Transverse Schwarzschild field

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For Schwarzschild's static spherically symmetric external field, a coordinate system is determined in which the metric field is the transverse field satisfying the coordinate conditions of Arnowitt, Deser, and Misner.

I. INTRODUCTION

We assume here that Dirac's gravitational Lagrangian density is used,[1] so that the primary coordinate constraints reduce to $p^{\mu 0}=0$ $(\mu=0,1,2,3)$.

It has been suggested that for dynamic quantization of the gravitational field one then should use a coordinate system in which the spatial components g_{ij} of the metric, and their canonical conjugates p^{ij}, would satisfy the four coordinate conditions[2]

$$g_{ij,j}=0 \quad (i=1,2,3)\ , \tag{1}$$

$$p^{ii}{}_{,jj}-p^{ij}{}_{,ij}=0\ . \tag{2}$$

These conditions, which are not invariant under coordinate transformations, are supposed to be for the gravitational field what the Coulomb-gauge condition

$$A_{j,j}=0 \tag{3}$$

is for special-relativistic Maxwell theory. Also the latter condition is not even Lorentz invariant, so that a transformation to a moving coordinate system requires a simultaneous gauge transformation, if (3) is to be maintained. Similarly, after a transformation to a moving coordinate system, in general (1) and (2) would not remain automatically valid. An additional coordinate transformation is required, if (1) and (2) are to be reestablished.

The purpose of postulating the validity of these conditions and their time derivatives is to make all but the dynamic components of the gravitational field functionals of the sources of the gravitational field [just like (3) and its time derivative make the electric potential and the longitudinal part of the electric field strength functionals of the electric charge distribution at equal time]. This should leave only the dynamic components of the gravitational field as quantities independent of other fields at the same time, allowing canonical quantization of these dynamic components, while treating the other components like derived variables.[3]

We shall not discuss here the question whether (1) and (2) satisfactorily serve this purpose in curved space. We will here merely answer the question how in one simple case a coordinate system satisfying (1) and (2) is related to a coordinate system that is more familiar.

II. THE STATIC SPHERICALLY SYMMETRIC EXTERNAL FIELD

In special-relativistic Maxwell theory, the external static spherically symmetric field is given by $A_0=-q/r$, $A_i=0$. Condition (3) then is automatically satisfied, and no gauge transformation is needed for achieving this.

In gravitational theory, the external static spherically symmetric field is known in various coordinate systems. It is best known in Schwarzschild coordinates. It is known in isotropic coordinates,[4] and it is known in harmonic coordinates.[5] However, none of these coordinate systems fulfills condition (1).

Therefore, we want to solve here the following problem. Let r,θ,φ be Schwarzschild's spatial coordinates. Let x,y,z be the coordinates in a system satisfying (1), related conventionally to polar coordinates r,θ,φ. We want to determine r as a function of r.

We introduce dimensionless quantities by $X=r/2m$, $R=\mathrm{r}/2m$, $R^i=\mathrm{x}^i/2m$, and $dS=d\sigma/2m$ for the spatial line element given by $dS^2=g_{ij}dR^i dR^j$. (Here, $m=GM/c^2$.) Let also $d\psi^2=d\theta^2+\sin^2\theta\, d\varphi^2$. Then, from Schwarzschild's equation

$$dS^2=(1-X^{-1})dX^2+X^2 d\psi^2\ , \tag{4}$$

by $dX=(dX/dR)(R^i dR^i/R)$ and $R^2 d\psi^2=dR^i dR^i-dR^2$, we find

$$g_{ij}=[(1-X^{-1})^{-1}(dX/dR)^2-X^2/R^2]R_iR_j/R^2+(X^2/R^2)\delta_{ij}\ . \tag{5}$$

We now want

$$\begin{aligned}0&=g_{ij,j}\\&=\frac{4R^i}{R^2}\left[\frac{1}{1-1/X}\left[\frac{dX}{dR}\right]^2-\frac{X^2}{R^2}\right]\\&\quad+R^iR\frac{d}{dR}\left[\frac{(dX/dR)^2}{R^2(1-1/X)}-\frac{X^2}{R^4}\right]\\&\quad+\frac{R^i}{R}\frac{d}{dR}\left[\frac{X^2}{R^2}\right]\equiv QR^i/R^2\ .\end{aligned} \tag{6}$$

We find

$$Q=\frac{2}{1-1/X}\left[\frac{dX}{dR}\right]^2-\frac{2X^2}{R^2}+\frac{2R}{(1-1/X)}\frac{dX}{dR}\frac{d^2X}{dR^2}-\frac{R}{X^2}\frac{(dX/dR)^3}{(1-1/X)^2}\ , \tag{7}$$

so that

$$Q=\frac{1}{R}\frac{d}{dR}\left[\frac{R^2(dX/dR)^2}{1-1/X}\right]-\frac{2X^2}{R^2}\ . \tag{8}$$

Therefore, $Q=0$ gives

$$2X^2=R\frac{d}{dR}\left[\frac{(RdX/dR)^2}{1-1/X}\right]\ . \tag{9}$$

We now put $\eta=\ln R$. This gives

$$\begin{aligned}2X^2&=\frac{d}{d\eta}\left[\frac{(dX/d\eta)^2}{1-1/X}\right]\\&=\frac{2(dX/d\eta)(d^2X/d\eta^2)}{1-1/X}-\frac{(dX/d\eta)^3}{(1-1/X)^2X^2}\ ,\end{aligned} \tag{10}$$

so

$$\frac{d^2X}{d\eta^2}=\frac{X(X-1)}{dX/d\eta}+\frac{(dX/d\eta)^2}{2X(X-1)}\ . \tag{11}$$

Putting also $\xi=\ln X$, we find

$$\begin{aligned}\frac{d^2\xi}{d\eta^2}&=\frac{d}{d\eta}\left[\frac{1}{X}\frac{dX}{d\eta}\right]\\&=(X-1)\frac{d\eta}{dX}+\frac{1}{X^2}\left[\frac{1}{2X-2}-1\right]\left[\frac{dX}{d\eta}\right]^2\\&=(1-e^{-\xi})\left[\frac{d\eta}{d\xi}\right]+\left[\frac{1}{2(e^{\xi}-1)}-1\right]\left[\frac{d\xi}{d\eta}\right]^2.\end{aligned} \tag{12}$$

Thence,

$$\begin{aligned}\frac{d^2\eta}{d\xi^2}&=\frac{d\eta}{d\xi}\frac{d}{d\eta}\left[\frac{1}{d\xi/d\eta}\right]\\&=-\left[\frac{d\eta}{d\xi}\right]^3\frac{d^2\xi}{d\eta^2}\\&=\left[1-\frac{e^{-\xi}}{2(1-e^{-\xi})}\right]\frac{d\eta}{d\xi}\\&\quad-[1-e^{-\xi}]\left[\frac{d\eta}{d\xi}\right]^4\ .\end{aligned} \tag{13}$$

This nonlinear second-order differential equation therefore represents the coordinate condition (1) for the external Schwarzschild field.

Condition (2) is automatically satisfied, because Dirac's gravitational Lagrangian density provides

$$p^{ij}=-(2\kappa c)^{-1}(-g)^{-1/2}E^{ijnm}\Gamma^0{}_{mn}\ . \tag{14}$$

Here, $\kappa=8\pi G/c^4$, and $E^{ijnm}=e^{ij}e^{nm}-e^{im}e^{nj}$, where the matrix $||e^{ij}||$ is the inverse of the 3×3 matrix $||g_{ij}||$, while

$$\Gamma^0{}_{mn}=\tfrac{1}{2}g^{0\lambda}(g_{\lambda m,n}+g_{\lambda n,m}-g_{mn,\lambda})\ . \tag{15}$$

In the Schwarzschild field, $g^{0i}=0$ and $g_{0m}=0$ and $g_{mn,0}=0$, so that $\Gamma^0{}_{mn}=0$ and $p^{ij}=0$. We assume here with Dirac that the matter Lagrangian does not contain gravitational field velocities.[6]

III. FIRST INTEGRATION OF THE COORDINATE CONDITION

Inserting (11) in

$$\frac{d^2\eta}{dX^2}=\left[\frac{d\eta}{dX}\right]\frac{d}{d\eta}\left[\left[\frac{dX}{d\eta}\right]^{-1}\right]=-\left[\frac{d\eta}{dX}\right]^3\frac{d^2X}{d\eta^2}\ , \tag{16}$$

we obtain

$$\frac{d^2\eta}{dX^2}=-X(X-1)\left[\frac{d\eta}{dX}\right]^4-\frac{1}{2X(X-1)}\frac{d\eta}{dX}\ . \tag{17}$$

Introducing the abbreviations

$$p=d\eta/dX,\quad \vartheta=(1-1/X)^{1/2}\ ,$$

so that

$$d\vartheta/dX=1/(2\vartheta X^2)\ , \tag{18}$$

we may write (17) as

$$\frac{dp}{dX}=-X^2\vartheta^2p^4-\frac{p}{2X^2\vartheta^2}$$
$$=-X^2\vartheta^2p^4-\frac{p}{\vartheta}\frac{d\vartheta}{dX}\ . \quad (19)$$

Putting

$$s=\ln(p\vartheta)\ ,$$

so that

$$p\vartheta=e^s\ , \quad (20)$$

we obtain from (19),

$$\frac{ds}{dX}=-X^2\vartheta^2p^3=-\frac{X^2}{\vartheta}e^{3s}\ , \quad (21)$$

so that

$$\frac{ds}{d\vartheta}=-2X^4e^{3s}=-\frac{2}{(1-\vartheta^2)^4}e^{3s}\ . \quad (22)$$

Thence,

$$d(e^{-3s})=6X^4d\vartheta\ . \quad (23)$$

Let

$$F=(X^3+\tfrac{5}{4}X^2+\tfrac{15}{8}X)\vartheta+\tfrac{15}{16}\ln\left[\frac{1+\vartheta}{1-\vartheta}\right]. \quad (24)$$

Then,

$$dF=\left[X^3+\tfrac{5}{4}X^2+\tfrac{15}{8}X+\frac{15/8}{1-\vartheta^2}\right]d\vartheta$$
$$+(3X^2+\tfrac{5}{2}X+\tfrac{15}{8})\vartheta dX$$
$$=(X^3+\tfrac{5}{4}X^2+\tfrac{15}{4}X)d\vartheta$$
$$+2\left[1-\frac{1}{X}\right](3X^4+\tfrac{5}{2}X^3+\tfrac{15}{8}X^2)d\vartheta$$
$$=6X^4d\vartheta\ , \quad (25)$$

so that (23) is integrated by

$$e^{-3s}=F+3K\ , \quad (26)$$

where K is an integration constant. It follows that

$$p\vartheta=(F+3K)^{-1/3}\ , \quad (27)$$

so that

$$\frac{d\eta}{dX}=\vartheta^{-1}(F+3K)^{-1/3} \quad (28)$$

and

$$\frac{d\eta}{d\xi}=\frac{X}{\vartheta}(F+3K)^{-1/3}$$
$$=e^{\xi}(1-e^{-\xi})^{-1/2}(3K+F)^{-1/3}\ . \quad (29)$$

IV. ASYMPTOTIC SOLUTION AT LARGE r

For large X and ξ, we will put here

$$\epsilon=\frac{1}{X}=e^{-\xi}\ ,$$

so that

$$\ln\epsilon=-\xi\ . \quad (30)$$

Then, expansion in powers (and logarithms) of ϵ gives

$$\vartheta=(1-\epsilon)^{1/2}=1-\tfrac{1}{2}\epsilon+O(\epsilon^2) \quad (31)$$

and, from (29),

$$\left[\frac{d\eta}{d\xi}\right]^{-3}=\epsilon^3\vartheta^3(F+3K)$$
$$=(1-\epsilon)^2(1+\tfrac{5}{4}\epsilon+\tfrac{15}{8}\epsilon^2)$$
$$+\tfrac{15}{16}\epsilon^3\ln\frac{4}{\epsilon}+3K\epsilon^3+O(\epsilon^4)$$
$$=1-\tfrac{3}{4}\epsilon+\tfrac{3}{8}\epsilon^2-\tfrac{5}{2}\epsilon^3+\tfrac{15}{16}\epsilon^3(2\ln2+\xi)+3K\epsilon^3$$
$$+O(\epsilon^4)\ . \quad (32)$$

Thence,

$$\frac{d\eta}{d\xi}=1+\tfrac{1}{4}\epsilon-\tfrac{1}{8}\epsilon^2+\epsilon^3(\tfrac{5}{6}-\tfrac{5}{8}\ln2-\tfrac{5}{16}\xi-K)$$
$$+\frac{2}{9}(\tfrac{9}{16}\epsilon^2-\tfrac{9}{16}\epsilon^3+\cdots)$$
$$-\tfrac{14}{81}(-\tfrac{27}{64}\epsilon^3+\cdots)+\cdots$$
$$=1+\tfrac{1}{4}e^{-\xi}-\tfrac{5}{16}\xi e^{-3\xi}$$
$$+(\tfrac{25}{32}-\tfrac{5}{8}\ln2-K)e^{-3\xi}+\cdots\ . \quad (33)$$

Integration gives

$$\eta=\xi-\tfrac{1}{4}e^{-\xi}+\tfrac{5}{48}\xi e^{-3\xi}-\tfrac{65}{288}e^{-3\xi}$$
$$+(\tfrac{5}{24}\ln2)e^{-3\xi}+\frac{K}{3}e^{-3\xi}+\cdots\ , \quad (34)$$

where there is no second integration constant, if we postulate that $R/X=e^{\eta-\xi}\to 1$ for $X\to\infty$.

From (34) we now find for R asymptotically,

$$\begin{aligned}R&=e^{\eta}\\&=e^{\xi}\exp(-\tfrac{1}{4}\epsilon+\tfrac{5}{48}\xi\epsilon^3-\text{etc.})\\&=X(1-\tfrac{1}{4}X^{-1}+\text{etc.})\\&=X-\tfrac{1}{4}+\tfrac{1}{32}X^{-1}+\tfrac{5}{48}X^{-2}\ln X\\&\quad+\left[\tfrac{5}{24}\ln 2+\frac{K}{3}-\tfrac{263}{1152}\right]X^{-2}+\cdots\,,\end{aligned}\tag{35}$$

so that $R\to X-\frac{1}{4}$ for $X\to 0$.

Solving for X from (35) in successive approximations, we find

$$\begin{aligned}X&=R+\tfrac{1}{4}-\tfrac{1}{32}R^{-1}-\tfrac{5}{48}R^{-2}\ln R\\&\quad+\left[\tfrac{17}{72}-\frac{K}{3}-\tfrac{5}{24}\ln 2\right]R^{-2}+\cdots\,.\end{aligned}\tag{36}$$

It is now easily verified that, up to terms of relative order ϵ^3, (36) satisfies Eq. (9), and (34) satisfies Eqs. (13) and (29).

V. SOLUTIONS WITH NEGATIVE K

If $K<0$, there exist points with $X>1$, for which $F<3|K|$, so that $F+3K<0$. Then, by (29), $d\eta/d\xi<0$, and r would *increase* upon further decrease of r. At the point where $d\eta/d\xi$ jumps from $-\infty$ to $+\infty$, r as a function of r shows a cusp at $r=r_c$ where r has its minimal value r_c.

It is questionable whether condition (1) can be deemed satisfied at the cusp by solutions of this kind. Therefore, and for the lack of elegance of a radial coordinate that would increase for circles of decreasing circumference $2\pi r$ around the origin, solutions with $K<0$ are undesirable when the external Schwarzschild field prevails down to those values of r where $d\eta/d\xi$ would become negative. If, however, there is a matter distribution around the origin which extends to a spherical boundary *outside* the points where the external solution would make $dR/dX<0$, solutions with $K<0$ outside that matter distribution would be acceptable, if they can be fitted at the boundary of matter to an acceptable internal solution that satisfies condition (1).

When, below, we discuss solutions with $K<0$ also at $r<r_c$, we should remember that these external solutions lose their physical meaning inside the matter distribution which we expect to extend beyond r_c when negative K is used at all. So, the behavior of these external solutions at and inside r_c is indicated for mathematical curiosity only.

The sphere where dr/dr flips sign is given by $F+3K=0$, so that $X_c=r_c/2m$ is related to K by

$$\begin{aligned}K=&-[\tfrac{1}{3}(X_c{}^2+\tfrac{5}{4}X_c+\tfrac{15}{8})X_c\vartheta_c+\tfrac{15}{16}\ln X_c\\&+\tfrac{5}{8}\ln(1+\vartheta_c)]\,,\end{aligned}\tag{37}$$

where $\vartheta_c{}^2=1-1/X_c$. Close to this sphere, at $X=X_c+\Delta$, we find, by (25) and (18),

$$\begin{aligned}F+3K&=F-F_c=6X_c{}^4(\vartheta-\vartheta_c)\\&=3X_c{}^2\Delta/\vartheta_c\,,\end{aligned}\tag{38}$$

so, by (28),

$$d\eta/dX=(3X_c{}^2/\vartheta_c{}^2)^{-1/3}\Delta^{-1/3}\,.\tag{39}$$

Integration gives

$$\eta-\eta_c=\tfrac{3}{2}(3X_c{}^2/\vartheta_c{}^2)^{-1/3}\Delta^{2/3}\,.\tag{40}$$

Below, we will also use the radial coordinate

$$\phi=\log\xi=\log\ln(r/2m)\,,\tag{41}$$

so that

$$dX=Xd\xi=(X\xi\ln 10)d\phi\,.\tag{42}$$

Then, at $\phi=\phi_c+\delta$,

$$\eta=\eta_c+C\delta^{2/3}\tag{43}$$

with

$$\begin{aligned}C&=\tfrac{3}{2}(3X_c{}^2/\vartheta_c{}^2)^{-1/3}(X_c\xi_c\ln 10)^{2/3}\\&=\tfrac{1}{2}[3(\xi_c/\vartheta_c)\ln 10]^{2/3}\,.\end{aligned}\tag{44}$$

For instance, a cusp in $R(X)$ at $X=X_c=1.5625=\frac{25}{16}$ is found, according to (37), for $K=-(\frac{8025}{4096}+\frac{5}{8}\ln 2)=-2.392\,445\,50$, and, near it, by (43) and (44), $\eta=\eta_c+1.4888(\phi-\phi_c)^{2/3}$. This is a good approximation for $|\phi-\phi_c|<10^{-5}$. It enables us to compute η (or R) as a function of ϕ (or ξ or X) through the neighborhood of the cusp, where Simpson integration of (28) or (29) or of $d\eta/d\phi$ would fail because of the singularity in the integrand.

VI. SOLUTIONS WITH NON-NEGATIVE K NEAR THE SCHWARZSCHILD RADIUS

For $K\geq 0$ we now consider the limit $X\to 1$, $\xi\to 0$, $\vartheta\to 0$. For $K\neq 0$ we may expand in powers of (ϑ/K). The case $K=0$ must be treated separately.

For $\vartheta \to 0$ we find from (25), by $X=(1-\vartheta^2)^{-1}$,

$$F=6\vartheta+8\vartheta^3+12\vartheta^5+\tfrac{120}{7}\vartheta^7+\cdots \,. \quad (45)$$

Since, by (28) and (18),

$$\frac{d\eta}{d\vartheta}=\frac{d\eta}{dX}\frac{dX}{d\vartheta}=2(F+3K)^{-1/3}(1-\vartheta^2)^{-2}\,, \quad (46)$$

we find, for $K=0$ and $\vartheta \ll 1$,

$$\frac{d\eta}{d\vartheta}=2F^{-1/3}(1-\vartheta^2)^{-2} = \left[\tfrac{4}{3}\right]^{1/3}\vartheta^{-1/3}(1+\tfrac{14}{9}\vartheta^2+\tfrac{149}{81}\vartheta^4+\cdots)\,. \quad (47)$$

Integration gives

$$\eta=\eta_0+\left[\tfrac{9}{2}\right]^{1/3}\vartheta^{2/3}(1+\tfrac{7}{18}\vartheta^2+\tfrac{149}{567}\vartheta^4+\cdots)\,, \quad (48)$$

where η_0 is η at the Schwarzschild radius (at $r=2m$, $X=1$, $\xi=0$, $\vartheta=0$).

For $K\neq 0$ and $|\vartheta| \ll |K|$ we find

$$\frac{d\eta}{d\vartheta}=2(3K)^{-1/3}\left[1+\frac{F}{3K}\right]^{-1/3}(1-\vartheta^2)^{-2}$$
$$=(3K)^{-1/3}\left[2+4\vartheta^2+6\vartheta^4+\cdots-\frac{4\vartheta}{3K}-\frac{40\vartheta^3}{9K}+\cdots+\frac{16\vartheta^2}{9K^2}+\frac{224\vartheta^4}{27K^2}+\cdots -\frac{224\vartheta^3}{81K^3}+\cdots+\frac{1120\vartheta^4}{243K^4}+\cdots\right]. \quad (49)$$

Integration gives

$$\eta=\eta_0+(3K)^{-1/3}\left[2\vartheta-\frac{2\vartheta^2}{3K}+\frac{4}{3}\vartheta^3+\frac{16\vartheta^3}{27K^2} -\frac{10\vartheta^4}{9K}-\frac{56\vartheta^4}{81K^3}+\frac{6}{5}\vartheta^5 +\frac{224\vartheta^5}{135K^2}+\frac{224\vartheta^5}{243K^4}+\cdots\right]. \quad (50)$$

VII. GENERAL NUMERICAL INTEGRATION

For $\xi > 8$, we may use (34) for finding η as a function of ξ. For ξ between 0 and 10, we then could obtain $\eta(\xi)$ by integrating (29) numerically from ξ up to 10, and subtracting the result from $\eta(10)$ calculated by (34).

There is a problem with this method when ξ becomes small as we come close to the Schwarzschild radius. By (18),

$$\frac{d\eta}{d\xi}=X\frac{d\eta}{dX}=(2\vartheta X)^{-1}\frac{d\eta}{d\vartheta}\,. \quad (51)$$

For $\xi \to 0$ we have

$$X\to 1 \text{ and } \vartheta=(1-e^{-\xi})^{1/2}\to \xi^{1/2}\,. \quad (52)$$

Since, according to (47) and (49), $d\eta/d\vartheta$ for $\xi\to 0$, $\vartheta\to 0$ becomes proportional to $\vartheta^{-1/3}$ for $K=0$ and becomes constant for $K\neq 0$, it follows from (51) and (52) that, for $\xi \to 0$,

$$d\eta/d\xi \propto \vartheta^{-4/3} \propto \xi^{-2/3} \text{ for } K=0\,, \quad (53a)$$

$$d\eta/d\xi \propto \vartheta^{-1} \propto \xi^{-1/2} \text{ for } K\neq 0\,. \quad (53b)$$

This divergence of $d\eta/d\xi$ toward the Schwarzschild radius makes ξ there a poor integration variable for numerical calculations.

Since, in both cases (53a) and (53b), $\xi d\eta/d\xi=d\eta/d(\ln\xi)$ remains finite, ln ξ or log ξ is a good integration variable. We therefore use ϕ of (41). From (28) and (42) we now get

$$d\eta/d\phi=(F+3K)^{-1/3}(X\xi/\vartheta)\ln 10\,. \quad (54)$$

For given K and X or ξ or ϕ we now calculate η by

$$\eta=\eta(10)-\int_\phi^1 (F+3K)^{-1/3}X\xi\vartheta^{-1}(\ln 10)d\phi \quad (55)$$

with $\eta(10)$ from Eq. (34).

TABLE I. Calculating R_0 for chosen K. We use here the notation $\eta[\phi]=\eta(\xi)$, $\eta_0=\eta(0)=\eta[-\infty]$.

K	$-2.392\,445\,03$	0	$0.712\,632\,814$	$8.644\,843\,172$	$27.873\,343$	Eqs. used
$\eta[-6]-\eta_0$	$-0.001\,037\,0$	$0.016\,509\,6$	$0.001\,551\,8$	$0.000\,675\,6$	$0.000\,457\,3$	(48) and (50)
$\eta[1]-\eta[-6]$	$8.955\,103\,1$[a]	$11.132\,246\,2$	$10.691\,584\,0$	$9.999\,313\,0$	$9.594\,066\,2$	(55)
$\eta[1]-\eta_0$	$8.954\,066\,1$	$11.148\,755\,8$	$10.693\,135\,8$	$9.999\,988\,6$	$9.594\,523\,5$	+
$\eta[1]$	$9.999\,988\,6$	$9.999\,988\,6$	$9.999\,988\,6$	$9.999\,988\,6$	$9.999\,988\,6$	(34) and $\xi=10$
η_0	$1.045\,922\,5$	$-1.148\,767\,2$	$-0.693\,147\,2$	$-0.000\,000\,0$	$0.405\,465\,1$	–
R_0	$2.846\,022\,8$	$0.317\,027\,4$	$0.500\,000\,0$	$1.000\,000\,0$	$1.500\,000\,0$	(56)

[a]Here, use is made of Eq. (40) across the cusp at $X=1.5625$, $R=0.8962$.

For given positive K we now find the value η_0 of η at $X=1$, $\xi=0$, by calculating $[\eta(10^{-6})-\eta_0]$ from (48) (for $K=0$) or (50) [for $K\neq 0$] with $\vartheta=[1-\exp(-10^{-6})]^{1/2}$, and by subtracting the result from $\eta(10^{-6})$ as calculated by (55) with $\phi=-6$.

Finally, from η_0 we obtain

$$R_0=r_0/2m=e^{\eta_0}\,. \tag{56}$$

Some results thus obtained are listed in Table I. More results, showing R_0 as a function of K, are shown in Fig. 1. These (and following) results were obtained using a Hewlett-Packard 41C calculator with "Quad-mod" memory module added.

In Fig. 2 we show curves for $(R-X)$ against X, for the K values of Table I. This includes the negative K value mentioned at the end of Sec. V. Horizontally we use a logarithmic scale. As $R-X\rightarrow-\frac{1}{4}$ for $X\rightarrow\infty$, all curves approach each other on the right.

The cusp in the curve for $K=-2.392\,445\,03$ lies at $\phi=\log\ln\frac{25}{16}=-0.350\,385\,663\,8$. For integrating downward through this cusp, we used

$$\eta_2=\eta_1+\int_{\phi_1}^{\phi_2}(F+3K)^{-1/3}X\xi\vartheta^{-1}(\ln 10)d\phi \tag{57}$$

with for ϕ_n the successive values ..., -0.349, -0.350, -0.3503, -0.35038, -0.350385. While each next interval of integration here is smaller than the previous one, we also make the steps in the Simpson integration each time 10 times smaller. Beyond $-0.350\,385$, the integrations would start losing their accuracy, but at this point Eq. (40) gives already accurate values for η, and gives us η_c at the cusp, from η at $\phi=-0.350\,385$, by

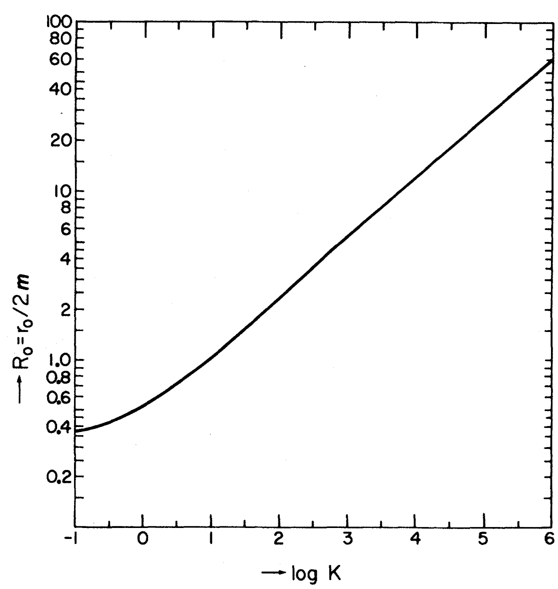

FIG. 1. $R_0=r_0/2m$ (on logarithmic scale) versus $\log K$ (on linear scale) for $0.1\leq K\leq 10^6$.

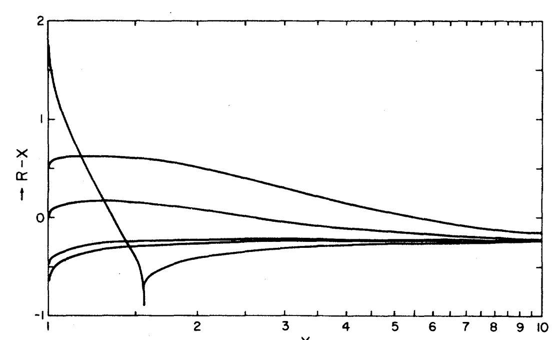

FIG. 2. $R-X=(r-r)/2m$ (on linear scale) versus $X=r/2m$ (on logarithmic scale) for $1\leq X\leq 10$, plotted for five values of K. At right from top to bottom, $K=27.873\,343$, $K=8.644\,843\,172$, $K=0.712\,632\,814$, $K=0$, and $K=-2.392\,445\,03$, corresponding to $R_0=$ 1.5, 1.0, 0.5, 0.317 027 4, and 2.846 022 8 at $X_0=1$ (at $r_0=2m$). To the left of its cusp at $X=1.5625$, the bottom curve is undesirable because of $dR/dX<0$. For large X, all curves show $R\rightarrow X-\frac{1}{4}$.

$$\eta_c = \eta[-0.350\,385] - 0.000\,113\,29$$
$$= -0.109\,543.$$

We then use (40) again for finding η at $\phi = -0.350\,386$. This gives

$$\eta[-0.350\,586] = \eta[-0.350\,385] - 0.000\,041\,306\,6\ .$$

From there, we start again using Simpson integration of (57), between successive ϕ values -0.350386, -0.35039, -0.3504, -0.351, -0.352, ..., now each time increasing the size of the Simpson integration steps. Soon we can use steps as large as those used for positive values of K, as we continue by (57) to $\phi = -6$, from where we find η_0 at $\phi = -\infty$ by using Eq. (50). (Only, $x^{\pm 1/3}$ for $x < 0$ must be computed as $-|x|^{\pm 1/3}$.)

VIII. DISCUSSION

The purpose of the coordinate condition (1) is to eliminate the longitudinal metric field. This presumes the possibility of Fourier expansion of the field, with coefficients determined by integration over space. A pure (vacuum) Schwarzschild field, as it removes from ordinary space a region inside the Schwarzschild radius, would play havoc with such integrations over space.

Therefore, the usefulness of the coordinate condition (1) would seem dubious, unless the presence of a matter distribution extending beyond the Schwarzschild radius would remove the hole in space and replace the external solution by an internal solution, where the external solution could cause trouble.

It therefore would be useful to study coordinate systems that satisfy condition (1) in the presence of a matter distribution; for instance, the static spherically symmetric incompressible perfect fluid, for which we know an internal Schwarzschild solution which at the boundary of matter continuously goes over into the external solution discussed in the present paper. The freedom in the solution presented here, by the possibility of choosing the first integration constant K, might then perhaps be useful for joining our external coordinate system satisfying (1) to an internal coordinate system satisfying (1) and behaving properly at the origin.

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