - 2(\vec{\nabla} \psi) \cdot (\vec{\nabla} f) = 0 , \end{aligned} \tag{A14}$$

and one has the Ernst equation for $\epsilon = (f + i\psi)$.

Transforming to cylindrical coordinates and taking account of differences of conventions and notations one finds now Manton's ansatz.[5,38]

To relate our ansatz (3.8) to the R gauge one has to start with, in terms of (τ,ρ) defined by (3.2),

$$\begin{aligned} &y = \tan\frac{\theta}{2} e^{i\varphi} , \quad z = \tfrac{1}{2}(\rho + i\tau) , \\ &\bar{y} = \tan\frac{\theta}{2} e^{-i\varphi} , \quad \bar{z} = \tfrac{1}{2}(\rho - i\tau) . \end{aligned} \tag{A15}$$

More generally for curved spaces with static spherical symmetry the "spherical" choice can be generalized smoothly.[21,27] For such cases the metric being no longer diagonal in Cartesian coordinates, the standard choice (A3) becomes too complicated, since eventually one has to use the $g_{\mu\nu}$'s.

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