Magnetic fields and spontaneous neutron-antineutron transitions

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The effect of a constant-plus-time-periodic magnetic field on neutron-antineutron oscillations is considered. The underlying self-adjoint system of linear differential equations and the corresponding Hill's equation are studied. An approximate analytic solution is found for all times. By optimizing the growth of the antineutron probability, two free parameters of the time-varying field are determined automatically.

In a recent paper by Arndt, Prasad, and Riazuddin[1] the phenomenology of neutron-antineutron oscillations in the presence of magnetic fields was developed. They determined by numerical methods the optimum experimental conditions for these fields. Here we solve this problem analytically.

Let the neutron be in a homogeneous time-dependent magnetic field of the form

$$B(t)=B_0(1-r\sin\omega t)\ . \tag{1}$$

In the presence of a fundamental baryon-mixing force, parametrized in terms of a small frequency ω_m, the time evolution of the neutron (antineutron) wave function is found by solving the following self-adjoint system of linear differential equations with periodic coefficients,

$$dn/dt\equiv\dot{n}=-i\omega_B(t)n-i\omega_m\bar{n}\ ,$$

and (2)

$$\dot{\bar{n}}=-i\omega_m n+i\omega_B(t)\bar{n}\ ,$$

where $\omega_B(t)=\omega_B(1-r\sin\omega t)$ and $c=\hbar=1$. Applying unitary transformations,

$$n=N\exp\left[-i\int_0^t\omega_B(t')dt'\right]$$

and (3)

$$\bar{n}=\bar{N}\exp\left[i\int_0^t\omega_B(t')dt'\right]$$

the system is reduced to

$$\dot{N}=i\omega_m F_+(t)\bar{N}\ ,$$

and (4)

$$\dot{\bar{N}}=-i\omega_m F_-(t)N\ ,$$

where

$$F_\pm(t)=\exp\left[\pm 2i\int_0^t\omega_B(t')dt'\right]\ .$$

At $t=0$ the beam contains only neutrons, so $n(0)=N(0)=1$ and $\bar{n}(0)=\bar{N}(0)=0$. Thus the implicit solution of Eq. (4) is

$$N(t)=1+(-i\omega_m)^2\int_0^t dt_1F_+(t_1)\int_0^{t_1}dt_2F_-(t_2)N(t_2)\ ,$$

and (5)

$$\bar{N}(t)=(-i\omega_m)\int_0^t dt_1F_-(t_1)\left[1+(-i\omega_m)\int_0^{t_1}dt_2F_+(t_2)\bar{N}(t_2)\right]\ .$$

The basic system (2) may be transformed into the corresponding second-order differential equation in terms of dimensionless time $x=\omega t$,

$$\bar{n}''(x)+Q(x)\bar{n}(x)=0\ , \tag{6}$$

where

$$Q(x)=\lambda^2+ia\cos x-(2a^2/r)\sin x-(a^2/2)\cos 2x\ ,\quad \lambda^2=(\omega_m{}^2+\omega_B{}^2)/\omega^2,\quad a=\omega_B r/\omega\ .$$

Since $Q(x)$ is periodic, the solutions will be Hill's functions.[2]

With the system represented by a Hill's equation, many powerful theorems may be brought to bear on its solution. In particular, the Floquet-Lyapunov theorem[2] requires that the general solution of Eq. (6) has the form

$$\bar{n}(x)=e^{\alpha x}p_1(x)+e^{-\alpha x}p_2(x) , \tag{7}$$

where p_1 and p_2 are bounded functions whose periodicity is the same as that of Q. The generally complex characteristic exponent α determines the stability of the solutions. Conservation of the sum of neutron and antineutron probabilities [the self-adjoint property of the matrix of coefficients of the system of Eq. (2)] causes the characteristic exponent to be purely imaginary, so the solution will be double oscillatory. The trivial periodicity is that of Q, namely, $2\pi/\omega$.

The nontrivial periodicity is due to the characteristic exponent α. Its determination is as difficult as finding the explicit solution of the Hill equation. We expect $|\alpha| \sim 2\pi/\omega_m \sim 10^4$ sec. As will be seen later, the frequency ω must be of the order of the static-field frequency ω_B. In the earth's magnetic field $\omega_B \sim 10^4\ \text{sec}^{-1}$, and therefore $(2\pi/\omega) \sim 10^{-4}$ sec. We will refer to those two kinds of oscillations as those of large and small Hill's periodicity. They will become manifest in our approximate solutions for large and small times, respectively.

Let us first consider the small-time region and calculate the antineutron probability to the lowest order in $(\omega_m t)$. By taking the first term of Eq. (5), after straightforward manipulations we obtain

$$P_{\bar{n}}(t)=(\omega_m t)^2 \,|\, I(x,z,\nu) \,|^2 , \tag{8}$$

where

$$xI(x,z,\nu)=\int_0^x dx' \exp(i\nu x'+iz\cos x') ,$$

with $x=\omega t$, $z=\nu r$, and $\nu=2\omega_B/\omega$. The frequencies ω_m and ω_B, $\omega_m \ll \omega_B$, are considered as known, while ν and z [r and ω in Eq. (1)] are considered as free parameters.

The integral of Eq. (8) becomes resolvable if we apply the Bessel differential operator with respect to the parameter z,

$$\left[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}+(z^2-\nu^2)\right]I(x,z,\nu)$$
$$=\int_0^x dx\left[\frac{d^2}{dx^2}-2\nu i\frac{d}{dx}\right]\exp(i\nu x+iz\cos x)$$
$$=i(\nu+z\sin x)\exp(i\nu x+iz\cos x)-i\nu\exp(iz) . \tag{9}$$

For integer values of ν but arbitrary z, the inhomogeneity in Eq. (9) vanishes whenever $x=2\pi N$, where N is any non-negative integer. Therefore the solution, up to a multiplicative constant, is a Bessel function,

$$I(n,z,x)=C(n,x)J_n(z) . \tag{10}$$

Comparing this result with the exact solution of the integral of Eq. (9) in the limit of small z, we find $C=i^n x$, so that we have

$$I(n,z,x)=i^n xJ_n(z) . \tag{11}$$

Thus, for a sequence of times $t_N=(2\pi/\omega)N$ $=(n\pi/\omega_B)N$, where n is the integer to be determined, the antineutron probability becomes

$$P_{\bar{n}}(t)=(\omega_m t)^2 J_n{}^2(z) . \tag{12}$$

It will be maximal for the choice[3] $n=z=0$ (zero external field), which is of no interest. Since $|J_n(z)|$ falls off rapidly with increasing n, the best choice is, clearly, $n=1$, where $J_1(z)$ assumes the extremal value of 0.68 at $z=nr=1.86$, in agreement[4] with the values found by a numerical analysis by Arndt, Prasad, and Riazuddin.[1] According to Eq. (9) the ratio of the static-field frequency to the time-varying frequency, $\nu=2\omega_B/\omega$, may propitiously be an integer. If this is the case, the solution (11) of Eq. (9) is sufficient for practical purposes. The inhomogeneity just adds small oscillations around the time-varying quadratic growth in Eq. (12). The complete solution of the inhomogeneous equation was obtained in the form of a series of Bessel functions and we do not include it here. It is represented by the wiggling curve in Fig. (1). The oscillations seen are those of

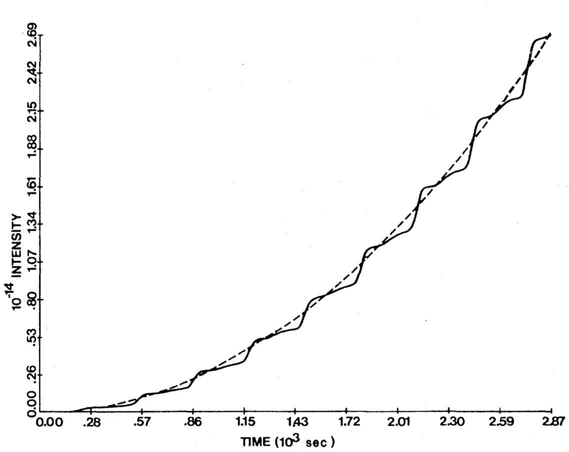

FIG. 1. Antineutron probability for small times. The wiggling curve is the complete evaluation of the integral by means of a series of Bessel functions of Eq. (8). The smooth curve is the integral approximated by Eq. (11).

the small Hill's period and are the only oscillations that exist in the perturbative solution to the lowest order in $(\omega_m t)$.

To obtain an approximate solution valid for all times, we iterate Eq. (5) to get

$$\bar{N}(t)=\sum_{k=0}^{\infty}\bar{N}_{2k+1}(t)\ , \tag{13}$$

$$N(t)=\sum_{k=0}^{\infty}N_{2k}(t)\ . \tag{14}$$

Making use of the expansion of $\exp(\pm inr\cos x)$ in terms of Bessel functions,[3] we obtain

$$\bar{N}_{2k+1}=(-i\omega_m/\omega)^{2k+1}e^{inr}\sum_{l_1,\ldots,l_{2k+1}} i^{l_1+l_2+\cdots+l_{2k+1}}J_{l_1}(-nr)J_{l_2}(nr)J_{l_3}(-nr)\cdots J_{l_{2k+1}}(-nr)$$
$$\times\int_0^x dx_1 e^{ix_1(l_1-n)}\int_0^{x_1}dx_2 e^{ix_2(l_2+n)}$$
$$\times\int_0^{x_2}dx_3 e^{ix_3(l_3-n)}\cdots\int_0^{x_{2k}}dx_{2k+1}e^{ix_{2k+1}(l_{2k+1}-n)}\ .$$

We note that when the integers $l_1, l_2, \ldots, l_{2k+1}$ run from $-\infty$ to $+\infty$, and $\mathrm{lm}\neq n$, the product of the exponential functions oscillates wildly but averages to a value roughly equal to zero. The essential contribution to the integral comes from the resonant configuration when all exponents vanish. The integrals thus reduce to

$$[x^{2k+1}/(2k+1)!]\delta_{l_1,n}\delta_{l_2,-n}\delta_{l_3,n}\cdots\delta_{l_{2k+1,n}}\ ,\quad x=\omega t\ ,$$

so that we obtain

$$\bar{N}_{2k+1}=i^n e^{inr}J_n{}^{2k+1}(nr)[(-i\omega_m t)^{2k+1}/(2k+1)!]\ . \tag{15}$$

Now the absolute convergent series of Eq. (13) may be summed to give

$$\bar{N}(t)=e^{i(n(r-\pi/2)-\pi/2)}\sin[\omega_m J_n(nr)t]\ . \tag{16}$$

By the same procedure we obtain the neutron wave function

$$N(t)=\cos[\omega_m J_n(nr)t]\ . \tag{17}$$

In the zero-field limit $\omega_B=0$, solutions of the equations reduce to $-i\sin(\omega_m t)$ and $\cos(\omega_m t)$, respectively. These are precisely the solutions of Eq. (2) for the case $\omega_B=0$. The sum of the neutron and antineutron probabilities is manifestly time independent and equal to 1.

We see that the antineutron probability *at any time* (shown in Fig. 2)

$$P_{\bar{n}}(t)=\sin^2[\omega_m J_n(nr)t]\ , \tag{18}$$

is optimized by the same parameters, $n=2\omega_B/\omega$ and r, determined at small times. Further, the antineutron probability does not depend on the static-field frequency explicitly; only on its ratio to the frequency of the time-varying field. Having in view the numerical properties of Bessel functions,[3] there is, of course, a series of extremal time-varying configurations:

$$(n,r)=(1,1.85),(2,1.55),(3,1.40),\ldots\ .$$

The corresponding ratio of Eq. (18) and the free-oscillation probability, $\sin^2\omega_m t$, at small times is

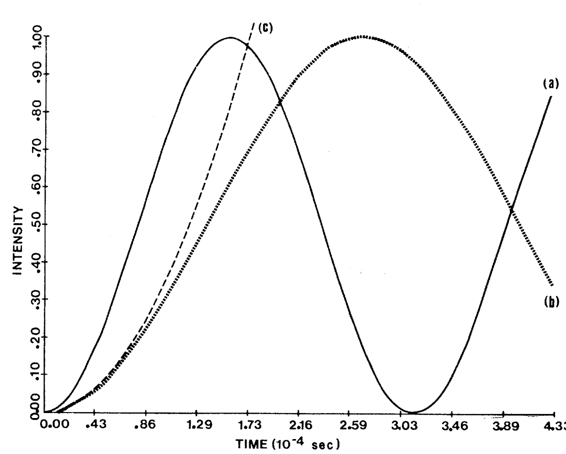

FIG. 2. Antineutron probability for (a) zero magnetic field, (b) constant-plus-time-periodic magnetic field, and (c) small-time solution, Eq. (12).

given by

$$J_n{}^2(nr)=(0.58)^2,(0.49)^2,(0.43)^3,\ldots, \text{ respectively .}$$

We should also like to point out that the solution to Eq. (18) does not allow a static-field limit, which might be compared with the exact solution of the basic system of Eq. (2) for $B(t)=B_0$, namely,

$$P_{\bar{n}}^0(t)=(\omega_m/\omega_0)^2\sin^2(\omega_0 t),\quad \omega_0{}^2=\omega_m^2+\omega_B{}^2 .$$

This is obviously a consequence of the fact that the solution depends only on the ratio $n=2\omega_B/\omega$; the static field gets "absorbed" into the average value of the time-varying field. Thus within the neutrons lifetime the time-varying magnetic field can shield the static field such that the antineutron probability can be as large as one third of that for the field-free case.

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[1]R. A. Arndt, V. B. Prasad, and Riazuddin, Phys. Rev. D 24, 1431 (1981).
[2]V. A. Yakubovich, *Linear Differential Equations with Periodic Coefficients* (Wiley, New York, 1975).
[3]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics series, No. 55 (Dover, New York, 1972), p. 390.
[4]We think that the value 0.34 in Eq. (11) of Ref. 2 should be replaced by $J_1(1.85)=0.58$.
[5]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1965), p. 973.