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# Gravitational repulsion in the Schwarzschild field

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To the distant observer, who uses measuring instruments not affected by gravity, gravitational repulsion can occur anywhere in the Schwarzschild field. It depends on the relationship between the transverse and radial Schwarzschild velocities. On the other hand, local observers, whose measuring instruments are affected by gravity, cannot detect a positive value for the acceleration of gravity.

## I. INTRODUCTION

Newton's theory of gravitation states that mass experiences an attractive gravitational force. Newton's theory, however, is only applicable when the gravitational fields are weak and the particle velocities are small compared to the speed of light $c$. For arbitrary fields and velocities Einstein's theory of gravitation (general relativity) is valid. As we show in the following, according to Einstein's theory, particles with certain relativistic Schwarzschild velocities are not attracted but repelled by gravitating point masses.

## II. HISTORICAL BACKGROUND

The history of gravitational repulsion is characterized by extreme controversy. Seven years after Einstein founded general relativity Bauer[1] and, independently, Hilbert[2] noticed that a massive particle in radial motion can be repulsed in the Schwarzschild field. Bauer concluded that repulsion can only take place near the Schwarzschild radius, $r=\alpha$. On the other hand McVittie[3] found that repulsion can occur anywhere in the Schwarzschild field and that it takes place if the total particle velocity is greater than $c/\sqrt{2}$. Independently, Jaffe and Shapiro[4] arrived at the same conclusions as McVittie. However, Cavalleri and Spinelli[5] strongly criticized the conclusions of Jaffe and Shapiro on the basis that they used an improper velocity, instead of the locally measured particle velocity.

Cavalleri and Spinelli also maintained that a particle's locally measured velocity at the Schwarzschild radius is $c$. This result agrees with Newtonian gravitational theory, where freely falling particles with zero initial velocity at infinity arrive at $r=\alpha$ with the speed $c$. Independently, Markley[6] found that the particle velocity at $r=\alpha$ is $c$, showing that the gravitational force is always attractive in general relativity too. Next, Zeldovich and Novikov[7] explicitly repudiated the notion of gravitational repulsion. They asserted that the decrease in the Schwarzschild velocity of a particle only occurs near $r=\alpha$ and that it can be completely attributed to the fact that the Schwarzschild time interval $dt$ differs from the locally measured time interval $dT$. Later Landau and Lifshitz[8] came to similar conclusions as Markley[6] and Zeldovich and Novikov.[7]

Opposed to the above authors, Janis,[9] without using the concept of gravitational repulsion, concluded that particles cross the Schwarzschild radius with speeds less than $c$. Cavalleri and Spinelli[10] claimed to refute Janis's results, but, Janis[11] later rejected Cavalleri and Spinelli's arguments and reaffirmed his earlier results. Meanwhile, Treder and Firtze,[12] without mentioning the other authors, confirmed the results of Bauer and Hilbert. They considered radial motion only and emphasized that repulsion takes place at a well-defined distance from the gravitating point mass.

From the historical background we see that the answers to four interrelated questions are in dispute: 1. Can gravitational repulsion take place in the Schwarzschild field? If it can: 2. Does it only occur for distances near the Schwarzschild radius? 3. Is it attributable solely to the difference between the Schwarzschild time interval $dt$ and the locally measured time interval $dT$? 4. Is it dependent on the radial velocity only or on the total particle velocity?

 

It is curious that nobody to date has considered the relationship between gravitational acceleration and the components of the total velocity. This point is crucial because it is well known that particles in transverse (nonradial) motion are deflected toward, that is attracted by, a large central body.[13] In the following we prove that gravitational repulsion actually occurs in the Schwarzschild field and we obtain the conditions for massive particles in arbitrary radial and transverse motion to experience it. We accomplish this by studying the velocity dependency of the Schwarzschild acceleration of gravity.

## III. SCHWARZSCHILD ACCELERATION

We now derive the Schwarzschild radial acceleration $a_S$ from the radial equation of motion. In spherical coordinates ($x^0=t, x^1=r, x^2=\theta$, $x^3=\phi$) the Schwarzschild line element $ds$ is

$$ds^2=B\,dt^2-A\,dr^2-r^2d\theta^2-r^2\sin^2\theta\,d\phi^2\,, \quad (1)$$

where

$$A=(1-\alpha/r)^{-1}\,, \quad (2)$$

$$B=(1-\alpha/r)\,, \quad (3)$$

$c=1$, and $\alpha=2GM$. The radial equation of motion[14] for a particle in the Schwarzschild field is

$$\frac{d^2r}{dp^2}+\frac{A'}{2A}\left[\frac{dr}{dp}\right]^2-\frac{r}{A}\left[\frac{d\theta}{dp}\right]^2-\frac{r\sin^2\theta}{A}\left[\frac{d\phi}{dp}\right]^2+\frac{B'}{2A}\left[\frac{dt}{dp}\right]^2=0\,, \quad (4)$$

where a prime denotes differentiation with respect to $r$. We also need the relation between $dt$ and $dp$, where $p$ is a parameter describing the particle trajectory. It is[14]

$$dt/dp=1/B\,. \quad (5)$$

To obtain the Schwarzschild radial acceleration $a_S$ as a function of the Schwarzschild radial velocity $\dot{r}=dr/dt$ and the Schwarzschild transverse velocity $v_S=(r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)^{1/2}$, we first insert Eq. (5) in the identities

$$dx^i/dp=(dx^i/dt)(dt/dp)\,, \quad i=1,2,3\,. \quad (6)$$

Now, combining Eqs. (6) and (4) and rearranging yields

$$\ddot{r}-\frac{r\dot{\theta}^2}{A}-\frac{r\dot{\phi}^2\sin^2\theta}{A}=\left[\frac{B'}{B}-\frac{A'}{2A}\right]\dot{r}^2-\frac{B'}{2A}\,. \quad (7)$$

Using Eqs. (2) and (3) in (7) and rearranging leads to the Schwarzschild radial acceleration $a_S$ where $g=GM/r^2$:

$$a_S=\ddot{r}-r\dot{\theta}^2-r\dot{\phi}^2\sin^2\theta=g\left[\frac{3\dot{r}^2}{1-\alpha/r}-2v_S{}^2-(1-\alpha/r)\right]. \quad (8)$$

For purely radial motion Eq. (8) reduces to the radial acceleration according to Treder and Fritze.[12]

Next, we discuss the Schwarzschild acceleration of gravity $a_S$. Newton's theory is valid for small particle velocities ($\dot{r}<<1$ and $v_S<<1$) in weak fields ($r>>\alpha$). Consequently, Eq. (8) becomes $a_S=-g$. In Newton's theory the acceleration of gravity is always negative, indicating gravitational attraction. In Einstein's theory, however, Eq. (8) shows that for

$$\dot{r}^2>\frac{(1-\alpha/r)^2}{3}+\tfrac{2}{3}v_S{}^2(1-\alpha/r) \quad (9)$$

we have $a_S>0$, implying gravitational repulsion.

We now turn to consider how each component of the velocity influences the Schwarzschild acceleration of gravity. For transverse motion ($\dot{r}=0$) Eq. (8) becomes

$$a_S=\ddot{r}-r\dot{\theta}^2-r\dot{\phi}^2\sin^2\theta=-g\,[2v_S{}^2+(1-\alpha/r)]\,. \quad (10)$$

Equation (10) shows us that transverse particle motion does not lead to gravitational repulsion. We note that the acceleration of gravity for highly relativistic particles ($v_S\approx 1$) in weak fields is

$$a_S=-3g\,. \quad (11)$$

For purely radial motion ($\dot{\theta}=\dot{\phi}=0$) Eq. (8) yields

$$a_S=\ddot{r}=g\left[\frac{3\dot{r}^2}{1-\alpha/r}-(1-\alpha/r)\right]. \quad (12)$$

We see that the gravitational acceleration of extreme relativistic particles in weak fields is

$$a_S = +2g \ . \tag{13}$$

In general, the acceleration of gravity is positive if

$$\dot{r} > \frac{1-\alpha/r}{\sqrt{3}} \ . \tag{14}$$

Inequality (14) was first derived by Hilbert[2] and later confirmed by Treder.[15] It tells us that for $\dot{r} > 1/\sqrt{3}$ a particle at every point in the Schwarzschild field is repelled.

## IV. LOCALLY MEASURED ACCELERATIONS

Schwarzschild coordinates, velocities, and accelerations refer to quantities measured by a distant observer who is not located in the gravitational field, whereas locally measured quantities are determined by observers situated in the gravitational field. Thus, the measuring instruments of the distant observer are not affected by gravity, while the instruments of the local observer are affected by the gravitational field, because the measured time and radial space intervals differ from the corresponding Schwarzschild quantities. Consequently, it is to be expected that locally measured velocities and accelerations will not be the same as the Schwarzschild quantities. We now turn to the derivation of locally measured radial accelerations.

First we also consider the radial equation of motion, Eq. (4), which can be written

$$\begin{aligned} a_p &= \frac{d^2r}{dp^2} - r\left[\frac{d\theta}{dp}\right]^2 - r\sin^2\theta\left[\frac{d\phi}{dp}\right]^2 \\ &= g\left[\left[\frac{dr^2}{dp^2} - 1\right](1-\alpha/r)^{-1} - 2v_p{}^2\right] , \end{aligned} \tag{15}$$

where

$$v_p{}^2 = r^2\left[\frac{d\theta}{dp}\right]^2 + r^2\sin^2\theta\left[\frac{d\phi}{dp}\right]^2 \ .$$

Now, $dr/dp < 1$; consequently, it follows from Eq. (15) that $a_p < 0$. We see that $a_p$ does not indicate the existence of gravitational repulsion. But, one must realize that $dp$ is measured by local observers. It is to within a constant multiple the proper time measured along the particle's trajectory. Finally, we note that as $r \to \infty$, we have $dp \to dt$, and for highly relativistic particles in transverse motion ($dr/dp \approx 0$ and $v_p \approx 1$) we obtain $a_p = a_S = -3g$.

Next, we study the totally locally measured radial acceleration determined completely by meter sticks and clocks, which are situated in and affected by the gravitational field. We rewrite the line element Eq. (1):

$$ds^2 = dT^2 - dR^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 \ , \tag{16}$$

where

$$dT^2 = B\, dt^2 \ , \tag{17}$$

$$dR^2 = A\, dr^2 \ . \tag{18}$$

$dT$ and $dR$ are the locally measured time and space (radial) intervals. The locally measured radial velocity is $dR/dT$. From Eqs. (2) and (3) we see that $A = 1/B$; consequently

$$dR/dT = \sqrt{A}\, dr/\sqrt{B}\, dt = \dot{r}/B \ . \tag{19}$$

Equation (19) tells us that $dR/dT > \dot{r}$.

Now, to find the totally locally measured radial acceleration we must return to the radial equation of motion, Eq. (15). First, we must express $dr/dp$ and $d^2r/dp^2$ in terms of the locally measured radial velocity $dR/dT$ and the locally measured radial acceleration $d^2R/dT^2$. Equations (5), (17), and (18) lead to

$$dr/dp = dR/dT \tag{20}$$

and

$$d^2r/dp^2 = (1/B^{1/2})(d^2R/dT^2) \ . \tag{21}$$

Thus, the totally locally measured radial velocity $dR/dT$ is equal to $dr/dp$. Now, we place equations (17), (20), and (21) into (15) to obtain

$$\begin{aligned} \frac{d^2R}{dT^2} - \left[r\left[\frac{d\theta}{dT}\right]^2 + r\sin^2\theta\left[\frac{d\phi}{dT}\right]^2\right](1-\alpha/r)^{1/2} \\ = g\left[\left[\frac{dR}{dT}\right]^2 - 1\right](1-\alpha/r)^{-1/2} \ ; \end{aligned} \tag{22}$$

or with

$$v_l = [r^2(d\theta/dT)^2 + r^2\sin^2\theta(d\phi/dT)^2]^{1/2} \ ,$$

the totally locally measured transverse velocity, we have

$$\begin{aligned} a_l &= \frac{d^2R}{dT^2} - r\left[\frac{d\theta}{dT}\right]^2 - r\sin^2\theta\left[\frac{d\phi}{dT}\right]^2 \\ &= g\left[\left[\frac{dR}{dT}\right]^2 - v_l{}^2 - 1\right] + O\left[\frac{1}{r^3}\right] . \end{aligned} \tag{22a}$$

Equation (22a) tells us that $a_l < 0$ because for massive particles $dR/dT < 1$. Thus, local observers cannot detect gravitational repulsion, apparently because their measuring instruments are affected by gravity, whereas distant observers, whose instruments are not affected by the gravitational field, can measure a positive value for the acceleration of gravity. Equation (22a) also tells us that for highly relativistic particles in transverse motion ($v_l \approx 1$ and $dR/dT \approx 0$) in weak fields we have $a_l = -2g$.

## V. CONCLUSIONS

We now answer the questions posed in Sec. II. In Sec. III we proved that if the Schwarzschild velocities obey inequality (9), then the Schwarzschild acceleration of gravity is positive, whereas in Sec. IV we proved that the locally measured gravitational acceleration can never be positive. We conclude that gravitational repulsion can occur in the Schwarzschild field; but, it can only be detected by an observer whose meter sticks and clocks are not affected by gravity.

Next, we consider the Schwarzschild radial acceleration $a_S$ as $r \to \infty$. Equation (8) reduces to

$$a_S = g(3\dot{r}^2 - 2v_S^2 - 1) \,. \tag{23}$$

Thus, even if the fields are very weak the Schwarzschild acceleration of gravity can be positive if

$$\dot{r}^2 > \tfrac{1}{3} + 2v_S^2/3 \tag{24}$$

is satisfied. We conclude that gravitational repulsion can occur at distances which are not near the Schwarzschild radius.

A positive value for the Schwarzschild acceleration of gravity cannot be completely attributed to the fact that locally measured time intervals $dT$ differ from the Schwarzschild time intervals $dt$ as maintained by Zeldovich and Novikov.[7] Equation (7) shows that it is the radial velocity term which is responsible for the occurrence of gravitational repulsion. That is, it may occur if

$$\left[\frac{B'}{B} - \frac{A'}{2A}\right]\dot{r}^2 - \frac{B'}{2A} > 0 \,. \tag{25}$$

Now, $dT^2 = B\,dt^2$; consequently to prove our assertion, we simply place $B=1$ into equation (25). From equation (2) we see that $A' < 0$, meaning inequality (25) is still satisfied if $\dot{r} > 0$. Thus, the Schwarzschild acceleration can be positive even if $dT$ does not differ from $dt$. We conclude that gravitational repulsion cannot come about merely because in the Schwarzschild field $dT$ differs from $dt$.

Finally, we point out that it is not possible to correlate gravitational repulsion with the total particle velocity as McVittie[3] and Jaffe and Shapiro[4] claim, because the transverse and radial velocities affect the acceleration of gravity differently (Sec. III). We conclude that gravitational repulsion is not a function of the total particle velocity or energy; rather, its occurrence depends on the relationship between the transverse and the radial velocity.

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[1]H. Bauer, *Mathematische Einführung in die Gravitationstheorie Einsteins* (Leipzig, 1922).

[2]D. Hilbert, Math. Ann. 92, 1 (1922).

[3]G. C. McVittie, *General Relativity and Cosmology* (Chapman and Hall, London, 1956).

[4]J. Jaffe, and I. I. Shapiro, Phys. Rev. D 6, 405 (1972).

[5]G. Cavalleri and G. Spinelli, Lettt. Nuovo Cimento 6, 5 (1973).

[6]F. Markley, Am. J. Phys. 41, 45 (1973).

[7]Ya. B. Zeldovich and I. D. Novikov, *Relativistic Astrophysics* (University of Chicago Press, Chicago, 1971), Vol. 1.

[8]L. Landau and E. Lifshitz, *The Classical Theory of Fields,* 3rd ed. (Addison-Wesley, Reading, Mass., 1971).

[9]A. I. Janis, Phys. Rev. D 8, 2360 (1973).

[10]G. Cavalleri and G. Spinelli, Phys. Rev. D 15, 3065 (1977).

[11]A. I. Janis, Phys. Rev. D 15, 3068 (1977).

[12]H. J. Treder and K. Fritze, Astron. Nachr. 296, 109 (1975).

[13]M. P. Silvermann, Am. J. Phys. 48, 72 (1980).

[14]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[15]H. J. Treder, *Die Relativität der Trägheit* (Akademie, Berlin, 1972).