Angular-momentum—angle commutation relations and minimum-uncertainty states

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We extend the canonical commutation relations (CCR) in quantum mechanics to the case where appropriate dynamical variables are angular momenta and angles. It is found that projection operators of the resultant Weyl algebra provide us with a new and powerful way of characterizing minimum-uncertainty states, including those obtained by Carruthers and Nieto. The uniqueness theorem of the Schrödinger representation remains valid for extended CCR in a simple case. Finally, a wide range of applicability of our method is suggested.

I. INTRODUCTION

The problem of quantizing a dynamical system which is primarily described by angular momenta and angles occurs in a number of cases in modern particle theories. The angular momenta mentioned here need not be taken literally, but may be understood in a much wider sense. They include dynamical variables corresponding to internal symmetries, whether they are represented by continuous groups or discrete ones, as far as some restrictions are satisfied. It is well known that the Pauli exclusion principle on certain types of fields (fermions) led Jordan and Wigner to propose anticommutation relations for these fields, without seeking any deeper reasons for such algebraic relations.[1] It worked extremely well, as we know today from the partial success of quantum electrodynamics. Likewise, it is quite conceivable that some basic constraints from the internal symmetry and selection rules can be dynamically described by simply extending the conventional (anti-) commutation relations to appropriate ones. Our attempt to extend canonical commutation relations (CCR) to angular-momentum—angle variables is motivated by this reason. It seems to be natural in view of the recent trend in particle physics where isospin groups or non-Abelian gauge groups play a fundamental role.[2] Indications of such an attempt are already implicit in the literature.[3] It is important here to propose *reasonable* CCR for non-Cartesian dynamical variables. This requires an extra insight into CCR, because the conventional CCR, $[q,p]=i$, applies only to a Cartesian coordinate q and conjugate momentum p, but not to general dynamical variables. This subject has been studied by a number of authors in connection with minimum-uncertainty states, which minimize the uncertainty product of mutually conjugate variables.[4] These states are particularly useful in clarifying physical meanings of CCR. They are as classical as possible. The minimum-uncertainty states or coherent states have been extensively applied to the infrared problem in quantum electrodynamics.[5] One also may be interested in finding a non-Abelian analog of coherent states by applying the method presented in this work. Indeed, there is an attempt to apply a similar technique to infrared problems of non-Abelian gauge theories.[6]

With these motivations in mind, we try to extend CCR to angular-momentum—angle variables in Weyl form. This is the main theme of the present paper. Then we will present a novel method of characterizing minimum-uncertainty states, which will be helpful in understanding the physical picture of extended CCR. It is accomplished by finding suitable projection operators of Weyl algebra and is motivated by an earlier work of von Neumann.[7] It seems to have a wide range of applicability. For the conventional CCR, the minimum-uncertainty states are characterized as coherent states, i.e., eigenstates of the annihilation operator.[8–12] However, the minimum-uncertainty states for an extended CCR discussed in this paper are not to be confused with spin coherent states[13] or generalized coherent states.[14] These are not directly relevant to minimum-uncertainty states considered here. We notice that they had been introduced earlier also by Mackey as the *system of imprimitivity* for a given group representation.[15] Similar comments will apply to related works[16] too, although most of them mentioned here are undoubtedly useful for our purpose. We restrict ourselves to systems with rotational degrees of freedom only. The application to more realistic cases has to be done subsequently. In Sec. II, CCR for angular momenta and angles is set up in Weyl form for both the Abelian rotation group and SU(2). In Sec. III, the Weyl algebra and some projection operators of conventional CCR are constructed at first. Then a corresponding analysis is done for an Abelian rotation group. Unfortunately, a more interesting case of SU(2) will be left unsolved. In Sec. IV, we study minimum-uncertainty states for an extended CCR. In Sec. V, we mention the uniqueness theorem of the Schrödinger representation for extended CCR. Section VI contains concluding remarks.

II. EXTENDED CCR IN WEYL FORM

According to Weyl, the conventional CCR for a one-dimensional quantum system is cast into the form[17]

$$S(\alpha,\beta)\equiv U(\alpha)V(\beta)\exp\left[-\frac{i}{2}\alpha\beta\right] =\exp\left[\frac{i}{2}\alpha\beta\right]V(\beta)U(\alpha)\ , \tag{1}$$

where $U(\alpha)\equiv\exp(i\alpha p)$ and $V(\beta)\equiv\exp(i\beta q)$ are one-parameter families of unitary operators and depend on real parameters α and β ranging from $-\infty$ to $+\infty$. We are interested in a system which has only a rotational degree of freedom around a fixed axis. Equation (1) suggests a straightforward generalization of CCR to this case in the form

$$S(\alpha,l)\equiv U(\alpha)V(l)\exp\left[-\frac{i}{2}\alpha l\right] =\exp\left[\frac{i}{2}\alpha l\right]V(l)U(\alpha)\ , \tag{2}$$

where $U(\alpha)\equiv(i\alpha L_z)$ and $V(l)\equiv[\exp(i\phi)]^l$ are families of unitary operators in a Hilbert space H. Here, L_z is an infinitesimal generator of rotations around a fixed (z) axis. Equation (2) contains only $\exp(i\phi)$, but not ϕ itself, as a quantum angle variable and therefore does not cause difficulties related to periodicity. Parameters α and l are restricted to $0\leq\alpha\leq2\pi$, and $l=0,\pm1,\pm2,\ldots$, respectively. Equation (2) can be cast into a more familiar form by defining $\cos\phi=[\exp(i\phi)+\exp(-i\phi)]/2$, etc.[18]:

$$[L_z,\cos\phi]=i\sin\phi\ ,\qquad [L_z,\sin\phi]=-i\cos\phi\ . \tag{3}$$

The physical implications of Eq. (3) have been studied previously and no controversies were found,[4] so we shall assume it in the following.

Incidentally, we add that the similar relation for the number and phase variables in quantum mechanics is more complicated than the one given above, because the phase operator $\exp(i\phi)$ is not unitary in this case. Consequently, there exists no self-conjugate operator ϕ. This is due to the fact that the number operator N is then required to have non-negative eigenvalues, while there is no such restriction to the angular momentum operator L_z. Indeed the unitarity of $\exp(i\phi)$ and the commutativity of $\sin(\phi)$ and $\cos(\phi)$ are easily shown in our case.[4] Equations (1) and (2) have a simple interpretation: the coordinates $\exp(i\beta q)$ and $[\exp(i\phi)]^l$ are transformed under a finite unitary transformation $U(\alpha)$ into $\exp[i\beta(q+\alpha)]$ and $[\exp(i\phi)\exp(i\alpha)]^l$, respectively. By the same reasoning, one can expect to find the analog of Eq. (2) when $U(\alpha)$ is replaced by a unitary rotation of SU(2). However, a naive replacement of $V(l)$ in Eq. (2) by $[\exp(i\phi)]^l\,[\exp(i\theta)]^m\,[\exp(i\psi)]^n$, where ϕ, θ, and ψ are quantum Euler angles, does not lead to any simple CCR for SU(2), if $U(\alpha)$ is one of SU(2) rotations. A very natural extension of Eq. (2) is obtained by observing that $\{V(l)\,|\,l=0,\pm1,\pm2,\ldots\}$ constitutes a complete set of irreducible representations of the group $\{U(\alpha)\,|\,0\leq\alpha\leq2\pi\}$. Thus, for the group SU(2), we make the following substitutions:

$$U(\alpha)\rightarrow\exp(i\vec{n}\cdot\vec{J})\equiv\exp(i\gamma J_z)\exp(i\beta J_y)\exp(i\alpha J_z),\quad V(l)\rightarrow D^j(\phi\theta\psi)_{mm'}\ , \tag{4}$$

and arrive at an extended CCR for SU(2) in a remarkably concise form:

$$\exp(i\vec{n}\cdot\vec{J})D^j(\phi\theta\psi)\exp(-i\vec{n}\cdot\vec{J})D^j(\phi\theta\psi)^{-1}=D^j(\alpha\beta\gamma)\ , \tag{5}$$

where $j=0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ldots$. The matrix $D^j(\phi\theta\psi)$ is expressed as a set of product of $\exp(i\phi/2)$, $\exp(i\theta/2)$, and $\exp(i\psi/2)$, which are quantum Euler angles. Specifically, $D^j(\phi\theta\psi)$ in Eq. (5) takes the form, for $j=\frac{1}{2}$,

$$D^{1/2}(\phi\theta\psi)=\begin{bmatrix} \exp\left[\frac{i}{2}(\phi+\psi)\right]\cos\frac{\theta}{2} & \exp\left[\frac{i}{2}(\phi-\psi)\right]\sin\frac{\theta}{2} \\ -\exp\left[\frac{i}{2}(-\phi+\psi)\right]\sin\frac{\theta}{2} & \exp\left[-\frac{i}{2}(\phi+\psi)\right]\cos\frac{\theta}{2} \end{bmatrix} . \tag{5'}$$

The right-hand side of Eq. (5) is no longer a simple phase factor, but is expressed as $D^j(\alpha\beta\gamma)$ in terms of c-number parameters α, β, and γ. It is not difficult to see by expanding the left-hand side of Eq. (5) into power series that it is satisfied for $j=\frac{1}{2}$ by the Schrödinger representation for angle variables,[19] i.e.,

$$J_y=-i\left[-\sin\phi\cos\theta\frac{\partial}{\partial\phi}+\cos\phi\frac{\partial}{\partial\theta}+\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi}\right],\quad J_z=-i\frac{\partial}{\partial\phi} . \tag{6}$$

Notice the close similarity of Eq. (5) with conventional CCR in Weyl form, Eq. (1). The extension of $S(\alpha,\beta)$ or $S(\alpha,l)$ can be defined as follows:

$$S(\alpha\beta\gamma)^j_{mm'}\equiv[D^j(\alpha\beta\gamma)^{-1/2}\exp(i\vec{n}\cdot\vec{J})D^j(\phi\theta\psi)]_{mm'}=[D^j(\alpha\beta\gamma)^{1/2}D^j(\phi\theta\psi)\exp(i\vec{n}\cdot\vec{J})]_{mm'} \tag{7}$$

in a matrix notation. From this, it is evident that the Weyl algebra, which is defined as a set consisting of all possible linear combinations of $S(\alpha,\beta)$, or $S(\alpha,l)$, etc., can be defined for a wide class of groups. However, our analysis in the following sections is mainly devoted to a simpler case of the Abelian rotation group, Eq. (2).

Before concluding this section, we remark that the explicit form of CCR for SU(2) given above is new, although a general group-theoretical abstract form of CCR has been given by Mackey.[15]

III. PROJECTION OPERATORS OF WEYL ALGEBRA

Let us assume that the physical quantities are represented by self-adjoint operators of the algebra defined in a previous section.[17,20] The equation of motion for observables can be set up by giving a Hamiltonian of the system as an element of the algebra. Our analysis in the following is, however, independent of an explicit form of the Hamiltonian. As we shall see, the true physical merit of this approach becomes clear when minimum-uncertainty states for Eqs. (1) and (2) are considered. The central idea is not new, but little known. So we first discuss the conventional CCR and its minimum-uncertainty states in the new light. Let us recall that, for $[q,p]=i$, minimum-uncertainty states which satisfy $\Delta q\Delta p=\frac{1}{2}$ are represented by Gaussian wave functions and may be characterized in several different ways. Indeed, they have been extensively studied as coherent states. Yet there is an alternative way of characterizing them. To see this, following von Neumann, let us define a Hermitian operator[7]:

$$E_0=\frac{1}{2\pi}\int\int_{-\infty}^{\infty}d\alpha\, d\beta\,\xi_0(\alpha,\beta)S(\alpha,\beta) , \tag{8}$$

where $\xi_0(\alpha,\beta)=\exp[-(\alpha^2+\beta^2)/4]$. By using a composition rule for $S(\alpha,\beta)$, which follows from Eq. (1),

$$S(\alpha,\beta)S(\alpha',\beta')=\exp\left[\frac{i}{2}(\alpha\beta'-\alpha'\beta)\right]\times S(\alpha+\alpha',\beta+\beta') , \tag{9}$$

it is possible to show that

$$E_0{}^2=E_0,\quad E_0^\dagger=E_0 . \tag{10}$$

So, E_0 is a projection operator of Weyl algebra for (1). If the Schrödinger representation ($p=-id/dq$) is assumed in $S(\alpha,\beta)$, then for any function $f(q)$ of q, we have

$$S(\alpha,\beta)f(q)=\exp\left[i\beta\left[q+\frac{\alpha}{2}\right]\right]f(q+\alpha) . \tag{11}$$

Now, let us suppose that $f(q)$ is a nonvanishing eigenfunction of E_0. Its eigenvalue must be equal to one, because E_0 is a projection operator. The equation $E_0f(q)=f(q)$ means, after normalization,

$$f(q)=\pi^{-1/4}\exp(-q^2/2) . \tag{12}$$

It represents, therefore, one of the minimum-uncertainty states. This observation due to von Neumann is quite remarkable and prompts us to see whether similar projection operators to minimum-uncertainty states can be found for the Weyl algebra of extended CCR, as in Eq. (2) or Eq. (5). Before answering this question, let us study the case of conventional CCR a little further. Our observations are summarized as follows.

(I) By explicit calculation, it is relatively easy to show that the weight function

$$\xi(\alpha,\beta)=\exp\left[-\frac{1}{4}\left[\eta\alpha^2+\frac{\beta^2}{\eta}\right]+i(-\alpha v+\beta u)\right] \tag{13}$$

gives a projection operator for arbitrary real values of u, v, and η ($\eta\neq 0$). The corresponding eigenfunction in the Schrödinger representation is written as

$$f_{u,v}(q)=S(u,v)f_{0,0}(q)\ , \tag{14}$$

where $f_{0,0}(q)=(\eta/\pi)^{1/4}\exp(-\eta q^2/2)$ represents a minimum-uncertainty state. Equation (14) is nothing but a well-known expression for coherent-state wave functions, which are characterized by a pair of real variables u and v (η is assumed to be fixed). So, all coherent states are reproduced by choosing simple weight functions such as (13).

(II) It is also possible to find an alternative family of projection operators. Let us choose $\xi_0(\alpha,\beta)=\exp(-\rho)$, $\xi_1(\alpha,\beta)=(1-2\rho)\exp(-\rho)$, $\xi_2(\alpha,\beta)=(1-4\rho+2\rho^2)\exp(-\rho),\ldots$, where $\rho\equiv(\alpha^2+\beta^2)/4$. These weight functions give through Eq. (8) a series of mutually orthogonal projection operators $E_0, E_1, E_2, \ldots$. Eigenfunctions of these operators are identical to harmonic-oscillator wave functions (up to multiplicative constants) $\exp(-q^2/2)H_n(q)$ with $n=0,1,2,\ldots$, where $H_n(q)\equiv\exp(q^2)(-d/dq)^n\exp(-q^2)$. Although we do not prove it here, an infinite series $E_0, E_1, E_2, \ldots$, exists and its eigenfunctions span the entire Hilbert space.

Thus it is evident that, corresponding to a choice of a complete system of bases in Hilbert space, there exists a freedom in choosing the set of projection operators. It is known that for those bases discussed in (I) no two are mutually orthogonal, whereas bases in (II) are mutually orthogonal. In case (II), the minimum-uncertainty state is reproduced only for E_0. Therefore, as a means of characterizing all possible minimum-uncertainty states of conventional CCR, the expression (13) is superior to those given in (II).

Following the same method, we are led to find an analog of Eq. (13) for angular-momentum—angle commutation relations. If it exists, then it may correspond to a state which is as classical as possible. So, let us turn our attention to the algebra generated by $S(\alpha,l)$ of Eq. (2). We denote the projection operator of this algebra by E and write it as follows:

$$E=\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi}d\alpha\,\xi(\alpha,l)S(\alpha,l)\ . \tag{15}$$

Then, conditions $E^\dagger=E$, $E^2=E$, imply

$$\xi^*(\alpha,l)=\xi(-\alpha,-l),\quad \xi(\alpha\pm2\pi,l)=(-1)^l\xi(\alpha,l)\ ,$$
$$\xi(\alpha,l)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi}d\beta\exp\left[\pm\frac{i}{2}(\alpha k-\beta l)\right]\xi(\alpha-\beta,l-k)\xi(\beta,k)\ . \tag{16}$$

In deriving the second equation of (16), a periodicity condition with respect to α was applied to the integrand of Eq. (15). One solution of Eq. (16) can be expressed in terms of modified Bessel functions as follows:

$$\xi(\alpha,l)=\frac{I_l[2c\cos(\alpha/2)-2ic'\sin(\alpha/2)]}{I_0(2c)}\times e^{-im\alpha+il\psi}\ , \tag{17}$$

where c and c' are arbitrary real constants, $m=0$, $\pm1, \pm2, \ldots$, and $-\pi\le\psi\le\pi$. This is most easily verified by employing an integral representation for $I_l(x)$, i.e.,[21]

$$I_l(x)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\,e^{il\theta+x\cos\theta}\ . \tag{18}$$

Projection operators for a more interesting case of SU(2), Eq. (7), are as yet unavailable. In the next section, we study properties of the solution (17).

IV. MINIMUM-UNCERTAINTY STATES

Let us assume the Schrödinger representation for an angle variable (i.e., $L_z=-id/d\phi$) in $S(\alpha,l)$. Then the eigenfunction of the projection operator with the weight function (17) is written as

$$f_{\psi,m}(\phi)=S(\psi,m)f_{0,0}(\phi)\ , \tag{19}$$

where

$$f_{0,0}(\phi)=[2\pi I_0(2c)]^{-1/2}$$
$$\times\exp(c\cos\phi+ic'\sin\phi)\ , \tag{20}$$

including a normalization factor. Equation (19) should be compared with Eq. (14) of conventional CCR. Furthermore, we notice that the expression (19) with $c'=0$, and $\psi=0$ or $\pm\pi/2$, represents exactly the same state that was obtained by Carruthers and Nieto as the minimum-uncertainty state of Eq. (3). Indeed, by defining as usual $(\Delta L_z)^2=\langle L_z{}^2\rangle-\langle L_z\rangle^2$, etc., it is easily confirmed for $f_{\psi,m}(\phi)$ $(c'=0)$ that

$$\begin{aligned}&(\Delta L_z)^2(\Delta\cos\phi)^2/\langle\sin\phi\rangle^2=\tfrac{1}{4}\quad(\psi=0)\ ,\\&(\Delta L_z)^2(\Delta\sin\phi)^2/\langle\cos\phi\rangle^2=\tfrac{1}{4}\quad\left[\psi=\pm\frac{\pi}{2}\right]\ .\end{aligned}\tag{21}$$

Figure 1 shows the weight function $\xi(\alpha,l)$ and the corresponding eigenfunction $f_{0,0}(\phi)$ for the choice $c'=0$, $c=1$, which has an expansion around $\phi=0$:

$$f_{0,0}(\phi)=[2\pi I_0(2)]^{1/2}\left[1-\frac{\phi^2}{2}+\frac{\phi^4}{6}-\cdots\right]\ . \tag{22}$$

For comparison's sake, we also show $\xi_0(\alpha,\beta)$ and the eigenfunction of E_0, whose expansion

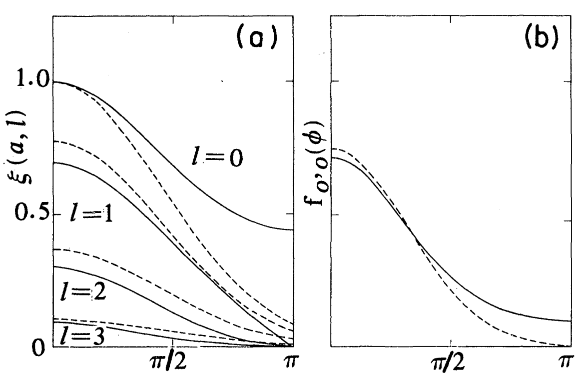

FIG. 1. The weight function $\xi(\alpha,l)$ of the projection operator (15) is shown in (a) as a function of α and l, with the choice $c=1$, $c'=m=\psi=0$, $l=0$, 1, 2, 3 (solid lines). Dashed curves show $\xi_0(\alpha,\beta)$ with $\beta=0$, 1, 2, 3 (from above to below in this order) for comparison's sake. The corresponding eigenfunction (20) [(12)] of the projection operator is shown in (b) by a solid [dashed] curve. The abscissa in (b) refers to ϕ (or q).

around $q=0$ is given by

$$f(q)=\pi^{-1/4}\left[1-\frac{q^2}{2}+\frac{q^4}{8}-\cdots\right]\ . \tag{23}$$

The similarity of these two cases is rather impressive. This is also true for the formal aspect of Eq. (17) with $c'=0$ as compared with Eq. (13). Equation (19) with $c'=0$ clearly exhausts all possible minimum-uncertainty states for Eq. (2). By recalling the argument of the previous section, we find that there is a family of simple projection operators of Weyl algebra which reproduces all possible minimum-uncertainty states. However, it is not yet clear whether this situation persists in the SU(2) case. In some cases, the Weyl algebra contains only a finite number of independent unitary operator basis elements. This happens when the group $\{U(\alpha)\}$ is actually a finite group. Beautiful examples of prime order groups have been studied by Schwinger in a series of papers.[22] Even in these cases, our algebraic method will be useful in decomposing the Weyl algebra into its irreducible constituents, which replace minimum-uncertainty states. Apparently, it may not be possible to characterize them as states which are as classical as possible.

V. UNIQUENESS OF THE SCHRÖDINGER REPRESENTATION

In deriving Eqs. (19) to (22), we assumed the Schrödinger representation of extended CCR(2) for the angle variable , i.e., $L_z=-id/d\phi$. According to a well-known theorem of Stone and von Neumann, the Schrödinger representation of CCR(1) is both unique and irreducible.[23] It is interesting to know whether a similar theorem is valid in our case. By following a method parallel to the original proof, it is not difficult to see that this is actually the case.[24] The only difference in the present case is that possible values of parameters α and l are restricted to $0\le\alpha\le 2\pi$ and $l=0,\pm1,\pm2,\ldots$, respectively, in contrast to α and β of Eq. (1) which can take all real values from $-\infty$ to $+\infty$. Consequently, the projection operator (15) with the weight function (17) $(c'=0)$ is different from (8). However, the key point is that eigenfunctions of both projection operators are one dimensional in the Schrödinger representation, as is indicated in Eqs. (12) and (20); namely, they admit only one linearly independent function as eigenfunction. This implies that the Schrödinger representation is irreducible for (2) too. In contrast, if eigenfunctions of projection operator (15) span a multidimensional space, it is shown that the representation is then a direct sum of a finite or countably infinite number of irreducible representations, each one of which is equivalent to the Schrödinger representation. We emphasize that these arguments were made possible only by an explicit use of the projection operator (15), which replaces Eq. (8) in our case. The advantage of this way of proving the above-mentioned theorem lies in the fact that it is related to the minimum-uncertainty states for angular-momentum—angle commutation relations, and therefore the parallelism with the conventional way is easier to understand. We remark that the uniqueness theorem discussed here is related to a more general theorem due to Mackey, and that we proved it in Ref. 24 only for a special case.

VI. CONCLUDING REMARKS

In previous sections, we learned that CCR can be extended to very general cases in Weyl form. Furthermore, the resultant Weyl algebra is found to give a novel way of characterizing minimum-uncertainty states, not only for the conventional CCR, but also for angular-momentum—angle CCR corresponding to an Abelian rotation group. These properties were confirmed by explicit constructions. If the method described here is applied to systems with many (or countably infinite) degrees of freedom,[25] it will be particularly suitable in studying semiclassical aspects of field theories.

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