Predictions of supersymmetric grand unified theories

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Renormalization effects are analyzed for a class of supersymmetric grand unified theories which contain the standard $SU(3)_c \times SU(2)_L \times U(1)$ model. Predictions for $\sin^2\hat{\theta}_W(m_W)$ and the proton lifetime are obtained as functions of $\Lambda_{\overline{MS}}$ ($\overline{MS}$ is the modified minimal-subtraction scheme), N_H (number of relatively light Higgs doublets), and μ (the scale of supersymmetry breaking). For realistic input parameters we find $0.23 \leqslant \sin^2\hat{\theta}_W(m_W) \leqslant 0.26$ and $10^{38} \gtrsim \tau_p \gtrsim 10^{29}$ yr. Loop effects that could render the larger predicted values of $\sin^2\hat{\theta}_W(m_W)$ consistent with experiment are described.

Grand unified theories (GUT's) of strong and electroweak interactions which break down to $SU(3)_c \times SU(2)_L \times U(1)$ via a single superheavy mass scale m_S [such as the Georgi-Glashow SU(5) model[1]] predict[2,3]

$$\sin^2\hat{\theta}_W(m_W) = 0.216 + 0.004(N_H - 1) - 0.006 \ln(\Lambda_{\overline{MS}}/0.1\ \text{GeV}) \qquad (1)$$

for the weak mixing angle [defined in the modified minimal-subtraction ($\overline{MS}$) scheme at mass scale m_W]. In Eq. (1) N_H is the number of relatively light Higgs doublets with mass $\simeq m_W$ and $\Lambda_{\overline{MS}}$ is the mass scale of perturbative quantum chromodynamics (QCD). For the minimal case $N_H = 1$, and using $\Lambda_{\overline{MS}} = 0.1$ GeV obtained from Υ decay,[4] Eq. (1) gives $\sin^2\hat{\theta}_W(m_W) = 0.216$. This prediction is in remarkably good agreement with the average experimental value[3,5]

$$\sin^2\hat{\theta}_W(m_W) = 0.215 \pm 0.014 \ . \qquad (2)$$

Such theories also predict that the proton decays with a lifetime[3]

$$\tau_p \simeq 4 \times 10^{28 \pm 1} \times 10^{-0.76(N_H - 1)} (\Lambda_{\overline{MS}}/0.1\ \text{GeV})^4\ \text{yr} \ , \qquad (3)$$

where the ± 1 in the exponent represents a conservative estimate of the theoretical uncertainty. For $N_H = 1$ and $\Lambda_{\overline{MS}} \simeq 0.1$ GeV this prediction is already somewhat below the present experimental bound

$$\tau_p^{\text{exp}} \gtrsim 10^{30}\ \text{yr} \ . \qquad (4)$$

A possible way of increasing the proton lifetime is to impose a supersymmetry constraint on the theory. One assumes that every boson (fermion) of the standard model has a supersymmetric fermion (boson) partner which (presumably) has not yet been observed.[6,7] In such supersymmetric extensions of grand unified theories, coupling-constant renormalizations are changed; hence the unification mass m_S may be significantly altered.[8] (The proton lifetime is proportional to $m_S{}^4$ and thus very sensitive to changes in m_S.) Indeed, an estimate by Dimopoulos, Raby, and Wilczek[8] (DRW) found (neglecting Higgs multiplets) $\tau_p \approx 10^{45}$ yr while $\sin^2\hat{\theta}_W(m_W)$ was essentially unchanged. Since that work first appeared, several groups have tried to construct realistic supersymmetric grand unified theories.[9,10] In so doing, it was noted that m_S and hence τ_p exhibits a strong dependence on the number of relatively light Higgs isodoublets, N_H, present in the model.[10,11] Since in realistic supersymmetric theories $N_H = 2, 4, \ldots$ (an even number because of an anomaly-cancellation requirement for their fermionic partners), τ_p generally turns out to be much smaller than the 10^{45}-yr DRW estimate and $\sin^2\hat{\theta}_W(m_W)$ becomes somewhat larger than the nonsupersymmetric prediction in Eq. (1).

To ascertain more precisely the predictions of the class of supersymmetric GUT's outlined above, we have carried out a detailed investigation of coupling-constant renormalizations in such theories. Because most of our formal analysis is the same as the nonsupersymmetric case which has been described in detail elsewhere[2,3] and the full supersymmetric two-loop β functions which we employ have been given in a recent paper by Einhorn and Jones,[11] we will merely outline our assumptions and present the final results.

We assume that the standard $SU(3)_c \times SU(2)_L \times U(1)$ model with N_H relatively light weak isodoublets and n_g generations of quarks and leptons is the correct low-energy theory. Supersymmetry is imposed on the spectrum of particles by requiring the gauge bosons to have spin-$\frac{1}{2}$ fermion partners, the ordinary spin-$\frac{1}{2}$ quarks and leptons to have scalar partners, and the Higgs scalars to have spin-$\frac{1}{2}$ fermionic partners. To simplify our analysis, we assume that all the added supersymmetric partners have equal mass $\mu \geqslant m_W$ and allow μ to vary.

Given the above assumptions, we compute

$\hat{\alpha}_i(\mu)$, $i=1,2,3$ (the effective low-energy couplings defined by $\overline{\text{MS}}$) for $\mu \geq m_W$ by integrating the standard $SU(3)_c \times SU(2)_L \times U(1)$ β functions[3,11] up to μ. Then the supersymmetric two-loop β functions are employed to evolve the couplings from μ to m_S, the unification mass scale. We obtain

$$\hat{\alpha}_1^{-1}(\mu)=\hat{\alpha}_1^{-1}(m_S)-\frac{1}{2\pi}(-2n_g-\tfrac{3}{10}N_H)\ln\frac{m_S}{\mu}$$
$$+\frac{1}{4\pi}\left[\frac{-\frac{88}{15}n_g}{9-2n_g}\ln\frac{\hat{\alpha}_3(m_S)}{\hat{\alpha}_3(\mu)}+\frac{-\frac{6}{5}n_g-\frac{9}{10}N_H}{6-2n_g-\frac{1}{2}N_H}\ln\frac{\hat{\alpha}_2(m_S)}{\hat{\alpha}_2(\mu)}+\frac{-\frac{38}{15}n_g-\frac{9}{50}N_H}{-2n_g-\frac{3}{10}N_H}\ln\frac{\hat{\alpha}_1(m_S)}{\hat{\alpha}_1(\mu)}\right], \tag{5a}$$

$$\hat{\alpha}_2^{-1}(\mu)=\hat{\alpha}_2^{-1}(m_S)-\frac{1}{2\pi}(6-2n_g-\tfrac{1}{2}N_H)\ln\frac{m_S}{\mu}$$
$$+\frac{1}{4\pi}\left[\frac{-8n_g}{9-2n_g}\ln\frac{\hat{\alpha}_3(m_S)}{\hat{\alpha}_3(\mu)}+\frac{24-14n_g-\frac{7}{2}N_H}{6-2n_g-\frac{1}{2}N_H}\ln\frac{\hat{\alpha}_2(m_S)}{\hat{\alpha}_2(\mu)}+\frac{-\frac{2}{5}n_g-\frac{3}{10}N_H}{-2n_g-\frac{3}{10}N_H}\ln\frac{\hat{\alpha}_1(m_S)}{\hat{\alpha}_1(\mu)}\right], \tag{5b}$$

$$\alpha_3^{-1}(\mu)=\hat{\alpha}_3^{-1}(m_S)-\frac{1}{2\pi}(9-2n_g)\ln\frac{m_S}{\mu}$$
$$+\frac{1}{4\pi}\left[\frac{54-\frac{68}{3}n_g}{9-2n_g}\ln\frac{\hat{\alpha}_3(m_S)}{\hat{\alpha}_3(\mu)}+\frac{-3n_g}{6-2n_g-\frac{1}{2}N_H}\ln\frac{\hat{\alpha}_2(m_S)}{\hat{\alpha}_2(\mu)}+\frac{-\frac{11}{15}n_g}{-2n_g-\frac{3}{10}N_H}\ln\frac{\hat{\alpha}_1(m_S)}{\hat{\alpha}_1(\mu)}\right]. \tag{5c}$$

Finally, using the relationship

$$\hat{\alpha}^{-1}(\mu)=\hat{\alpha}_2^{-1}(\mu)+5\hat{\alpha}_1^{-1}(\mu)/3 \tag{6}$$

and the boundary conditions[3,12]

$$\hat{\alpha}_1^{-1}(m_S)=\hat{\alpha}_2^{-1}(m_S)-\frac{1}{6\pi}=\hat{\alpha}_3^{-1}(m_S)-\frac{1}{4\pi} \tag{7}$$

we find

$$\frac{\hat{\alpha}(\mu)}{\hat{\alpha}_3(\mu)}=\frac{3}{8}\left[1-\frac{\hat{\alpha}(\mu)}{2\pi}(18+N_H)\ln\left(\frac{m_S}{\mu}\right)+\frac{\hat{\alpha}(\mu)}{2\pi}\right.$$
$$\left.+\frac{\hat{\alpha}(\mu)}{4\pi}\left(\frac{144-\frac{384}{9}n_g}{9-2n_g}\ln\frac{\hat{\alpha}_3(m_S)}{\hat{\alpha}_3(\mu)}+\frac{-24+8n_g+5N_H}{6-2n_g-\frac{1}{2}N_H}\ln\frac{\hat{\alpha}_2(m_S)}{\hat{\alpha}_2(\mu)}+\frac{\frac{8}{3}n_g+\frac{3}{5}N_H}{-2n_g-\frac{3}{10}N_H}\ln\frac{\hat{\alpha}_1(m_S)}{\hat{\alpha}_1(\mu)}\right)\right]. \tag{8}$$

Given a value for μ, we compute $\hat{\alpha}_i(\mu)$, and then determine $\hat{\alpha}_i(m_S)$ and m_S/μ by iterating Eqs. (5)–(8). Having obtained those parameters we next compute $\sin^2\hat{\theta}_W(m_W)$ by iterating the formulas[3,13]

$$m_W=38.5\text{ GeV}/\sin\hat{\theta}_W(m_W) \tag{9}$$

and

$$\sin^2\hat{\theta}_W(m_W)=\frac{3}{8}\left[1-\frac{\hat{\alpha}(m_W)}{2\pi}(10-\tfrac{1}{3}N_H)\ln\frac{m_S}{m_W}-\frac{\hat{\alpha}(m_W)}{2\pi}\left(\frac{20+2N_H}{9}\right)\ln\frac{\mu}{m_W}+\frac{5\hat{\alpha}(m_W)}{18\pi}+\frac{\hat{\alpha}(m_W)}{4\pi}\right.$$
$$\times\left(\frac{-\frac{32}{9}n_g}{9-2n_g}\ln\frac{\hat{\alpha}_3(m_S)}{\hat{\alpha}_3(\mu)}+\frac{40-\frac{64}{3}n_g-\frac{13}{3}N_H}{6-2n_g-\frac{1}{2}N_H}\ln\frac{\hat{\alpha}_2(m_S)}{\hat{\alpha}_2(\mu)}+\frac{\frac{32}{9}n_g-\frac{1}{5}N_H}{-2n_g-\frac{3}{10}N_H}\ln\frac{\hat{\alpha}_1(m_S)}{\hat{\alpha}_1(\mu)}\right.$$
$$\left.\left.+\frac{-\frac{16}{9}n_g}{11-\frac{4}{3}n_g}\ln\frac{\hat{\alpha}_3(\mu)}{\hat{\alpha}_3(m_W)}+\frac{\frac{680}{9}-\frac{236}{9}n_g-\frac{19}{9}N_H}{\frac{22}{3}-\frac{4}{3}n_g-\frac{1}{6}N_H}\ln\frac{\hat{\alpha}_2(\mu)}{\hat{\alpha}_2(m_W)}+\frac{\frac{16}{9}n_g-\frac{1}{5}N_H}{-\frac{4}{3}n_g-\frac{1}{10}N_H}\ln\frac{\hat{\alpha}_1(\mu)}{\hat{\alpha}_1(m_W)}\right)\right]. \tag{10}$$

The relationships given in Eqs. (5)–(10) include all leading and next-to-leading logarithmic corrections. In addition they contain all ordinary $O(\alpha)$ corrections.

Carrying out the iterative analysis described above for $n_g=3$, $\mu=m_W$, $N_H=2$ or 4, $\alpha^{-1}(0)=137.035\,993$, and a range of $\Lambda_{\overline{MS}}$ values, we find the numerical results given in Table I.[14] The proton-lifetime predictions in Table I were obtained by slightly modifying the standard SU(5) prediction[3] to take into account the increase in $\hat{\alpha}_i(m_S)$, i.e., we used

$$\tau_p \simeq 3\times 10^{-29\pm 1}\times (m_S \text{ in GeV})^4 \text{ yr} . \qquad (11)$$

Of course in specific supersymmetric models, τ_p may be substantially smaller than our estimate if superheavy-Higgs-boson-mediated amplitudes are significant. Also, as pointed out by Weinberg,[15] there may be higher-order amplitudes of effective dimension 5 which give rise to proton lifetimes proportional to $m_S{}^2$ rather than $m_S{}^4$ (a potential disaster). We assume that such amplitudes are forbidden by some additional symmetry.[15]

What if $\mu > m_W$? In the leading-logarithmic approximation, one finds for $\mu > m_W$ that $\sin^2\hat{\theta}_W(m_W)$ decreases by

$$\Delta\sin^2\hat{\theta}_W(m_W) = -\frac{\hat{\alpha}(m_W)}{\pi}\frac{8N_H}{54+3N_H}\ln\mu/m_W \qquad (12)$$

and τ_p is reduced by a factor

$$(m_W/\mu)^{(48-8N_H)/(54+3N_H)} . \qquad (13)$$

So we see that for $N_H=4$, varying μ has a more substantial effect on $\sin^2\hat{\theta}_W(m_W)$ (about twice the $N_H=2$ case); but it is less important for τ_p.

We have also examined the $n_g=4$ case (assuming that the charged-particle members of the fourth generation have mass $\simeq m_W$). We find that m_S increases by about 25% relative to the $n_g=3$ case. However, because the unification coupling $\hat{\alpha}_i(m_S)$ increases by almost a factor of 2, the final $n_g=4$ prediction for τ_p is actually $\approx 30\%$ smaller than the three generation result. The extra generation is found to increase $\sin^2\hat{\theta}_W(m_W)$ by about $+0.001$.

From the above numerical analysis we learn that supersymmetric GUT predictions are very sensitive to the Higgs content of the theory but not very sensitive to changes in μ (unless it is $>>1$ TeV) or whether $n_g=3$ or 4 (for $n_g\geq 5$ and $\mu\simeq m_W$ the couplings diverge before they can reach a unification point). If $N_H=2$, τ_p is potentially observable (in planned experiments which will probe up to $\approx 10^{33}$ yr) if $\Lambda_{\overline{MS}}$ is near (or below) the Mackenzie-Lepage value of 0.1 GeV. It also helps somewhat if $\mu > m_W$. On the other hand, for $N_H=4$, the τ_p predictions are very similar to the ordinary SU(5) predictions in Eq. (3). Hence the observation of proton decay in the range $\tau_p\simeq 10^{30}$–10^{33} yr can be compatible with supersymmetry.

Perhaps a more definite prediction of supersymmetric GUT's is that $\sin^2\hat{\theta}_W(m_W)$ is larger than the standard SU(5) prediction in Eq. (1). Indeed, one might conclude that the experimental constraint in Eq. (2) already rules out $N_H=4$ and disagrees somewhat with the $N_H=2$ results (see Table I) if $\Lambda_{\overline{MS}}\simeq 0.1$ GeV. Those of course are the interesting cases in which τ_p is small enough to observe. There is, however, a possible way to circumvent the constraint in Eq. (2) which we now explain.

TABLE I. Supersymmetric GUT predictions for $\mu=m_W$, $n_g=3$, and $N_H=2$ and 4.

$\Lambda_{\overline{MS}}$ (GeV)	m_W (GeV)	$\sin^2\hat{\theta}_W(m_W)$	$\hat{\alpha}_3(m_W)$	$\hat{\alpha}^{-1}(m_W)$	$\hat{\alpha}_i(m_S)$	m_S (GeV)	τ_p (yr)
$N_H=2$							
0.05	78.2	0.242	0.093	127.68	0.040	2.1×10^{15}	$6\times10^{32\pm1}$
0.10	78.8	0.239	0.102	127.65	0.041	4.8×10^{15}	$2\times10^{34\pm1}$
0.20	79.4	0.235	0.113	127.62	0.042	1.1×10^{16}	$4\times10^{35\pm1}$
0.30	79.7	0.233	0.122	127.59	0.043	1.7×10^{16}	$3\times10^{36\pm1}$
0.40	80.0	0.232	0.128	127.58	0.044	2.4×10^{16}	$1\times10^{37\pm1}$
$N_H=4$							
0.05	75.0	0.263	0.093	127.75	0.042	1.3×10^{14}	$8\times10^{27\pm1}$
0.10	75.5	0.260	0.103	127.72	0.043	2.6×10^{14}	$1\times10^{29\pm1}$
0.20	75.8	0.258	0.114	127.70	0.045	5.5×10^{14}	$3\times10^{30\pm1}$
0.30	76.1	0.256	0.122	127.67	0.046	8.5×10^{14}	$2\times10^{31\pm1}$
0.40	76.3	0.255	0.129	127.66	0.046	1.2×10^{15}	$5\times10^{31\pm1}$

Deep-inelastic ν_μ scattering experiments measure $R_\nu \equiv \sigma(\nu_\mu + N \rightarrow \nu_\mu + X)/\sigma(\nu_\mu + N \rightarrow \mu + X)$ which in the standard $SU(2)_L \times U(1)$ model depends on two parameters ρ and $\sin^2\hat{\theta}_W(m_W)$. A two-parameter fit to R_ν and $R_{\bar{\nu}}$ data gives[16]

$$\rho = 1.010 \pm 0.020 \ , \tag{14a}$$

$$\sin^2\hat{\theta}_W(m_W) = 0.236 \pm 0.030 \ . \tag{14b}$$

A one-parameter fit to R_ν data alone holding ρ fixed (and near 1) yields[3,5]

$$\sin^2\hat{\theta}_W(m_W) = 0.226 \pm 0.014 - 0.49(1-\rho^2) \ . \tag{15}$$

Since radiative corrections in the standard model reduce ρ to ≈ 0.99, the quoted result in Eq. (2) follows. In a bigger, more complicated theory there may be new unaccounted for contributions to ρ. If one takes the view that they may be present at the level of a few percent [as allowed by Eq. (14a)], then only the much less stringent constraint in Eq. (14b) is applicable and it is certainly compatible with the predictions for $\sin^2\hat{\theta}_W(m_W)$ given in Table I.

What could cause a small increase in ρ and thus raise the experimental value of $\sin^2\hat{\theta}_W(m_W)$? New higher-dimensional Higgs multiplets are a potential source; however, since Eq. (14a) tells us that ρ is close to 1, new unaccounted for radiative corrections would seem to be more natural candidates. For example, one generally assumes in the radiative corrections that $m_t \simeq 20$ GeV (or at least that $m_t^2 << m_W^2$).[3,5] A very large t-quark mass would increase ρ by[17]

$$\Delta\rho \simeq \frac{3\alpha}{16\pi \sin^2\theta_W} \frac{m_t^2}{m_W^2} \ .$$

[For $m_t \simeq 240$ GeV one finds $\Delta\rho \simeq +0.017$ which shifts the central value for $\sin^2\hat{\theta}_W(m_W)$ in Eq. (2) up by about $+0.017$. Of course a fourth generation of fermions with large mass splittings among doublet partners would also add small positive corrections to ρ.[17] Furthermore, in the supersymmetric theories we are considering there are assumed to be a sizeable number of additional fermions and scalars with mass $\mu \gtrsim m_W$ (the supersymmetric partners of ordinary particles). Mass splittings between weak-isodoublet members (so far we have taken them to be degenerate[18]) will also contribute positive increments to ρ. In total, all such mass-splitting loop effects may add up and shift ρ by $\simeq +(2-3)\%$.

In conclusion, our main result is that GUT's with supersymmetry breaking at $\mu \simeq m_W - 1$ TeV predict a proton lifetime in the range $10^{38} \gtrsim \tau_p \gtrsim 10^{29}$ yr and a weak mixing angle $0.23 \leqslant \sin^2\hat{\theta}_W(m_W) \leqslant 0.26$. The actual predictions depend rather sensitively on the Higgs content of the theory. The lower (experimentally observable) lifetimes correspond to larger values of $\sin^2\hat{\theta}_W(m_W)$ which disagree with deep-inelastic ν_μ scattering results unless $\rho > 1$. A more precise determination of ρ or an independent precise measurement of $\sin^2\hat{\theta}_W(m_W)$ is clearly required to clarify the validity of supersymmetric GUT's and pinpoint their predictions.

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