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Classical model of the electron and the definition of electromagnetic field momentum

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The classical model of the electron in a vacuum is discussed in terms of a simple procedure for its assembly. The description of the assembly in two different inertial frames clarifies the traditional Lorentz-transformation difficulties of the model and confirms the appropriateness of the standard definition of the electromagnetic momentum density $\vec{g}=(1/4\pi c)\vec{E}\times\vec{B}$. Recent suggestions for alternative definitions of electromagnetic momentum are seen to destroy the conceptual simplicity of classical electrodynamics as revealed in the example.

The proper definition of the linear momentum in the electromagnetic field provides one of the recurring problems in physics. It appears most frequently in studies of the classical electron model in a vacuum.[1] After more than a half century of use of the standard electromagnetic momentum density $\vec{g}=(1/4\pi c)\vec{E}\times\vec{B}$, several prominent textbook writers[2] have called for a change. I believe this change is an error. In this paper the assembly of the classical electron is discussed in a simple example which clarifies the traditional Lorentz-transformation difficulties of the classical electron model and which again confirms the standard definition of the electromagnetic momentum density as the correct choice for classical electromagnetic theory.

The classical model of the electron consists of a spherically symmetric distribution of electric charge; for simplicity in this discussion, we will specialize the distribution to a thin spherical shell of radius a, total charge e, and mechanical mass m. The naive discussions of the classical model of the electron consider the mechanical and electromagnetic aspects of the model while completely ignoring the additional nonelectromagnetic forces, the Poincaré stresses, which stabilize the model. We are interested in clarifying the Lorentz-transformation difficulties of this naive view and hence will *define* our system to include only the mechanical and electromagnetic aspects; the nonelectromagnetic stabilizing forces are accordingly *external* to our system.

In the unprimed inertial frame S in which the spherical shell is at rest, the system, consisting of the shell mass and its electromagnetic fields, has a total energy

$$U_{\rm tot}=U_{\rm mech}+U_{\rm em}=mc^2+e/2a \qquad (1)$$

consisting of the mechanical rest energy $U_{\rm mech}=mc^2$ and the electromagnetic energy

$$U_{\rm em}=\frac{1}{8\pi}\int d^3r(E^2+B^2)=\frac{e^2}{2a} \qquad (2)$$

in the electrostatic field. The mass of the sphere is at rest and there is no magnetic field, $\vec{B}=0$, so that the momentum vanishes: $\vec{P}_{\rm tot}=\vec{P}_{\rm mech}+\vec{P}_{\rm em}=0$.

We now wish to examine this same system from a second primed inertial frame S' moving with velocity $-\vec{V}=-\hat{i}V$ relative to S so that the shell moves with velocity $+\vec{V}$ relative to S'. If we use the traditional definition $\vec{p}=m\gamma\vec{v}$ with $\gamma=(1-v^2/c^2)^{-1/2}$ for the momentum associated with the mechanical mass and the usual definition $\vec{g}=(1/4\pi c)\vec{E}\times\vec{B}$ for the density of momentum of the electromagnetic field, then here the total momentum of our system in S' is

$$\begin{aligned}\vec{P}'_{\rm tot}&=\vec{P}'_{\rm mech}+\vec{P}'_{\rm em}\\ &=m\gamma\vec{V}+(1/4\pi c)\int d^3x'(\vec{E}'\times\vec{B}')\,. \end{aligned}\qquad (3)$$

The evaluation of the integral for the electromagnetic momentum is carried out in the Appendix and yields

$$\begin{aligned}\vec{P}'_{\rm em}&=(1/4\pi c)\int d^3x'(\vec{E}'\times\vec{B}')\\ &=\tfrac{4}{3}\vec{V}\gamma U_{\rm em}/c^2\,,\end{aligned}\qquad (4)$$

where $U_{\rm em}=e^2/2a$ is the electrostatic energy in (2). Now if we expected the energy $U_{\rm em}$ and the momentum $\vec{P}_{\rm em}$ in the electromagnetic field to be a Lorentz four-vector, then we would anticipate from a Lorentz transformation between S and S'

$$\vec{P}'_{\rm em}=\vec{V}\gamma U_{\rm em}/c^2 \; . \qquad (5)$$

Clearly the discrepancy in the factor of $\frac{4}{3}$ between the expressions of Eqs. (4) and (5) frustrates this expectation. Why is that $\frac{4}{3}$ there? What is involved? Should we redefine the density of electromagnetic field momentum so as to remove the factor of $\frac{4}{3}$?

The problem of the factor of $\frac{4}{3}$ appearing in Eq. (4) is an old one.[3] It has been approached from a number of points of view, but apparently never from the assembly of the classical electron which is for me the simplest and clearest. Accordingly, a simple example of the assembly of a charged spherical shell as seen in two different inertial frames is outlined below.

The assembly of the classical model of the electron is imagined in terms of a thin spherical shell of total mechanical mass m and charge e sent rushing inwards from spatial infinity with the initial kinetic energy $mc^2(\gamma_v-1)$ at spatial infinity equal to the final electrostatic potential energy $U_{\rm em}=e^2/2a$. Since the shell is perfectly spherically symmetric, there is no radiation loss and all the initial kinetic energy at spatial infinity is converted into electrostatic potential energy when the shell comes momentarily to rest at radius a. Just at this instant when the spherical shell comes to rest, the stabilizing forces are applied. These forces prevent the reexpansion of the shell. The external forces are applied simultaneously at zero velocity and hence transfer neither energy nor momentum to the spherical shell. We have thus assembled our classical electron as a thin-shell charge of energy

$$U_{\rm tot}=mc^2+e^2/2a$$

and vanishing momentum $\vec{P}_{\rm tot}=0$.

The above description of the assembly of the classical electron is given from the point of view of an observer in the frame S in which the total momentum of the electron is zero. Let us view the same assembly process from the primed frame S' moving with velocity $-\vec{V}=-\hat{i}V$ relative to the initial frame S. When the massive charged shell has infinite radius, the electric and magnetic fields $\vec{E}'$, $\vec{B}'$ vanish, and all the particle energy and momentum is that associated with mechanical mass. Now the behavior of noninteracting mechanical mass is well known in special relativity, and the energy and momentum transform as a Lorentz four-vector. Hence, initially the system momentum, which is all mechanical momentum, is given by

$$\vec{P}'_{\rm tot}=\vec{P}'_{\rm mech}(t'\to-\infty)=\vec{V}\gamma U_{\rm tot}/c^2 \; .$$

As time passes the electromagnetic fields increase from their initial zero values and part of the mechanical momentum is converted into electromagnetic momentum. However, since there are no external forces on the system for times less than some t'_s, the total momentum is conserved and

$$\vec{P}'_{\rm tot}=\vec{P}'_{\rm mech}+\vec{P}'_{\rm em}$$
$$=\vec{V}\gamma U_{\rm tot}/c^2, \;\; t'<t'_\delta \; ; \qquad (6)$$

the proper Lorentz transformation properties still hold. The crucial change comes when the external stabilizing forces are applied beginning at $t'=t'_\delta$. In the frame S these stabilizing forces are applied simultaneously; consequently the net external force on the system vanishes and there is no change in the momentum of the system. Contrastingly in the S' frame the external stabilizing forces are applied at different times to different parts of the spherical shell. Thus starting with the application of the first force and until the moment in the S' frame when all the external forces have been applied, there is a net external force on the shell, and hence net momentum is transferred to the shell. The total momentum of the shell is thus increased from the value

$$\vec{P}'_{\rm tot}=\vec{V}\gamma U_{\rm tot}/c^2$$

which held before the external forces were applied. The change in momentum $\Delta\vec{P}'$ of the charge shell as seen in the S' frame is equal to the net impluse $\vec{I}'$ delivered by the external stabilizing forces as seen in the S' frame. The net impulse $\vec{I}'$ can be computed as in the Appendix with the value for $\Delta\vec{P}'=\vec{I}'$ given by

$$\Delta\vec{P}'=\tfrac{1}{3}\vec{V}\gamma U_{\rm em}/c^2 \; . \qquad (7)$$

But this is precisely the discrepancy associated with the factor of $\frac{4}{3}$. The total system momentum $\vec{P}'^{\rm (after)}_{\rm tot}$ after all the external stabilizing forces have been applied has been changed in the S' frame from the value in Eq. (6) over to

$$\vec{P}'^{\rm (after)}_{\rm tot}=\vec{V}\gamma U_{\rm tot}/c^2+\tfrac{1}{3}\vec{V}\gamma U_{\rm em}/c^2$$
$$=\vec{V}\gamma m+\tfrac{4}{3}\vec{V}\gamma U_{\rm em}/c^2 \qquad (8)$$

corresponding to the momentum of the mechanical mass m and exactly the electromagnetic momentum (4) involving the integral over the traditional field momentum density

$$\vec{g}\,'=(1/4\pi c)\vec{E}\,'\times\vec{B}\,' .$$

The factor of $\frac{4}{3}$ in the electromagnetic momentum is by no means extraneous; it is needed crucially to maintain the validity of the force-momentum balance in the S' frame.

A physical particle or system will in general involve contributions to the total momentum from both the electromagnetic fields and other sources. In our example the mechanical momentum of the shell at spatial infinity is converted into electromagnetic momentum as the shell rushes inward. As Poincaré pointed out in 1906 only the total energy and momentum can be expected to satisfy covariant behavior when transformed between different inertial frames.

Various authors[4] have taken a view which is opposite to that expressed here and have suggested that the factor of $\frac{4}{3}$ above is an embarrassment which should be removed. One method for removing the factor involves redefining the electromagnetic momentum of the system so that it is not the integral of the density

$$\vec{g}=(1/4\pi c)\vec{E}\times\vec{B}$$

as used above, but rather is[5]

$$\vec{P}_{em}=(\gamma/4\pi c)\int d^3x[\vec{E}\times\vec{B}+\vec{v}\cdot\vec{E}\vec{E}+\vec{v}\cdot\vec{B}\vec{B}-\tfrac{1}{2}\vec{v}(E^2+B^2)] , \qquad (9)$$

where

$$\gamma=(1-v^2/c^2)^{-1/2}$$

and $\vec{v}$ is the velocity of the inertial frame relative to some preferred inertial frame.

This redefinition, I believe, is an error. The usual ideas of force, energy, and momentum hold together properly with the traditional definition and not with the use of the density function given in Eq. (9) which eliminates the factor of $\frac{4}{3}$. The example given above is one illustration of this; if the laws of physics are to hold for all inertial frames in this open system in which nonelectromagnetic external forces are applied, then the electromagnetic field momentum should not transform as a Lorentz four-vector and the factor of $\frac{4}{3}$ is a consistent reflection of this fact. My opinion is that these other authors err in taking seriously as a model for a point charged particle the electromagnetic energy and momentum behavior of the classical model of the electron despite the nonelectromagnetic forces required for the classical model's stability. The nonelectromagnetic stabilizing forces play a crucial role and the attempts to circumvent the role played by these forces by redefining the electromagnetic momentum density only destroy the conceptual simplicity of the traditional view of classical electrodynamics.

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APPENDIX

1. Electromagnetic momentum for a spherical charge shell

The fields $\vec{E}\,'$ and $\vec{B}\,'$ needed in the integrand of Eq. (4) are easily found by Lorentz transformation[6] from the electrostatic fields in the S frame. Also if we change the variables of integration from x', y', z' at fixed time t' over to x, y, z at t', and use the spherical symmetry of the fields in S to replace ${E_y}^2+{E_z}^2$ by $\frac{2}{3}E^2$, then we have

$$\begin{aligned}\vec{P}\,'_{em}&=(1/4\pi c)\int d^3x'(\vec{E}\,'\times\vec{B}\,')\\&=(\hat{i}/4\pi c)\int (d^3x/\gamma)(V\gamma^2/c)({E_y}^2+{E_z}^2)\\&=\tfrac{4}{3}\vec{V}\gamma U_{em}/c^2 ,\end{aligned} \qquad (A1)$$

where $U_{em}=e^2/2a$ is the electrostatic energy in Eq. (2).

2. Net impulse of the external forces

The external forces applied at time t_0 in the S frame are exactly such as to balance the electrostatic forces of the sphere on itself:

$$f^{\mu}_{ext}(t,\vec{r})=-f^{\mu}_{em}(t,\vec{r})\theta(t-t_0) , \qquad (A2)$$

where f^{μ} stands for the force density and $\theta(t-t_0)$ is the unit step function. The electromagnetic force density $f^{\mu}_{em}(t,\vec{r})$ can be obtained from the symmetric electromagnetic stress-energy-momentum tensor[7] $f^{\mu}_{em}=-\partial_\nu\Theta^{\mu\nu}_{em}$, where $\Theta^{\mu\nu}_{em}$ involves simply the electrostatic field

$$\vec{E}=\theta(r-a)e\vec{r}/r^3 .$$

Thus it follows that

$$f^{\mu}_{em}=(0,(e/4\pi a^2)\delta(r-a)e\hat{r}/2a^2) . \qquad (A3)$$

The force density at a space-time point is a Lorentz four-vector which in the S' frame has three-components

$$f'_{\mathrm{ext}\,x}=\gamma f_{\mathrm{ext}\,x},\quad f'_{\mathrm{ext}\,y}=f_{\mathrm{ext}\,y}\,,\quad f'_{\mathrm{ext}\,z}=f_{\mathrm{ext}\,z}\,. \tag{A4}$$

By symmetry it is clear that the net impulse $\vec{\mathrm{I}}\,'$ has a component only in the x direction parallel to the relative velocities of the frames. Thus the resultant impluse $\vec{\mathrm{I}}\,'(t'_*)$ delivered by the external forces to the shell as measured in the S' frame is

$$\vec{\mathrm{I}}\,'(t'_*)=\hat{i}\int_{t'=-\infty}^{t'=t'_*}dt'\int d^3x' f'_{\mathrm{ext}\,x}(t',\vec{r}\,')\,. \tag{A5}$$

Now using the Lorentz invariance of the space-time volume element $dt'd^3x'=dt\,d^3x$, the Lorentz transformations for the coordinates, and the expressions (A2), (A3) and (A4), the integral for $\vec{\mathrm{I}}\,'(t'_*)$ becomes

$$\vec{\mathrm{I}}\,'(t'_*)=-\hat{i}\int d^3x\int_{t=-\infty}^{t=t'_*/\gamma-Vx/c^2}dt\,\gamma\theta(t-t_0)\times\delta(r-a)e^2x/8\pi a^5\,. \tag{A6}$$

If the time t'_* is sufficiently large that all the external forces have been applied, then the integration becomes

$$\begin{aligned}\vec{\mathrm{I}}\,'(t'_*)&=-\hat{i}\gamma\int_0^\infty r^2dr\int_{\theta=0}^{\pi}d\theta\sin\theta\int_{\phi=0}^{2\pi}d\phi\int_{t=0}^{t=t_*/\gamma-Vr\cos\theta/c^2}dt\,\delta(r-a)\cos\theta e^2/8\pi a^4\\&=-\hat{i}\gamma a^2\int_{\theta=0}^{\pi}d\theta\sin\theta 2\pi\left[\frac{t_*}{\gamma}-\frac{Va\cos\theta}{c^2}\right]\cos\theta\frac{e^2}{8\pi a^4}\\&=\tfrac{1}{3}\vec{\mathrm{V}}\gamma e^2/2ac^2\end{aligned} \tag{A7}$$

as required for Eq. (7).

Note added. My analysis above was written with two aims; first to provide a simple model for the formation of the classical electron model which I believe sharply clarifies the famous factor of $\frac{4}{3}$, and second to suggest that the example illustrates the validity of the traditional definition for momentum in the classical electromagnetic field and the inappropriateness of the new definition which is creeping into the advanced-textbook literature.[8]

My views are not shared by Professor F. Rohrlich. His rebuttal to my discussion appears in the following paper.[9]

It should be noted immediately that neither Professor Rohrlich nor I now disputes the accuracy of the other's calculations; at least I believe this is so. I do differ with Professor Rohrlich on two assertions of his reply (following paper[9]) and with his conclusion on the definition of electromagnetic momentum providing greatest conceptual clarity.

In the abstract to his article[9] Professer Rohrlich states: "The fundamental question is whether electromagnetic interactions can be separated from nonelectromagnetic ones in a Poincaré-invariant way. This question is answered in the affirmative." For me this is not at all the fundamental question. By suitable redefinitions in relatively moving frames, any quantity can be made Poincaré covariant, and Professor Rohrlich does this for the electromagnetic field momentum. I believe the fundamental question is what definitions are physically natural and conceptually useful. This difference in perspective may be one ground for the disagreement between Professor Rohrlich and me.

In the body of the article[9] Professor Rohrlich writes: "It must be emphasized that the separation (7) of the momentum into an electromagnetic and nonelectromagnetic part is not an observable separation but serves the convenience of the theory. It corresponds to the separation of the observed mass into an electromagnetic and nonelectromagnetic part." In my view this comment is appropriate for a system, such as a point charge, which cannot be regarded as composed of constituent pieces, but it is completely inappropriate for composite systems, such as colliding point charges. My strong impression is that Professor Rohrlich is always writing with the former example in mind and never from the more general perspective, and on this account he arrives at a conclusion different from my own.

If we look at the discussions provided by Professor Rohrlich and me, we see immediately that we are not discussing the same model but rather different ones. I assemble the charged sphere by sending a massive charged shell rushing inward from infinite radius. Before the application of the external stabilizing forces, the total energy-momentum $P^{\mu}_{\rm tot}=P^{\mu}_{\rm mech}+P^{\mu}_{\rm em}$ is a four-vector where the mechanical and electromagnetic momentum in any Lorentz frame have their natural definitions at a single time in that frame. Neither the mechanical part nor the electromagnetic part is separately a four-vector. This is just as in the collision between point charges where the total energy-momentum is a four-vector but we do not expect mechanical and electromagnetic energy-momentum to form separate four-vectors. I show that the ideas of momentum balance fit beautifully and naturally with the traditional definition of momentum in the classical electromagnetic field.

In contrast Professor Rohrlich assembles his sphere quasistatically. Thus, as he states above his Eq. (8), for him the mechanical momentum P^{μ}_{m} is separated and assumed to be a four-vector by itself. Thus Professor Rohrlich never discusses any interplay between mechanical momentum and electromagnetic momentum, but rather only the interplay between electromagnetic momentum and nonelectromagnetic forces where the unnaturalness of his definition of electromagnetic momentum for composite systems is not fully exposed. I believe the unnaturalness is easily exposed if we think in terms of Poynting's theorem.

Classical electromagnetism is a theory of considerable detail and beauty to which Professor Rohrlich has contributed significantly. In particular, classical electromagnetism allows the use of nonelectromagnetic external forces and nonelectromagnetic masses which are connected with the energy and momentum of the classical electromagnetic fields through Poynting's theorem[10] and its momentum analog[11] when using the traditional defintions of energy and momentum. One of the striking illustrations[12] of Poynting's theorem involves charged particles passing each other with arbitrary constant velocities, $v_i < c$. The nonelectromagnetic external forces, which are required to balance the interparticle Lorentz forces and so keep the particles moving with constant velocity, do not satisfy Newton's third law. Rather the work done by the external forces, the impulse supplied by the external forces, and the angular impulse supplied by the external forces lead exactly as a relativistic calculation with no approximation in every Lorentz frame to the appropriate changes of energy, linear momentum, and angular momentum in the electromagnetic field when the traditional definitions are made for the energy, momentum, and angular momentum in the electromagnetic field. I believe the conceptual simplicity of the traditional definitions of classical electrodynamics is given yet another striking illustration above in my example of the assembly of the classical model of the electron.

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[1]Discussions of the electromagnetic momentum in connection with the classical model of the electron appear in the following: H. A. Lorentz, *The Theory of Electrons*, 2nd ed. (Dover, New York, 1952), Secs. 24–28 (this is a republication of the 1915 edition); E. Fermi, Z. Phys. 24, 340 (1922); W. Wilson, Proc. Phys. Soc. London A48, 376 (1936); B. Kwal, J. Phys. Radium 10, 103 (1949); F. Rohrlich, Am. J. Phys. 28, 639 (1960); Phys. Today 15, 19 (1962); Am. J. Phys. 34, 987 (1966); 38, 1310 (1970); J. W. Zink, *ibid.* 34, 211 (1966); 36, 639 (1968); 39, 1403 (1971); J. W. Butler, *ibid.* 37, 1258 (1969); R. Benumof, *ibid.* 39, 392 (1971).

[2]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), Sec. 6-3; J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Sec. 17.5

[3]Apparently the factor of $\frac{4}{3}$ was found first by J. J. Thomson in 1881. See Rohrlich's account in Chap. 2 of the work listed in Ref. 2.

[4]See Ref. 2 and the articles by Rohrlich and Butler in Ref. 1.

[5]See, for example, F. Rohrlich, Am. J. Phys. 38, 1310 (1970), Eq. (3.24).

[6]See Jackson in Ref. 2, p. 552, Eq. (11.148).

[7]See Jackson in Ref. 2, Section 12.10b.

[8]See Jackson in Ref. 2, pp. 792–796.

[9]F. Rohrlich, following paper, Phys. Rev. D 25, 3251 (1982).

[10]See Jackson in Ref. 2, pp. 236–237.

[11]See Jackson in Ref. 2, pp. 237–239.

[12]T. H. Boyer, Am. J. Phys. 39, 257 (1971).