Gravitational perturbation of the hydrogen spectrum

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(Received 18 December 1981)

The perturbations of the energy levels of a freely falling one-electron atom in an arbitrary external gravitational field are considered. The energy-level shifts are calculated to first order in the Riemann tensor for the relativistic $2P_{3/2}$ levels and the nonrelativistic $3S$, $3P$, and $3D$ levels. These and earlier results are evaluated explicitly for atoms in radial orbits of the Schwarzschild and parametrized post-Newtonian metrics and for atoms in circular orbits of the Schwarzschild metric. Highly excited Rydberg atoms are also discussed.

I. INTRODUCTION

The general theory of relativity and other metric theories predict that gravitation is manifested as a curvature of spacetime. This curvature is characterized by the Riemann tensor $R^{\alpha}_{\ \beta\gamma\delta}$. It is of interest to know how the curvature of spacetime at the position of an atom affects its spectrum. Frequency shifts caused by local curvature are different for various spectral lines and are thus distinguishable from the familiar Doppler, gravitational, and cosmological red-shifts.

One can, in principle, use the atom as an instrument to detect possible regions of high curvature. Because the information about the curvature at the atom's location is carried by its electromagnetic spectrum, regions at large distances from us can be explored. In a recent experiment,[1] the wavelengths of the $2s$-$3p$ transitions in hydrogen and deuterium were measured to a precision of one part in 10^9. The magnitude of the energy-level perturbations produced in a freely falling atom by spacetime curvature increases as n^4, where n is the principal quantum number (see Appendix B). Highly excited or Rydberg atoms with $n \approx 10^2$ are studied in the laboratory and atoms with $n \approx 350$ are observed by radio astronomers. The characteristic radius of curvature D necessary to produce a wavelength shift of one part in 10^9 in a transition between two adjacent energy levels of hydrogen near $n = 100$ is $D \lesssim 0.7$ km. Near $n = 350$, a shift in wavelength of one part in 10^9 requires a radius of curvature of $D \lesssim 30$ km. For comparison, the characteristic radius of curvature of the spacetime near the surface of a typical neutron star is roughly $D \approx 30$ km. To observe the wavelength shift for atoms with $n = 350$ near a black hole of one solar mass would require an accuracy of about one part in 10^7.

For sources such as white dwarfs and neutron stars, it would be necessary to distinguish the gravitational perturbations of the spectrum from the perturbations produced by the electromagetic fields present near such sources. The results in Sec. VI show that in the Schwarzschild geometry, the level spacing of the gravitational effect is different from that of the well-known first-order (degenerate) Stark and Zeeman effects. One can show that the higher-order electromagnetic perturbations also produce patterns of energy-level shifts which are different from the lowest-order gravitational perturbations. Therefore, in principle, it would be possible to separate the electromagnetic and gravitational perturbations of the spectrum. Near a black hole, that problem may not arise because its electric and magnetic fields are both proportional to the black hole's electric charge, and a black hole in space is expected to be essentially uncharged.

One can also envision the possibility of using atomic spectra to detect gravitational waves near their sources. The frequencies of most sources of gravitational waves are small with respect to atomic frequencies. Therefore, the quasistatic approximation is valid and the results obtained here and in earlier papers[2–4] can be used, with $R_{\alpha\beta\gamma\delta}$ regarded as a function of time which oscillates at the frequency of the gravitational-wave background. The spectral-line shifts would oscillate at the frequency of the gravitational waves impinging on the atom. One might hope to observe such an effect in the absorption spectrum of cold gas (the sources of electromagnetic waves and of gravitational waves need not be the same). Because the atoms may be relatively close to the source of gravitational waves, the amplitude of those waves at the position of the atom can be much larger than at the Earth. Unambiguous information concerning the measurement of the wave by the atom would be transmitted in the atom's electromagnetic spectrum. (With T. Leen, we are currently investigating this topic.)

We use the notation and conventions of Ref. 3. In that reference, the Dirac equation in curved spacetime was reviewed, the Dirac Hamiltonian was evaluated in Fermi normal coordinates to first order in the Riemann curvature tensor of an arbitrary spacetime, the energy shifts of the relativistic $1S_{1/2}$, $2S_{1/2}$, and $2P_{1/2}$ levels were calculated, the nonrelativistic limit was discussed, and the shifts of the nonrelativistic $1S$, $2S$, and $2P$ levels were calculated. The hydrogen atom in certain cosmological spacetimes was considered by Audretsch and Schäfer,[5] who also discussed the previous literature. The mixing of opposite-parity states of an atom supported in a gravitational field was investigated by Fischbach, Freeman, and Cheng.[6] They showed that the separation of center-of-mass and relative coordinates is a complicated problem which in weak gravitational fields can evidently introduce corrections of the same magnitude as the gravitational perturbations. Atoms at rest in the Schwarzschild and Robertson-Walker metrices were also considered by Tourrenc and Grossiord.[7] In the present paper, we deal only with freely falling atoms. In a later paper, we will also discuss accelerated or supported atoms. We have not attempted a fully relativistic separation of center-of-mass and relative coordinates, which would introduce corrections beyond the use of the reduced mass for the mass of the electron.

Except for the perturbations of the nonrelativistic $2P$ levels, the energy-level shifts obtained in Ref. 3 vanish when $R_{\mu\nu}$ is zero (a vacuum spacetime in general relativity). Here we calculate the energy-level shifts of the relativistic $2P_{3/2}$ levels (Sec. II) and of the non-relativistic $3S$, $3P$, and $3D$ levels (Sec. III). All of these shifts involve the uncontracted Riemann tensor and need not vanish when $R_{\mu\nu}$ is zero. For the convenience of the reader, the energy-level shifts calculated in Ref. 3 and in the present paper are summarized in Sec. IV. The general expressions for these energy-level shifts are evaluated explicitly for atoms in circular and radial orbits in the Schwarzschild metric (Secs. V and VI). Finally, the energy shifts are evaluated for an atom falling radially in the spherically symmetric, lowest-order, parametrized post-Newtonian (PPN) metric (Appendix A). The results in that case depend on the initial velocity of the falling atom and on the PPN parameter γ. The order of magnitude of the energy-level perturbations for highly excited or Rydberg atoms is obtained in Appendix B.

II. RELATIVISTIC CALCULATIONS

In this section we calculate the energy shifts of the $2P_{3/2}$ states of the one-electron atom caused by an external gravitational field. For $n=2$ and $J=3/2$, ther are four eigenstates of the unperturbed Hamiltonian H_0 [Eq. (8.14) of Ref. 3], corresponding to the energy eigenvalue[8]

$$E_{2,3/2}=m[1+\zeta^2(4-\zeta^2)^{-1}]^{-1/2}\,, \tag{2.1}$$

with

$$\zeta=Ze^2 \tag{2.2}$$

(we use units with $\hbar=c=1$). The eigenstates are

$$\psi^M=\frac{1}{r}\begin{bmatrix} \frac{1}{\sqrt{3}}F(r)(3/2+M)^{1/2}Y_1^{M-1/2}(\theta,\phi) \\ \frac{1}{\sqrt{3}}F(r)(3/2-M)^{1/2}Y_1^{M+1/2}(\theta,\phi) \\ \frac{-i}{\sqrt{5}}G(r)(5/2-M)^{1/2}Y_2^{M-1/2}(\theta,\phi) \\ \frac{i}{\sqrt{5}}G(r)(5/2+M)^{1/2}Y_2^{M+1/2}(\theta,\phi) \end{bmatrix}, \tag{2.3}$$

with $M=\pm\frac{1}{2},\ \pm\frac{3}{2}$. (Note that the coefficient of the undefined Y_1^2 vanishes.) $F(r)$ and $G(r)$ are given by

$$F(r)=m^{1/2}N(1+w)^{1/2}(mr)^\gamma\exp(-\lambda mr)\,, \tag{2.4}$$

$$G(r)=-m^{1/2}N(1-w)^{1/2}(mr)^\gamma\exp(-\lambda mr)\,, \tag{2.5}$$

where

$$\gamma=(4-\zeta^2)^{1/2},\ \ w=\gamma/2,\ \ \lambda=\zeta/2\,, \tag{2.6}$$

and

$$N=\frac{\zeta^{\gamma+1/2}}{[2\Gamma(2\gamma+1)]^{1/2}}\,. \tag{2.7}$$

Then

$$G(r)=-\left[\frac{1-w}{1+w}\right]^{1/2}F\approx O(\zeta^2)F\,. \tag{2.8}$$

Therefore we can neglect G with respect to F for the purpose of calculating the energy shifts to lowest order in ζ.

The gravitational perturbation of the Hamiltonian to first order in the Riemann tensor is given in Ref. 3, Eq. (8.15); the largest term is

$$H_I = \frac{m}{2} R_{0l0m} x^l x^m \beta \ . \tag{2.9}$$

The energy shifts are determined by Eq. (5.15) of Ref. 3. Thus we need the matrix elements

$$H_{MM'} \equiv (\psi^M, H_I \psi^{M'}) \ . \tag{2.10}$$

Using the eigenfunctions (2.3) and neglecting G, the matrix elements reduce to

$$H_{MM'} = \frac{m}{2} R_{0l0m} \int dV \frac{x^l x^m}{3r^2} F^2(r) [(\tfrac{3}{2}+M)^{1/2} (\tfrac{3}{2}+M')^{1/2} Y_1^{M-1/2*} Y_1^{M'-1/2} + (\tfrac{3}{2}-M)^{1/2} (\tfrac{3}{2}-M')^{1/2} Y_1^{M+1/2*} Y_1^{M'+1/2}] \ . \tag{2.11}$$

Here R_{0l0m} is evaluated at the center of mass of the atom in a locally inertial proper frame with the spatial axes oriented along the Riemann tensor's principal directions. The product $x^l x^m$ can be expressed in terms of spherical harmonics. Therefore the angular part of (2.11) is the integral of three spherical harmonics and its value is obtained in terms of the Clebsch-Gordan coefficients or the 3-J symbols.[9] The radial part is also readily calculated. The matrix elements are given by

$$H_{-3/2,-3/2} = H_{3/2,3/2} = \zeta^{-2} m^{-1} (6R_{00} - 3R_{0z0z}) \ , \tag{2.12}$$

$$H_{-3/2,1/2} = H_{-1/2,3/2} = -\sqrt{3} \zeta^{-2} m^{-1} (R_{0x0x} - R_{0y0y}) \ , \tag{2.13}$$

$$H_{-1/2,-1/2} = H_{1/2,1/2} = \zeta^{-2} m^{-1} (4R_{00} + 3R_{0z0z}) \ . \tag{2.14}$$

The other matrix elements are obtained by symmetry,

$$H_{MM'} = H_{M'M} \ . \tag{2.15}$$

Now we can substitute Eqs. (2.12)–(2.15) into Eq. (5.15) of Ref. 3,

$$\det[H_{MM'} - E^{(1)} \delta_{MM'}] = 0 \tag{2.16}$$

to obtain the secular equation. Defining the following quantities,

$$p = \zeta^{-2} m^{-1} \ , \tag{2.17}$$

$$q = p(R_{00} - 3R_{0z0z}) \ , \tag{2.18}$$

$$s = -\sqrt{3} p (R_{0x0x} - R_{0y0y}) \ , \tag{2.19}$$

one finds that the secular equation is given by

$$(4q + 15pR_{0z0z} - E^{(1)})^2 (6q + 15pR_{0z0z} - E^{(1)})^2 - 2s^2 (6q + 15pR_{0z0z} - E^{(1)})(4q + 15pR_{0z0z} - E^{(1)}) + s^4 = 0 \ . \tag{2.20}$$

With the change of variable

$$\lambda = 5q + 15pR_{0z0z} - E^{(1)} \ , \tag{2.21}$$

Eq. (2.20) reduces to a quadratic equation in λ^2:

$$(\lambda - q^2)^2 - 2s^2 (\lambda^2 - q^2) + s^4 = 0 \ . \tag{2.22}$$

Therefore the solution to the fourth-degree secular equation is easily obtained. There are two double roots,

$$E_1^{(1)}(2P_{3/2}) = \zeta^{-2} m^{-1} \{ 5R_{00} + 2[R_{0x0x}{}^2 + R_{0y0y}{}^2 + R_{0z0z}{}^2 - (R_{0x0x} R_{0y0y} + R_{0x0x} R_{0z0z} + R_{0y0y} R_{0z0z})]^{1/2} \} \tag{2.23}$$

and

$$E_2^{(1)}(2P_{3/2})=\zeta^{-2}m^{-1}\{5R_{00}-2[R_{0x0x}{}^2+R_{0y0y}{}^2+R_{0z0z}{}^2-(R_{0x0x}R_{0y0y}+R_{0x0x}R_{0z0z}+R_{0y0y}R_{0z0z})]^{1/2}\}\ . \quad (2.24)$$

It is interesting that the average of these two energy shifts is the same as the energy shift $E^{(1)}(2P_{1/2})$ of the $2P_{1/2}$ levels [Eq. (11.22) of Ref. 3].

III. NONRELATIVISTIC CALCULATIONS

Here we calculate the perturbation to the energy of the third level of the nonrelativistic one-electron atom in the presence of an arbitrary gravitational field. The gravitational interaction obtained from Eq. (9.13) of Ref. 3 is

$$H_I=\tfrac{1}{2}mR_{0l0m}x^lx^m\ . \quad (3.1)$$

In the nonrelativistic regime one can calculate the matrix elements of the perturbation (3.1) between the nonrelativistic hydrogenic wave functions, using again a frame where R_{0l0m} is diagonal. Writing the third-level states $|3lm\rangle$ as $|lm\rangle$ for brevity, we find that the nonvanishing matrix elements involving the third-level eigenstates are

$$\langle 00|H_I|00\rangle=\frac{69}{2}\zeta^{-2}m^{-1}R_{00}\ , \quad (3.2)$$

$$\langle 00|H_I|2-2\rangle=\langle 00|H_I|22\rangle=\frac{15\sqrt{3}}{2}\zeta^{-2}m^{-1}(R_{0x0x}-R_{0y0y})\ , \quad (3.3)$$

$$\langle 00|H_I|20\rangle=-\frac{15\sqrt{2}}{2}\zeta^{-2}m^{-1}(R_{00}-3R_{0z0z})\ , \quad (3.4)$$

$$\langle 1-1|H_I|1-1\rangle=\langle 11|H_I|11\rangle=18\zeta^{-2}m^{-1}(2R_{00}-R_{0z0z})\ , \quad (3.5)$$

$$\langle 1-1|H_I|11\rangle=-18\zeta^{-2}m^{-1}(R_{0x0x}-R_{0y0y})\ , \quad (3.6)$$

$$\langle 10|H_I|10\rangle=18\zeta^{-2}m^{-1}(R_{00}+2R_{0z0z})\ , \quad (3.7)$$

$$\langle 2-2|H_I|2-2\rangle=\langle 22|H_I|22\rangle=9\zeta^{-2}m^{-1}(3R_{00}-2R_{0z0z})\ , \quad (3.8)$$

$$\langle 2-2|H_I|20\rangle=\langle 20|H_I|22\rangle=-3\sqrt{6}\zeta^{-2}m^{-1}(R_{0x0x}-R_{0y0y})\ , \quad (3.9)$$

$$\langle 2-1|H_I|2-1\rangle=\langle 21|H_I|21\rangle=9\zeta^{-2}m^{-1}(2R_{00}+R_{0z0z})\ , \quad (3.10)$$

$$\langle 2-1|H_I|21\rangle=-9\zeta^{-2}m^{-1}(R_{0x0x}-R_{0y0y})\ , \quad (3.11)$$

$$\langle 20|H_I|20\rangle=3\zeta^{-2}m^{-1}(5R_{00}+6R_{0z0z})\ . \quad (3.12)$$

The other matrix elements are obtained by symmetry.

$$\langle l'm'|H_I|lm\rangle=\langle lm|H_I|l'm'\rangle\ . \quad (3.13)$$

The secular equation

$$\det[\langle lm|H_I|l'm'\rangle-E^{(1)}\delta_{ll'}\delta_{mm'}]=0 \quad (3.14)$$

can be expressed compactly in terms of the following quantities:

$$A=\tfrac{69}{2}\zeta^{-2}m^{-1}R_{00}-E^{(1)}\ , \quad (3.15)$$

$$B=\frac{15\sqrt{3}}{2}\zeta^{-2}m^{-1}(R_{0x0x}-R_{0y0y})\ , \quad (3.16)$$

$$C=\frac{-15\sqrt{2}}{2}\zeta^{-2}m^{-1}(R_{00}-3R_{0z0z})\ , \quad (3.17)$$

$$D=18\zeta^{-2}m^{-1}(2R_{00}-R_{0z0z})-E^{(1)}\ , \quad (3.18)$$

$$E=18\zeta^{-2}m^{-1}(R_{00}+2R_{0z0z})-E^{(1)}\ , \quad (3.19)$$

$$G=9\zeta^{-2}m^{-1}(3R_{00}-2R_{0z0z})-E^{(1)}\ , \quad (3.20)$$

$$H=9\zeta^{-2}m^{-1}(2R_{00}+R_{0z0z})-E^{(1)}\ , \quad (3.21)$$

$$J=3\zeta^{-2}m^{-1}(5R_{00}+6R_{0z0z})-E^{(1)}\ . \quad (3.22)$$

With these definitions the secular equation is given by

$$EG(D^2-\tfrac{48}{25}B^2)(H^2-\tfrac{12}{25}B^2)\times[B^2(\tfrac{16}{25}A+2J+\tfrac{8}{5}\sqrt{2}C)+G(C^2-AJ)]=0\ . \tag{3.23}$$

This ninth-degree equation already contains factors of lower degree, which makes it possible to obtain its roots. The first six roots are

$$E_1^{(1)}=18\zeta^{-2}m^{-1}(R_{00}+2R_{0x0x})\ , \tag{3.24}$$

$$E_2^{(1)}=18\zeta^{-2}m^{-1}(R_{00}+2R_{0y0y})\ , \tag{3.25}$$

$$E_3^{(1)}=18\zeta^{-2}m^{-1}(R_{00}+2R_{0z0z})\ , \tag{3.26}$$

$$E_4^{(1)}=9\zeta^{-2}m^{-1}(3R_{00}-2R_{0x0x})\ , \tag{3.27}$$

$$E_5^{(1)}=9\zeta^{-2}m^{-1}(3R_{00}-2R_{0y0y})\ , \tag{3.28}$$

$$E_6^{(1)}=9\zeta^{-2}m^{-2}(3R_{00}-2R_{0z0z})\ . \tag{3.29}$$

In last three energy shifts are the roots of the third-degree equation

$$(E^{(1)})^3+a_1(E^{(1)})^2+a_2E^{(1)}+a_3=0\ , \tag{3.30}$$

with

$$a_1=-\tfrac{153}{2}\zeta^{-2}m^{-1}R_{00}\ , \tag{3.31}$$

$$a_2=\zeta^{-4}m^{-2}[1296R_{00}{}^2+1782(R_{0x0x}R_{0y0y}+R_{0x0x}R_{0z0z}+R_{0y0y}R_{0z0z})]\ , \tag{3.32}$$

$$a_3=\zeta^{-6}m^{-3}\left[-\frac{12\,393}{2}R_{00}{}^3-18\,954(R_{0x0x}{}^2R_{0y0y}+R_{0x0x}R_{0y0y}{}^2+R_{0x0x}{}^2R_{0z0z}+R_{0y0y}{}^2R_{0z0z}+R_{0x0x}R_{0z0z}{}^2+R_{0y0y}R_{0z0z}{}^2)-129\,762R_{0x0x}R_{0y0y}R_{0z0z}\right]\ . \tag{3.33}$$

This equation can be solved for any given Riemann tensor. We will not take the space here to write the general form of its roots. Even without solving Eq. (3.30) explicitly, it can be seen that the sum of the three roots is given by

$$E_7^{(1)}+E_8^{(1)}+E_9^{(1)}=\tfrac{153}{2}\zeta^{-2}m^{-1}R_{00}\ . \tag{3.34}$$

IV. SUMMARY OF RESULTS

For the convenience of the reader, we here summarize the energy shifts given in Ref. 3 and those obtained in this paper.

Whenever an energy shift can be expressed in the form

$$E^{(1)}=AR_{00}+BR+\sum_{i=1}^{3}C^{ii}R_{0i0i}\ , \tag{4.1}$$

we will just give the nonvanishing constants A, B, and C^{ii}. Here the curvature quantities R, R_{00}, and R_{i0i0} are evaluated at the center of mass of the atom in a locally inertial proper frame with the spatial axes oriented along the principal directions of R_{0i0j}. (Since we are assuming that $R_{\alpha\beta\gamma\delta}$ is quasistatic in Fermi coordinates, remaining nearly constant during the time of an atomic transition, the fact that the orthonormal tetrad is parallel transported along the geodesic does not prevent us from orienting the spatial axes along the principal directions of R_{0i0j}.) We first give the nonrelativistic results, and then the relativistic results. Nonrelativistic regime:

$$1S \text{ level: } A=\tfrac{1}{2}\zeta^{-2}m^{-1}\ , \tag{4.2}$$

$$2S \text{ level: } A=7\zeta^{-2}m^{-1}\ , \tag{4.3}$$

$2P$ levels:

$$(1)\ A=18\zeta^{-2}m^{-1},\ C^{11}=6\zeta^{-2}m^{-1}\ , \tag{4.4}$$

$$(2)\ A=3\zeta^{-2}m^{-1},\ C^{22}=6\zeta^{-2}m^{-1}\ , \tag{4.5}$$

$$(3)\ A=3\zeta^{-2}m^{-1},\ C^{33}=6\zeta^{-2}m^{-1}\ , \tag{4.6}$$

$3S$, $3P$, and $3D$ levels:

$$(1)\ A=18\zeta^{-2}m^{-1},\ C^{11}=36\zeta^{-2}m^{-1}\ , \tag{4.7}$$

$$(2)\ A=18\zeta^{-2}m^{-1},\ C^{22}=36\zeta^{-2}m^{-1}\ , \tag{4.8}$$

$$(3)\ A=18\zeta^{-2}m^{-1},\ C^{33}=36\zeta^{-2}m^{-1}\ , \tag{4.9}$$

$$(4)\ A=27\zeta^{-2}m^{-1},\ C^{11}=-18\zeta^{-2}m^{-1}\ , \tag{4.10}$$

$$(5)\ A=27\zeta^{-2}m^{-1},\ C^{22}=-18\zeta^{-2}m^{-1}\ , \tag{4.11}$$

$$(6)\ A=27\zeta^{-2}m^{-1},\ C^{33}=-18\zeta^{-2}m^{-1}\ , \tag{4.12}$$

where $\zeta=Ze^2$. The remaining three energy shifts of the $3S$, $3P$, and $3D$ levels are the roots of the third-degree equation (3.30), with coefficients given by (3.31)–(3.33). Relativisitic regime:

$$\begin{aligned}1S_{1/2}\text{ levels: }&A=\tfrac{1}{12}\zeta^{-2}\gamma(\gamma+1)(2\gamma+1)m^{-1}+\tfrac{1}{18}(2\gamma+1)m^{-1}\ ,\\ &B=\tfrac{1}{24}(2\gamma+1)m^{-1}\ ,\end{aligned} \tag{4.13}$$

$$2S_{1/2}\text{ levels: }A=7\zeta^{-2}m^{-1}\ , \tag{4.14}$$

$$2P_{1/2}\text{ levels: }A=5\zeta^{-2}m^{-1}\ , \tag{4.15}$$

$$\begin{aligned}2P_{3/2}\text{ levels: }E_1^{(1)}=\zeta^{-2}m^{-1}\{5R_{00}+2[&R_{0x0x}{}^2+R_{0y0y}{}^2+R_{0z0z}{}^2\\ &-(R_{0x0x}R_{0y0y}+R_{0x0x}R_{0z0z}+R_{0y0y}R_{0z0z})]^{1/2}\}\ ,\end{aligned} \tag{4.16}$$

$$E_2^{(1)}=\zeta^{-2}m^{-1}\{5R_{00}-2[R_{0x0x}{}^2+R_{0y0y}{}^2+R_{0z0z}{}^2-(R_{0x0x}R_{0y0y}+R_{0x0x}R_{0z0z}+R_{0y0y}R_{0z0z})]^{1/2}\}\ , \tag{4.17}$$

where $\zeta=Ze^2$ and $\gamma(1-\zeta^2)^{1/2}$. Equation (4.13) is good to all orders in ζ, while the other results are valid to lowest order in ζ.

V. FERMI NORMAL COORDINATES FOR RADIAL AND CIRCULAR ORBITS IN THE SCHWARZSCHILD METRIC

Fermi normal coordinates along a geodesic $\mathscr{P}(\tau)$, parametrized by the proper time τ, are characterized by a tetrad that satisfies the following conditions:[10,11]

$$(\text{i})\hat{e}_0=\frac{d\mathscr{P}(\tau)}{d\tau}\ , \tag{5.1}$$

i.e., $\hat{e}_0$ is the velocity vector of the particle on the geodesic $\mathscr{P}(\tau)$. Also

$$\hat{e}_\alpha\cdot\hat{e}_\beta=\eta_{\alpha\beta}\ . \tag{5.2}$$

(ii) The tetrad is parallel transported along the geodesic:

$$D\hat{e}_\alpha/d\tau=0\ . \tag{5.3}$$

The coordinates of an event Q near the geodesic $\mathscr{P}(\tau)$ are obtained by the following procedure.

For Q close enough to the geodesic $\mathscr{P}(\tau)$ there is a unique geodesic $\mathscr{R}(s)$ joining Q to $\mathscr{P}(\tau)$ and intersecting $\mathscr{P}(\tau)$ at a particular proper time τ. Let the tangent to $\mathscr{R}(s)$ at the point of intersection with $\mathscr{P}(\tau)$ be $\hat{n}=n^j\hat{e}_j$ and the proper distance along $\mathscr{R}(s)$ to the point Q be s. The Fermi normal coordinates of Q are given by

$$x^0(Q)=\tau\ , \tag{5.4}$$

$$x^i(Q)=n^i s \ . \tag{5.5}$$

One can now construct the Fermi normal tetrad for the radial and circular geodesics in the Schwarzschild metric.

The first integrals of motion for a particle in radial motion in the Schwarzschild metric are

$$X\dot{t}=\epsilon \ , \tag{5.6}$$

$$X\dot{t}^2-X^{-1}\dot{r}^2=1 \ , \tag{5.7}$$

where ϵ is the energy per unit mass at infinity and

$$X=1-\frac{2M}{r} \ . \tag{5.8}$$

The dot means derivative with respect to proper time. Using the above equations, one can verify that a Fermi normal basis for the radial geodesic is given by[10]

$$\hat{e}_0=\dot{t}(\tau)\frac{\partial}{\partial t}+\dot{r}(\tau)\frac{\partial}{\partial r} \ , \tag{5.9}$$

$$\hat{e}_1=X^{-1}\dot{r}(\tau)\frac{\partial}{\partial t}+X\dot{t}(\tau)\frac{\partial}{\partial r} \ , \tag{5.10}$$

$$\hat{e}_2=r^{-1}\frac{\partial}{\partial\theta} \ , \tag{5.11}$$

$$\hat{e}_3=(r\sin\theta)^{-1}\frac{\partial}{\partial\phi} \ , \tag{5.12}$$

where t,r,θ and ϕ are the Schwarzschild coordinates and τ is the proper time along the geodesic.

Given the tetrad we can calculate the components of the Riemann tensor that are necessary to obtain the energy shifts. In the Fermi frame the components of the curvature tensor are

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}=R_{\mu\nu\sigma\tau}(\hat{e}_\alpha)^\mu(\hat{e}_\beta)^\nu(\hat{e}_\gamma)^\sigma(\hat{e}_\delta)^\tau \ . \tag{5.13}$$

Here $R_{\mu\nu\sigma\tau}$ are the components of the Riemann tensor in the Schwarzschild coordinates. Using the conventions of Ref. 11, we have

$$R_{\hat{0}\hat{x}\hat{0}\hat{x}}=-2M/r^3 \ , \tag{5.14}$$

$$R_{\hat{0}\hat{y}\hat{0}\hat{y}}=R_{\hat{0}\hat{z}\hat{0}\hat{z}}=M/r^3 \ . \tag{5.15}$$

We note that in this frame $R_{\hat{0}\hat{i}\hat{0}\hat{j}}$ is diagonal for all times. This result is also given in Ref. 10.

For the circular orbits the first integrals of motion are ($\theta=\pi/2$; $r=$constant)

$$X\dot{t}=\epsilon \ , \tag{5.16}$$

$$\dot{\phi}=l/r^2 \ , \tag{5.17}$$

where ϵ is the energy per unit mass and l is the angular momentum per unit mass. These constants of motion are related by

$$X(1+l^2/r^2)=\epsilon^2 \ . \tag{5.18}$$

There are two circular orbits (one stable and one unstable) for a given angular momentum, corresponding to the extrema of the effective potential given by[11]

$$X(1+l^2/r^2) \ . \tag{5.19}$$

The radii of those orbits are the roots of the equation

$$Mr^2-l^2r+3Ml^2=0 \ . \tag{5.20}$$

For the stable orbit one has $r>6M$, and for the unstable point one has $3M<r\leq 6M$ (we are here using units in which $G,\hbar$ and c are numerically equal to unity).

A Fermi normal basis for the circular orbits is found using Eqs. (5.1)–(5.3) and Eqs. (5.16)–(5.20), with the result that

$$\hat{e}_0=\epsilon X^{-1}\frac{\partial}{\partial t}+lr^{-2}\frac{\partial}{\partial\phi} \ , \tag{5.21}$$

$$\hat{e}_1=-r^{-1}X^{-1/2}\sin(\alpha\phi)\left[l\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial\phi}\right] +X^{1/2}\cos(\alpha\phi)\frac{\partial}{\partial r} \ , \tag{5.22}$$

$$\hat{e}^2=r^{-1}\frac{\partial}{\partial\theta} \ , \tag{5.23}$$

$$\hat{e}_3=r^{-1}X^{-1/2}\cos(\alpha\phi)\left[l\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial\phi}\right] +X^{1/2}\cos(\alpha\phi)\frac{\partial}{\partial r} \ , \tag{5.24}$$

where

$$\alpha=(r-3M)^{1/2}r^{-1/2} \ . \tag{5.25}$$

Substituting Eqs. (5.21)–(5.24) into (5.13) we obtain the components of the Riemann tensor in the Fermi normal basis,

$$R_{\hat{0}\hat{i}\hat{0}\hat{j}}=Mr^{-3}(r-3M)^{-1}\{\, r\delta_i^2\delta_j^2+[(r-3M)\delta_i^3\delta_j^3-(2r-3M)\delta_i^1\delta_j^1]\cos^2\alpha\phi +[(r-3M)\delta_i^1\delta_j^1-(2r-3M)\delta_i^3\delta_j^3]\sin^2\alpha\phi-3rX(\delta_i^1\delta_j^3+\delta_j^1\delta_i^3)\sin\alpha\phi\cos\alpha\phi\,\} \ , \tag{5.26}$$

where δ^i_j is the Kronecker delta. We observe that only the last term in (5.26) is nondiagonal. It is not possible to diagonalize $R_{\hat{0}\hat{i}\hat{0}\hat{j}}$ for all times but it is diagonal for $\phi=0$, and the off-diagonal components will remain small with respect to the diagonal elements for some time interval. For the stable orbits, using (5.17) and (5.26) we find that the off-diagonal elements of (5.26) are much smaller than the diagonal elements for proper times satisfying the inequality

$$\tau \ll \frac{2r^{3/2}}{M^{1/2}}\left|\frac{r-3M}{3r-6M}\right| . \tag{5.27}$$

At $\tau=0$ ($\phi=0$) the relevant components of the Riemann tensor in the Fermi frame are

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}=-(2r-3M)(r-3M)^{-1}Mr^{-3} , \tag{5.28}$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}}=(r-3M)^{-1}Mr^{-2} , \tag{5.29}$$

$$R_{\hat{0}\hat{3}\hat{0}\hat{3}}=Mr^{-3} . \tag{5.30}$$

VI. ENERGY-LEVEL SHIFTS IN THE SCHWARZSCHILD GEOMETRY

Now we can use the calculations of the preceding section to obtain explicitly the energy shifts in the Schwarzschild spacetime. the nonrelativistic energy shifts for the atom falling on a radial orbit are obtained by substituting Eqs. (5.14) and (5.15) into Eqs. (4.2)–(4.12) and then into Eq. (4.1). The energy shifts are

$$E^{(1)}(1S)=E^{(1)}(2S)=0 , \tag{6.1}$$

$$E_1^{(1)}(2P)=-12Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.2}$$

$$E_2^{(1)}(2P)=E_3^{(1)}(2P)=6Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.3}$$

$$E_1^{(1)}(n=3)=-72Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.4}$$

$$E_2^{(1)}(n=3)+E_3^{(1)}(n=3)=E_4^{(1)}(n=3)=36Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.5}$$

$$E_5^{(1)}(n=3)=E_6^{(1)}(n=3)=-18Mr^{-3}\zeta^{-2}m^{-1} . \tag{6.6}$$

Substituting (5.14) and (5.15) into (3.31)–(3.32) gives the following coefficients for the third-degree equation (3.30):

$$a_1=0 , \tag{6.7}$$

$$a_2=-5\,346M^2r^{-6}\zeta^{-4}m^{-2} , \tag{6.8}$$

$$a_3=145\,800M^3r^{-4}\zeta^{-6}m^{-3} . \tag{6.9}$$

Solving (3.30) we obtain the last three energy shifts,

$$E_7^{(1)}(n=3)=36Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.10}$$

$$E_8^{(1)}(n=3)=(-18+\sqrt{4374})Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.11}$$

$$E_9^{(1)}(n=3)=(-18-\sqrt{4374})Mr^{-3}\zeta^{-2}m^{-1} . \tag{6.12}$$

The relativistic energy shifts are obtained by substituting (5.14) and (5.15) into (4.13)–(4.17). The energy shifts to lowest order are for an atom on a radial orbit,

$$E^{(1)}(1S_{1/2})=E^{(1)}(2S_{1/2})=E^{(1)}(2P_{1/2})=0 \tag{6.13}$$

and

$$E_1^{(1)}(2P_{3/2})=-E_2^{(1)}(2P_{3/2})=6Mr^{-3}\zeta^{-2}m^{-1} . \tag{6.14}$$

In a similar way, using now Eqs. (5.28)–(5.30) for the components of the Riemann tensor in the Fermi frame associated with the circular orbit we obtain the following nonrelativistic energy shifts:

$$E^{(1)}(1S)=E^{(1)}(2S)=0 , \tag{6.15}$$

$$E_1^{(1)}(2P)=-6(2r-3M)(r-3M)^{-1}Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.16}$$

$$E_2^{(1)}(2P)=6(r-3M)^{-1}Mr^{-2}\zeta^{-2}m^{-1} , \tag{6.17}$$

$$E_3^{(1)}(2P)=6Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.18}$$

$$E_1^{(1)}(n=3)=-36(2r-3M)(r-3M)^{-1}Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.19}$$

$$E_2^{(1)}(n=3)=-36(r-3M)^{-1}Mr^{-2}\zeta^{-2}m^{-1} , \tag{6.20}$$

$$E_3^{(1)}(n=3)=36Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.21}$$

$$E_4^{(1)}(n=3)=18(2r-3M)(r-3M)^{-1}Mr^{-3}\zeta^{-2}m^{-1} , \tag{6.22}$$

$$E_5^{(1)}(n=3)=-18(r-3M)^{-1}Mr^{-2}\zeta^{-2}m^{-1} ,\quad E_6^{(1)}(n=3)=-18Mr^{-3}\zeta^{-2}m^{-1} . \tag{6.23}$$

The coefficients of the cubic equation (3.30) are

$$a_1=0\ , \tag{6.24}$$

$$a_2=-1782\Delta M^2r^{-6}\zeta^{-4}m^{-2}\ , \tag{6.25}$$

$$a_3=72\,900(\Delta-1)M^3r^{-9}\zeta^{-6}m^{-3}\ , \tag{6.26}$$

where

$$\Delta=3(r^2-3Mr+3M^2)(r-3M)^{-2}\ . \tag{6.27}$$

Solving (3.30) with coefficients (6.24)–(6.26) gives the remaining energy shifts,

$$E_7^{(1)}(n=3)=2(594\Delta)^{1/2}Mr^{-3}\zeta^{-2}m^{-1}\cos\theta\ , \tag{6.28}$$

$$E_8^{(1)}(n=3)=2(594\Delta)^{1/2}Mr^{-3}\zeta^{-2}m^{-1}\cos(\theta+2\pi/3)\ , \tag{6.29}$$

$$E_9^{(1)}(n=3)=2(594\Delta)^{1/2}Mr^{-3}\zeta^{-2}m^{-1}\cos(\theta+4\pi/3)\ , \tag{6.30}$$

where

$$\theta=\tfrac{1}{3}\cos^{-1}[-50(\Delta-1)(3/22\Delta)^{3/2}]\ . \tag{6.31}$$

The relativistic energy shifts of the atom in circular orbit are found to be

$$E^{(1)}(1S_{1/2})=E^{(1)}(2S_{1/2})=E^{(1)}(2P_{1/2})=0 \tag{6.32}$$

and

$$E_1^{(1)}(2P_{3/2})=-E_2^{(1)}(2P_{3/2})=2(3\Delta)^{1/2}Mr^{-3}\zeta^{-2}m^{-1}\ . \tag{6.33}$$

Here the constant Δ is given in terms of the radial coordinate r of the circular orbit by Eq. (6.27).

ACKNOWLEDGMENTS

One of us (L. Parker) thanks Dr. Jacob Bekenstein for a helpful discussion and the National Science Foundation for support. The other (L. O. Pimentel) thanks CONACYT for support under Grant No. 11358.

APPENDIX A: PPN METRIC

In this section we calculate the energy shifts for an atom on a radial geodesic in the PPN (parametrized post Newtonian) metric and compare them with those obtained in Sec. VI.

We start with the spherically symmetric static metric

$$ds^2=-A(r)dt^2+B(r)dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ . \tag{A1}$$

Later we shall specialize (A1) to the PPN metric. The components of the Riemann tensor in the above coordinates are

$$R_{0101}=\frac{A''}{2}-\frac{A'^2}{4A}-\frac{A'B'}{B}\ , \tag{A2}$$

$$R_{0202}=\frac{rA'}{2B}\ , \tag{A3}$$

$$R_{0303}=R_{0202}\sin^2\theta\ , \tag{A4}$$

$$R_{1212}=\frac{rB'}{2B}\ , \tag{A5}$$

$$R_{1313}=R_{1212}\sin^2\theta\ , \tag{A6}$$

$$R_{2323}=r^2(1-B^{-1})\sin^2\theta\ , \tag{A7}$$

where the prime means derivative with respect to r.

For a radial geodesic the first integrals of the equations of motion are

$$A\dot{t}=\epsilon\ , \tag{A8}$$

$$A\dot{t}^2-B\dot{r}^2=1\ . \tag{A9}$$

Here the dot means derivative with respect to proper time and ϵ is an energy parameter.

A Fermi normal basis for this geodesic is given by

$$\hat{e}_0=\dot{t}\frac{\partial}{\partial t}+\dot{r}\frac{\partial}{\partial r}\ , \tag{A10}$$

$$\hat{e}_1=A^{-1/2}B^{1/2}\dot{r}\frac{\partial}{\partial t}+A^{1/2}B^{-1/2}\dot{t}\frac{\partial}{\partial r}\ , \tag{A11}$$

$$\hat{e}_2=\frac{1}{r}\frac{\partial}{\partial\theta}\ , \tag{A12}$$

$$\hat{e}_3=\frac{1}{r}\frac{\partial}{\partial\phi}\ , \tag{A13}$$

where we have chosen the orbit in the plane $\theta=\pi/2$.

The components of the Riemann tensor in the Fermi basis are obtained using Eq. (5.13):

$$R_{\hat{0}\hat{i}\hat{0}\hat{j}}=R_{0101}(AB)^{-1}\delta_i^1\delta_j^1+(\delta_i^2\delta_j^2+\delta_i^3\delta_j^3)\left[\frac{\epsilon^2R_{0202}}{r^2A^2}+\frac{(\epsilon^2-A)}{r^2AB}R_{1212}\right]\ , \tag{A14}$$

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}}=R_{\hat{1}\hat{3}\hat{1}\hat{3}}=\frac{\epsilon^2-A}{r^2A^2}R_{0202}+\frac{\epsilon^2}{r^2AB}R_{1212}\ , \quad \text{(A15)}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}=\frac{R_{2323}}{r^4}\ , \quad \text{(A16)}$$

where $\dot{t}$ and $\dot{r}$ have been eliminated using (A8) and (A9). $R_{\hat{0}\hat{i}\hat{0}\hat{j}}$ is already diagonal.

We can now specialize our calculations to the PPN metric by giving explicitly the functions A and B corresponding to that metric. To lowest order the PPN metric is characterized by[12]

$$A=1-\frac{2M}{r},\quad B=1+\frac{2\gamma M}{r}\ , \quad \text{(A17)}$$

where γ is a parameter that measures how much spatial curvature is produced by unit rest mass.[11] For general relativity $\gamma=1$ and for the Jordan-Brans-Dicke theory $\gamma=(1+\omega)(2+\omega)^{-2}$, ω being the free parameter of the theory. The Viking experiment[13] yielded $\gamma=1\pm2\times10^{-3}$.

The substitution of (A17) into (A2)–(A7) and into (A14)–(A16) gives, to lowest order, the following components of the Riemann tensor:

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}=-\frac{2M}{r^3}\ , \quad \text{(A18)}$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}}=R_{\hat{0}\hat{3}\hat{0}\hat{3}}=\frac{M[\gamma-\epsilon^2(\gamma-1)]}{r^3}\ , \quad \text{(A19)}$$

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}}=R_{\hat{1}\hat{3}\hat{1}\hat{3}}=\frac{M[\epsilon^2(1-\gamma)-1]}{r^3}\ , \quad \text{(A20)}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}=\frac{2\gamma M}{r^3}\ . \quad \text{(A21)}$$

It follows from these relations that to lowest order

$$R_{\hat{0}\hat{0}}=\frac{M[2(\gamma-1)(1-\epsilon^2)]}{r^3} \quad \text{(A22)}$$

and

$$R=0\ . \quad \text{(A23)}$$

Now we have the elements to calculate the energy shifts in the PPN metric. Substitution of (A18)–(A23) into Eqs. (4.1)–(4.17) gives us the following results.

Nonrelativistic energy shifts:

$$E^{(1)}(1S)=(\gamma-1)(1-\epsilon^2)Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A24)}$$

$$E^{(1)}(2S)=14(\gamma-1)(1-\epsilon^2)Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A25)}$$

$$E_1^{(1)}(2P)=12[3(\gamma-1)(1-\epsilon^2)-1]Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A26)}$$

$$E_2^{(1)}(2P)=E_3^{(1)}(2P)=6[(\gamma-1)(1-2\epsilon^2)+\gamma]Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A27)}$$

$$E_1^{(1)}(n=3)=18[2(\gamma-1)(1-\epsilon^2)-4]Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A28)}$$

$$E_2^{(1)}(n=3)=E_3^{(1)}(n=3)=36[(\gamma-1)(1-\epsilon^2)+\gamma]Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A29)}$$

$$E_4^{(1)}(n=3)=18[3(\gamma-1)(1-\epsilon^2)+2]Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A30)}$$

$$E_5^{(1)}(n=3)=E_6^{(1)}(n=3)=18[(\gamma-1)(1-\epsilon^2)-\gamma]Mr^{-3}\zeta^{-1}m^{-1}\ . \quad \text{(A31)}$$

The remaining three energy shifts can be calculated from the cubic equation (3.30) with coefficients given by (3.31)–(3.33) for any specific value of ϵ and γ.

Relativistic energy shifts:

$$E^{(1)}(1S_{1/2})=\frac{(\gamma-1)(1-\epsilon^2)}{18}[3\eta(\eta+1)(2\eta+1)\zeta^{-2}+2(2\eta+1)]Mr^{-3}m^{-1}\ , \quad \text{(A32)}$$

$$E^{(1)}(2S_{1/2})=14(\gamma-1)(1-\epsilon^2)Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A33)}$$

$$E^{(1)}(2P_{1/2})=10(\gamma-1)(1-\epsilon^2)Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A34)}$$

$$E_1^{(1)}(2P_{3/2})=\{\,10(\gamma-1)(1-\epsilon^2)+2(4+[\gamma-\epsilon^2(\gamma-1)][6-\gamma+\epsilon^2(\gamma-1)])^{1/2}\,\}Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A35)}$$

$$E_2^{(1)}(2P_{3/2})=\{\,10(\gamma-1)(1-\epsilon^2)-2(4+[\gamma-\epsilon^2(\gamma-1)][6-\gamma+\epsilon^2(\gamma-1)])^{1/2}\,\}Mr^{-3}\zeta^{-2}m^{-1}\ , \quad \text{(A36)}$$

where $\eta=(1-\zeta^2)^{1/2}$. We note that in this case the energy shifts depend on the initial velocity of the atom through ϵ. The relation between ϵ and the initial conditions follows from Eqs. (A8) and (A9):

$$\epsilon^2 = \frac{A^2(r_0)}{A(r_0) - B(r_0) v_{r0}^{\,2}} , \qquad \text{(A37)}$$

where r_0 and v_{r0} are the initial position and velocity, respectively. In principle the results of this appendix could be used to measure γ.

APPENDIX B: MAGNITUDE OF SHIFT IN HIGHLY EXCITED ATOMS

Here we consider highly excited or Rydberg atoms and relate the accuracy of measurement to the order of magnitude of the radius of curvature that can be detected by means of atomic spectra. Rydberg atoms are of interest becuase the energy perturbations caused by curvature increase as n^4, where n is the principal quantum number.

The leading term in the interaction Hamiltonian caused by spacetime curvature in the nonrelativistic limit is

$$H_I = \tfrac{1}{2} m R_{0l0m} x^l x^m . \qquad \text{(B1)}$$

The typical energy shift to the nth level of the atom is of order

$$E^{(1)}(n) \approx m D^{-2} \langle r^2 \rangle , \qquad \text{(B2)}$$

where D is the characteristic radius of curvature of the spacetime at the position of the atom and $\langle r^2 \rangle$ is given for state $| nlm \rangle$ by

$$\langle r^2 \rangle = \tfrac{1}{2}[5n^2 + 1 - 3l(l+1)] n^2 \zeta^{-2} m^{-2} . \qquad \text{(B3)}$$

Therefore, for large n we have

$$E^{(1)}(n) \approx 2.5 \zeta^{-2} m^{-1} D^{-2} n^4 , \qquad \text{(B4)}$$

with $\zeta = e^2$ (we are taking $Z = 1$ and $\hbar = c = 1$).

The unperturbed energy of the nth level is

$$E^{(0)}(n) = -2\pi \text{Ry} n^{-2} ,$$

where Ry is the Rydberg constant. Then one finds that for large n,

$$Q \equiv \frac{E^{(1)}(n+1) - E^{(1)}(n)}{E^{(0)}(n+1) - E^{(0)}(n)}$$

$$\approx 2.5 \zeta^{-2} (\pi m)^{-1} D^{-2} (\text{Ry})^{-1} n^6 . \qquad \text{(B5)}$$

This gives (in cgs)

$$D \approx \left[2.5 Q^{-1} \zeta^{-2} \left[\pi \frac{\hbar}{mc} \right]^{-1} (\text{Ry})^{-1} \right]^{1/2} n^3 , \qquad \text{(B6)}$$

or with $\text{Ry} = 1.097 \times 10^5 \text{ cm}^{-1}$,

$$D \approx Q^{-1/2} n^3 (2.2 \times 10^{-6} \text{ cm}) . \qquad \text{(B7)}$$

With $Q = 10^9$, corresponding to the accuracy of Ref. 1, one has

$$D \approx (7.0 \times 10^{-2} \text{ cm}) n^3 . \qquad \text{(B8)}$$

For $n = 100$, this gives

$$D \approx 7.0 \times 10^4 \text{ cm} , \qquad \text{(B9)}$$

and for $n = 350$,

$$D \approx 3.0 \times 10^6 \text{ cm} = 30 \text{ km} . \qquad \text{(B10)}$$

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