West's proof of confinement and two-dimensional models

T. E. Clark
Department of Physics, Purdue University, West Lafayette, Indiana 47907

S. T. Love
Max-Planck-Institut für Physik und Astrophysik, Munich, Federal Republic of Germany and Department of Physics, Purdue University, West Lafayette, Indiana 47907
(Received 13 November 1981)

West's proof of confinement by contradiction is studied for the unconfined SU(N) Gross-Neveu model to leading order in $1/N$ and for the confining Schwinger model. In both cases, we find that the contradiction found by West for four-dimensional QCD is averted.

I. INTRODUCTION

Quantum chromodynamics (QCD), the non-Abelian gauge theory describing the interactions of quarks and gluons, is an outstanding candidate for the theory of the strong interactions of hadronic matter. One of the most fundamental features of this theory is supposed to be the confinement of color nonsinglet states. A proof of this property in four dimensions has been as elusive as the quark itself. Attempts have always made use of the very technical details of the particular methodology employed, be it instanton or lattice physics. However, recently West[1] has outlined a proof of confinement by means of contradiction that relies on the very general field-theoretic features of the model such as its asymptotic freedom.

More specifically, West assumed the existence of the one-particle quark state $|p\rangle$ of momentum p^μ ($p^2=m^2$) and that the overlap between the quark field ψ and this state is nonvanishing so that

$$\langle 0|\psi|p\rangle\neq 0\ . \tag{1.1}$$

Then he introduced the electromagnetic current vertex function $\Gamma^\mu(p,q)$, defined by

$$\Gamma^\mu(p,q)u(p)=i(\not p'-m)\int dx\, e^{iqx}\langle 0|Tj^\mu(x)\psi(0)|p\rangle\ , \tag{1.2}$$

where j^μ is the current and $p'^\mu=p^\mu-q^\mu$. The current is assumed conserved so that the vertex function obeys the canonical Ward identity

$$q_\mu\Gamma^\mu(p,q)=(\not p'-m)\ . \tag{1.3}$$

In four-dimensional QCD, this vertex function is known only perturbatively. On the other hand, based on the summation of certain contributions from gluon exchanges, nonperturbative expressions for elastic form factors have been conjectured[2] for q^2 asymptotically large and spacelike. In order to exploit this, West incorporates the Ward-identity information in a dispersion relation for an elastic form factor $G(q^2,m)$ of $\Gamma^\mu(p,q)$ so that

$$1-G(q^2,m)=-\frac{1}{\pi}\int_{m^+}^{\infty}dW\frac{\mathrm{Im}G(q^2,W)}{W-m}\ . \tag{1.4}$$

The 1 on the left-hand side results directly from application of the Ward identity. For spacelike q^2, the imaginary part of G can be bounded using the Schwarz inequality by the product of the quark spectral function ρ_+ and the electromagnetic structure function W_L, to yield

$$|1-G(q^2,m)|\leq(-2mq^2)^{1/2}\int_{m^+}^{\infty}\frac{dW^2}{(W+m)[(W-m)^2-q^2]^{1/2}}\rho_+{}^{1/2}(W^2)W_L{}^{1/2}(q^2,W^2)\ . \tag{1.5}$$

For large spacelike q^2, asymptotic freedom implies that the right-hand side vanishes. However, the elastic form factor $G(q^2,m)$ does not approach 1; it decreases exponentially to zero.[2] Hence we have a contradiction. West then concludes that the original assumption $\langle 0|\psi|p\rangle \neq 0$ is at fault and the quark fields must be confined. The purpose of this paper is to study the details of his proof in the framework of specific two-dimensional models for which more than ordinary perturbative information is known.

In Sec. II the assumption of the nonzero-overlap matrix element is stated and elaborated upon.

The vertex function is defined and its canonical Ward identity is assumed. Finally, the dispersion relation for the vertex function is given and the bound for the form factors derived.

In Sec. III the assumptions are tested in the framework of the two-dimensional SU(N) Gross-Neveu model[3] to leading order in a $1/N$ perturbation expansion. This model is asymptotically free, but the fundamental Fermi fields are not confined.[4] Hence a contradiction is neither expected nor obtained. Even so, it demonstrates that the property of asymptotic freedom is not the only essential ingredient needed to obtain a contradiction from Eq. (1.5). It is also critical that the elastic form factors vanish for asymptotic spacelike q^2. This last property is model dependent, although it seems to be true in both four-dimensional QED (Ref. 5) and four-dimensional QCD.[2]

Finally in Sec. IV the assumptions are viewed in the light of the confining Schwinger model of massless two-dimensional quantum electrodynamics.[6] Once again, no contradiction is obtained from West's inequality since for asymptotic q^2 the form factor equals just the Born term. This results from the superrenormalizability of the model and it is conjectured that all superrenormalizable models, even though they might confine, have form factors equal to their Born term for asymptotic q^2. Hence they will not lead to a contradiction in West's inequality.

Appendix A contains notation, conventions, and some useful formulas.

II. THE FUNDAMENTAL INEQUALITY

In this section, we discuss the assumptions underlying the derivation of the current-vertex-function inequality which forms the basis of West's proof.

In short, the current Ward identity is combined with dispersion relations for the corresponding form factors and the Schwarz inequality in order to bound the current vertex function by the product of the current-current single-particle matrix element and a two-point spectral function. In order to proceed, we first assume that the two-dimensional theories under consideration allow for the existence of one-particle states $|p\rangle$ of momentum p^μ ($p^2=m^2$), which have the same quantum numbers as the fundamental Fermi fields ψ of the model. For example, in the SU(N) Gross-Neveu model, the states $|p\rangle$ are taken to lie in the fundamental representation of the group as do the fundamental Fermi fields ψ. Moreover, we must assume that there is a nonzero overlap between these states and the fundamental fields so that

$$\langle 0|\psi(x)|p\rangle \neq 0 . \tag{2.1}$$

Next, we assume that the model possesses a conserved current $j^\mu(x)$ so that $\partial_\mu j^\mu(x)=0$. This can be the electromagnetic current, the energy-momentum tensor, or some internal-symmetry current.

Following West,[1] we consider the time-ordered function

$$G^\mu(p,q)u(p) \equiv i\int d^2x\, e^{+iqx}\langle 0|Tj^\mu(x)\psi(0)|p\rangle , \tag{2.2}$$

where $u(p)$ is the momentum-space wave function for the one-particle state $|p\rangle$; for plane-wave solutions to the free Dirac theory, $u(p,x)\equiv e^{-ipx}u(p)$ with $(\not p - m)\,u(p)=0$. Since j^μ is conserved, G^μ obeys a Ward identity which is given by the equal-time commutator of ψ with j^0:

$$q_\mu G^\mu(p,q)u(p)=\int d^2x\, e^{+iqx}\delta(x^0)\langle 0|[j^0(x),\psi(0)]|p\rangle . \tag{2.3}$$

The models we study possess conserved U(1) currents whose zero component has an equal-time commutator with the fundamental field given by

$$\delta(x^0)[j^0(x),\psi(0)]=\delta^2(x)\psi(0) . \tag{2.4}$$

In this case, the Ward identity of Eq. (2.3) reduces to

$$q_\mu G^\mu(p,q)u(p)=\langle 0|\psi(0)|p\rangle . \tag{2.5}$$

Finally, it is assumed that G^μ has the usual analyticity properties so that dispersion relations for its form factors can be derived. The dispersion relations have the form

$$\gamma^\mu+\Gamma^\mu(p,q;p'^2=m^2) = \frac{-1}{\pi}\int_{m^+}^{\infty} dW \frac{\mathrm{Im}\Gamma^\mu(p,q;p'^2=W^2)}{W-m} \,. \tag{2.6}$$

Here the vertex function Γ^μ is defined by

$$\Gamma^\mu(p,q)u(p)\equiv(\not{p}'-m)G^\mu(p,q)u(p)\,, \tag{2.7}$$

with $p'^\mu=(p-q)^\mu$ and $m\geq 0$ being the mass of the fundamental Fermi field. Equation (2.6) is to be understood as dispersion relations for the form factors which are the scalar functions of q^2 and $W^2=p'^2$ obtained after Γ^μ is Lorentz decomposed. The matrix γ^μ is necessary so as to satisfy the Ward identity of Eq. (2.5). In the remainder of this section, a bound will be derived for the imaginary part of the vertex function Γ^μ.

Taking the imaginary part of Γ^μ, we find

$$\begin{aligned}2i[\mathrm{Im}\Gamma^\mu(p,q)]u(p)&=[\Gamma^\mu(p,q)-\Gamma^{\mu*}(p,q)]u(p)\\ &=i(\not{p}'-m)\int d^2x\, e^{+iqx}[\langle 0\,|\,j^\mu(x)\psi(0)\,|\,p\rangle+\langle 0\,|\,\psi(0)j^\mu(x)\,|\,p\rangle]\,,\end{aligned} \tag{2.8}$$

where we have used $\gamma_\mu^*=\gamma_\mu$ (see Appendix A for conventions and useful formulas) and the product ($\mathscr{P}\mathscr{T}$) of parity ($\mathscr{P}$) and time-reversal ($\mathscr{T}$) invariance of the theory to show that

$$\begin{aligned}&u^*(p)=u(p)\,,\\ &\langle 0\,|\,j^\mu(x)\psi(0)\,|\,p\rangle^*=\langle 0\,|\,j^\mu(-x)\psi(0)\,|\,p\rangle\,,\\ &\langle 0\,|\,\psi(0)j^\mu(x)\,|\,p\rangle^*=\langle 0\,|\,\psi(0)j^\mu(-x)\,|\,p\rangle\,.\end{aligned} \tag{2.9}$$

Inserting a complete set of states, $1=\sum_n |\,n\rangle\langle n\,|$, in the right-hand side and restricting the momentum q to be spacelike, $q^2<0$, so that the first term vanishes, we secure

$$\mathrm{Im}\Gamma^\mu(p,q)u(p)=\tfrac{1}{2}(\not{p}'-m)\sum_n(2\pi)^2\delta^2(p'-p_n)\langle 0\,|\,\psi(0)\,|\,n\rangle\langle n\,|\,j^\mu(0)\,|\,p\rangle\,. \tag{2.10}$$

As in Refs. 7 and 8, the imaginary part of Γ can be bounded by means of the Schwarz inequality. Defining

$$a_n\equiv(\not{p}'-m)\langle 0\,|\,\psi(0)\,|\,n\rangle\,,\quad b_n\equiv\langle n\,|\,j^\mu(0)\,|\,p\rangle\,, \tag{2.11}$$

and

$$\sum_n{}'\equiv\sum_n(2\pi)^2\delta^2(p'-p_n)\,,$$

we have

$$\mathrm{Im}\Gamma^\mu(p,q)u(p)=\tfrac{1}{2}\sum_n{}'a_nb_n^\mu\,. \tag{2.12}$$

Applying the Schwarz inequality

$$\left|\sum_n{}'a_nb_n^\mu\right|^2\leq\sum_n{}'\,|\,a_n\,|^2\sum_m{}'\,|\,b_m^\mu\,|^2 \tag{2.13}$$

yields

$$|\,\mathrm{Im}\Gamma^\mu(p,q)u(p)\,|^2\leq\left[\tfrac{1}{4}\sum_n(2\pi)^2\delta^2(p'-p_n)\,|\,a_n\,|^2\right]\left[\sum_m(2\pi)^2\delta^2(p'-p_m)\,|\,b_m^\mu\,|^2\right]\,, \tag{2.14}$$

where

$$\begin{aligned}\sum_m(2\pi)^2\delta^2(p'-p_m)\,|\,b_m^\mu\,|^2&=\sum_m(2\pi)^2\delta^2(p'-p_m)\langle p\,|\,j^\mu(0)\,|\,n\rangle\langle n\,|\,j^\mu(0)\,|\,p\rangle\\ &=\int d^2x\, e^{-iqx}\langle p\,|\,j^\mu(x)j^\mu(0)\,|\,p\rangle\quad\text{(no sum on }\mu)\end{aligned} \tag{2.15}$$

is the current-current single-particle matrix element and

$$\sum_n (2\pi)^2\delta^2(p'-p_n)\,|\,a_n\,|^2=\sum_n (2\pi)^2\delta^2(p'-p_n)(\not{p}'-m)\langle 0\,|\,\psi(0)\,|\,n\,\rangle\langle n\,|\,\bar{\psi}(0)\,|\,0\rangle(\not{p}'-m)\gamma^0 \ . \qquad (2.16)$$

For certain models, such as the Gross-Neveu model, we can further simplify this to obtain the Fourier transform of the two-point Wightman function,

$$\sum_n (2\pi)^2\delta^2(\not{p}'-p_n)\,|\,a_n\,|^2=(\not{p}'-m)\int d^2x\, e^{+ip'x}\langle 0\,|\,\psi(x)\bar{\psi}(0)\,|\,\rangle\rangle(\not{p}'-m)\gamma^0 \ . \qquad (2.17)$$

The fundamental inequality for the form factors can thus be written as

$$|\,[\gamma^\mu+\Gamma^\mu(p,q;p'^2=m^2)]u(p)\,|\le\frac{1}{2\pi}\int_{m^+}^{\infty}\frac{dW}{W-m}\left[\sum_n (2\pi)^2\delta^2(p'-p_n)\,|\,a_n\,|^2\right]^{1/2}$$
$$\times\left[\int d^2x\, e^{-iqx}\langle p\,|\,j^\mu(x)j^\mu(0)\,|\,p\,\rangle\right]^{1/2} . \qquad (2.18)$$

It is through this bound that the original assumptions can then be checked.

III. THE GROSS-NEVEU MODEL

The first model in which we test the inequality (2.18) is the SU(N) Gross-Neveu model[3] to leading order in a $1/N$ perturbation expansion. The model consists of a Fermi field ψ_a, $a=1,\ldots,N$, transforming as the fundamental representation and an auxiliary singlet scalar field σ, interacting with ψ_a via a Yukawa potential. The Lagrangian is given by

$$\mathscr{L}=\frac{i}{2}\bar{\psi}_a\overleftrightarrow{\not{\partial}}\psi_a-\frac{N}{2g}\sigma^2-\sigma\bar{\psi}_a\psi_a \ . \qquad (3.1)$$

On analysis of this model, one finds that a fermion mass term, $m\bar{\psi}_a\psi_a$, is generated when one operates in the correct vacuum which is defined by $\langle 0\,|\,\sigma\,|\,0\rangle=0$. Consistent with this, the renormalized Green's functions are calculated within the Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) momentum-space renormalization scheme. The $1/N$ Feynman rules are given in Fig. 1, where for the σ-field propagator the single-fermion-loop contributions to the self-energy have already been summed by the Schwinger-Dyson equation and should no longer be included. The fermion loop yields the function

$$\Delta(q^2)=\frac{1}{2}\left[1-\frac{4m^2}{q^2}\right]^{1/2}\ln\left[\frac{(1-4m^2/q^2)+1}{(1-4m^2/q^2)^{1/2}-1}\right] , \qquad (3.2)$$

so that $\Delta(0)=1$.

The renormalized U(1) current $j^\mu(x)$ is defined by means of Zimmermann's normal-product algorithm, symbolically written as

$$j^\mu(x)\equiv N_1[\bar{\psi}_a\gamma^\mu\psi_a(x)] \ . \qquad (3.3)$$

Thus to leading order in $1/N$, we find

$$G^\mu(p,q)u(p)=i\int d^2x\, e^{+iqx}\langle 0\,|\,Tj^\mu(x)\psi(0)\,|\,p\,\rangle$$
$$=\int d^2x\, d^2y\, e^{+iqx-ipy}\langle 0\,|\,TN_1[\bar{\psi}_a\gamma^\mu\psi_a(x)]\psi(0)\bar{\psi}(y)\,|\,0\rangle(\not{p}-m)u(p) \ . \qquad (3.4)$$

The two graphical contributions are shown in Fig. 2. Applying the Feynman rules and momentum subtractions, we find

$$G^\mu(p,q)u(p)=\frac{i}{\not{p}'-m}i\gamma^\mu u(p)-iN\int\frac{d^2k}{(2\pi)^2}(1-t_q^0)\mathrm{Tr}\left[\frac{i}{\not{k}-m}\gamma^\mu\frac{i}{\not{q}+\not{k}-m}\right]D(q^2)\frac{i}{\not{p}'-m}u(p)$$
$$=\frac{i}{\not{p}'-m}i\gamma^\mu u(p) \ . \qquad (3.5)$$

The loop integral is finite and zero even without the zero-order momentum Taylor subtraction t_q^0. In addition, we find the simple result to leading order in $1/N$ for the vertex function Γ^{μ},

$$\Gamma^{\mu}(p,q)u(p) \equiv (\not p' - m)G(p,q)u(p) = -\gamma^{\mu}u(p) \; . \tag{3.6}$$

We see immediately that the left-hand side of Eq. (2.18) vanishes and so the fundamental inequality is trivially satisfied:

$$0 \leq \frac{1}{2\pi}\int_{m^+}^{\infty}\frac{dW}{W-m}\left[\sum_n (2\pi)^2\delta^2(p'-p_n)\,|\,a_n\,|^2\right]^{1/2}\left[\int d^2x\, e^{-iqx}\langle p\,|\,j^{\mu}(x)j^{\mu}(0)\,|\,p\,\rangle\right]^{1/2} . \tag{3.7}$$

Thus, as one expected from the calculation of the exact S matrix[4] and exact form factors[9] in the Gross-Neveu model where the fundamental fields are shown not be be confined, West's use of the fundamental inequality equation (2.18) does not lead to a contradiction. The assumptions of Sec. II, in particular the assumption of nonzero overlap $\langle 0\,|\,\psi\,|\,p\,\rangle \neq 0$, for the fundamental Fermi fields and the one-particle state, are consistent. When West applied the inequality to four-dimensional QCD, he found a contradiction due to two critical properties of the model. First, for asymptotically large spacelike q^2, asymptotic freedom implied that the right-hand side of Eq. (2.18) went to zero. Similarly, the Gross-Neveu model is asymptotically free $[\beta(g) = -g^2/\pi]$. Thus using the renormalization group and light-cone expansion the right-hand side of (3.7) also goes to zero as q^2 approaches $-\infty$. Of course, in leading order in $1/N$, $\mathrm{Im}\Gamma^{\mu} = 0$ for all q^2 spacelike. Thus as far as this first critical property is concerned, QCD_4 and the Gross-Neveu model have the same behavior. The second critical property West used in obtaining a contradiction for QCD_4 was that for $q^2 \to -\infty$, the elastic form factor contained in $\Gamma^{\mu}(p,q;p'^2 = m^2)$ on the left-hand side of the dispersion relation (2.18) vanished. This results in the contradiction $(1 \leq 0)$ and hence leads to the conclusion that the set of initial assumptions are mutually inconsistent. West claims it is the existence of the single fermion state which is erroneous, thus obtaining quark confinement. The point is that this second property, the vanishing of the elastic form factor for asymptotic q^2, is an essential ingredient for obtaining a violation of the fundamental inequality. In nonasymptotically free QED_4 (Ref. 5) and (to a less well-established degree) in QCD_4,[2] this exponential damping results from a summing up of zero-mass gauge-field exchanges in the Feynman-diagram expansion for the vertex function. In the Gross-Neveu model, there are no massless fields to exponentiate and so the leading-order contribution to the vertex function remains the Born term γ^{μ}. Hence we see that two crucial properties are needed in order to obtain a contradiction by means of West's method; asymptotic freedom and asymptotic vanishing of the elastic form factor.

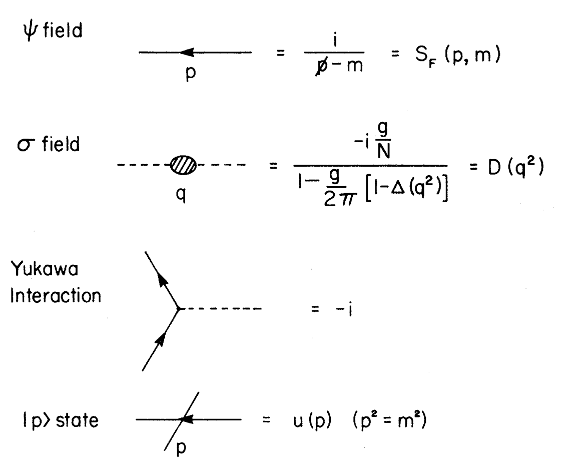

FIG. 1. $1/N$ Feynman rules.

IV. THE SCHWINGER MODEL

We next test the assumptions of Sec. II in a model which exhibits confinement of the fundamental Fermi fields—the Schwinger model of two-dimensional massless quantum electrodynam-

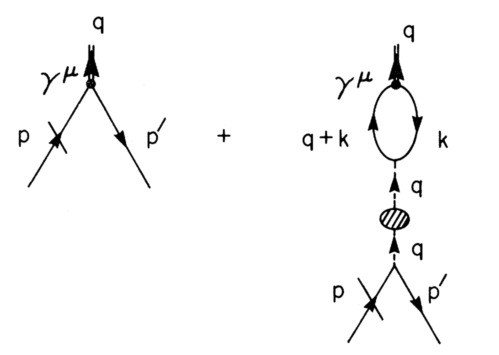

FIG. 2. Graphical contributions to $G^{\mu}(p,q)u(p)$ to leading order in $1/N$.

ics.[6] We first briefly recall the exact solution for the vertex function. The model is described by a (electron) Fermi field ψ and a (photon) vector field A_μ whose interaction is governed by the Lagrangian

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{2}\bar{\psi}\overleftrightarrow{\partial}\psi+ej^\mu A_\mu \ , \tag{4.1}$$

where the covariant field strength tensor is $F^{\mu\nu}=\partial^\mu A^\mu-\partial^\nu A^\mu$. The current is defined by Schwinger's symmetric limit as

$$j^\mu(x)\equiv\text{symm}\lim_{\epsilon\to 0}\bar{\psi}(x+\epsilon)\gamma^\mu\psi(x-\epsilon) \ , \tag{4.2}$$

where ϵ^μ is purely spacelike so that $\epsilon^\mu=(0,\epsilon)$. Following Brown[10] we work in the Coulomb gauge, defined by $n_\mu A^\mu=0$ with $n^\mu=(0,1)$. With this constraint, we can easily solve Maxwell's equations

$$\partial_\nu F^{\mu\nu}(x)=ej^\mu(x) \ , \tag{4.3}$$

with the help of the one-dimensional Green's function $\mathscr{D}(x)$ defined by

$$\frac{d^2}{dx^2}\mathscr{D}(x-y)=\delta(x-y) \ . \tag{4.4}$$

Using a principal-value boundary condition, this function takes the form

$$\begin{aligned}\mathscr{D}(x)&=\tfrac{1}{2}\,|\,x\,|\\ &=-\tfrac{1}{2}\int\frac{dk}{(2\pi)}e^{ikx}\\ &\quad\times\left[\frac{1}{(k+i\epsilon)^2}+\frac{1}{(k-i\epsilon)^2}\right] \ .\end{aligned} \tag{4.5}$$

Thus

$$F^{01}(x)=\frac{\partial}{\partial x^1}\int_{-\infty}^{+\infty}dy^1\mathscr{D}(x^1-y^1)ej^0(x^0,y^1) \tag{4.6}$$

and

$$A^0(x)=\int_{-\infty}^{+\infty}dy^1\mathscr{D}(x^1-y^1)ej^0(x^0,y^1) \ . \tag{4.7}$$

From the commutation relations and equations of motion for the Fermi fields, it can be shown that the electromagnetic current j^μ is conserved,

$$\partial_\mu j^\mu(x)=0 \ , \tag{4.8}$$

while the axial-vector current

$$\begin{aligned}j_5^\mu(x)&\equiv\text{symm}\lim_{\epsilon\to 0}\bar{\psi}(x-\epsilon)\gamma^\mu\gamma_5(x+\epsilon)\\ &=-\epsilon^{\mu\nu}j_\mu(x)\end{aligned} \tag{4.9}$$

obeys the anomalous conservation equation

$$\begin{aligned}\partial_\mu j_5^\mu(x)&=-\tfrac{1}{2}\frac{e}{\pi}\epsilon_{\mu\nu}F^{\mu\nu}(x)\\ &=\frac{e}{\pi}F^{01}(x) \ .\end{aligned} \tag{4.10}$$

These imply that the current $j^\mu(x)$ is a free massive field with mass $e/\sqrt{\pi}$ and satisfies the massive Klein-Gordon equation

$$\left[\partial^2+\frac{e^2}{\pi}\right]j^\mu(x)=0 \ . \tag{4.11}$$

Thus we are able to reduce any time-ordered product of operators involving the current and Fermi fields to just the time-ordered product of the Fermi fields. In particular, the time-ordered product of the electromagnetic current and one Fermi field we find

$$Tj^\mu(x)\psi(0)=\left\{-\partial_\nu\Delta_+\left[x;\frac{e^2}{\pi}\right]\gamma^\nu\gamma^\mu+\frac{e^2}{\pi}n^\mu\left[\int_{-\infty}^{+\infty}du^1\Delta_+\left[x^0,x^1-u^1;\frac{e^2}{\pi}\right]n_\mu\partial^\nu\mathscr{D}(u^1)\right]\right\}\psi(0) \ , \tag{4.12}$$

where Δ_+ is the propagator for a massive scalar field satisfying

$$\left[\partial_x{}^2+\frac{e^2}{\pi}\right]\Delta_+\left[x-y;\frac{e^2}{\pi}\right]=\delta^2(x-y) \tag{4.13}$$

and given explicitly by

$$\Delta_+\left[x;\frac{e^2}{\pi}\right]=\int_{-\infty}^{+\infty}\frac{d^2k}{(2\pi)^2}e^{-ikx}\frac{-1}{k^2-e^2/\pi+i\epsilon} \ . \tag{4.14}$$

We next take the vacuum—one-particle-state matrix element of this operator to obtain the vertex function

$$\Gamma^{\mu}(p,q)u(p)\equiv\not p'G(p,q)u(p)$$
$$\equiv\not p'\int d^2x\, e^{+iqx}\langle 0\,|\,Tj^{\mu}(x)\psi(0)\,|\,p\,\rangle$$
$$=-\left[\not p'\not q\gamma^{\mu}+\frac{e^2}{\pi}\not p'n^{\mu}n\cdot q\widetilde{\mathscr{D}}(q^1)\right]\widetilde{\Delta}_+\left[q;\frac{e^2}{\pi}\right]\langle 0\,|\,\psi(0)\,|\,p\,\rangle\ . \tag{4.15}$$

The vertex function obeys the Ward identity (2.5)

$$q_{\mu}\Gamma^{\mu}(p,q)u(p)=\not p'\langle 0\,|\,\psi(0)\,|\,p\,\rangle \tag{4.16}$$

explicitly.

Taking the asymptotic $q^2\to-\infty$ limit we find

$$\Gamma^{\mu}(p,q)u(p)=-\gamma^{\mu}\langle 0\,|\,\psi(0)\,|\,p\,\rangle\ , \tag{4.17}$$

which is the Born term. Hence the fundamental inequality equation (2.18) is *not* violated even though the fundamental Fermi fields are confined. As long as the elastic form factor reduces to the Born term in the asymptotic limit the fundamental inequality is trivially satisfied. The reason for this behavior is the superrenormalizability of two-dimensional QED. Four-dimensional gauge theories are renormalizable models so that in each order of perturbation theory the form factor has a dependence that goes as $[g\ln(-q^2/\mu^2)]^n$ (where μ is the renormalization point). As seen in Refs. 2 and 5 the perturbation series exponentiates to yield $e^{-g^2\ln^2(-q^2/\mu^2)}$ which goes to 0 as $q^2\to-\infty$, leaving 1 on the left-hand side of the fundamental inequality. However, in superrenormalizable theories each term in a perturbation expansion for the form factor goes as $\{(g^2/q^2)[\ln(-q^2/\mu^2)]^a\}^n$ for some constant a. Thus even if the series exponentiates, one finds

$$\exp\{(\pm g^2/q^2)[\ln(-q^2/\mu^2)]^a\}$$

which goes to 1 as $q^2\to-\infty$. In the Schwinger model the series sums to be Eq. (4.15) leaving only the Born term in the asymptotic limit. Thus the vertex function will always go to the Born term for a superrenormalizable theory making the fundamental inequality trivially satisfied.

ACKNOWLEDGMENTS

We would like to thank N. Nakagawa for numerous enjoyable discussions. Also we would especially like to thank G. B. West for many lengthy and quite fruitful conversations as well as for keeping us abreast of the latest developments in his work. This work was supported in part by the United States Department of Energy.

APPENDIX A: NOTATION, CONVENTIONS, AND USEFUL FORMULAS

The metric tensor $g_{\mu\nu}$ is defined so that its only nonvanishing elements are $g_{00}=1=-g_{11}$. The Levi-Civita tensor $\epsilon^{\mu\nu}$ is defined by $\epsilon^{\mu\nu}=-\epsilon^{\nu\mu}$ with $\epsilon^{01}=-\epsilon_{01}=+1$. Thus

$$\epsilon_{\mu\nu}\epsilon^{\lambda\rho}=\delta_{\mu}^{\rho}\delta_{\nu}^{\lambda}-\delta_{\mu}^{\lambda}\delta_{\nu}^{\rho}\ . \tag{A1}$$

The two-component Dirac matrices γ^{μ} are defined in terms of the Pauli matrices σ^i,

$$\sigma^1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma^2=\begin{bmatrix}0&-i\\+i&0\end{bmatrix},\quad \sigma^3=\begin{bmatrix}1&0\\0&-1\end{bmatrix} \tag{A2}$$

by

$$\gamma^0=\sigma^3,\quad \gamma^1=i\sigma^2,\quad \gamma_5=\gamma^0\gamma^1=\sigma^1 \tag{A3}$$

so that

$$\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu},\quad \gamma^0\gamma_{\mu}^T\gamma^0=\gamma_{\mu},\quad \gamma_{\mu}^{*}=\gamma_{\mu}\ , \tag{A4}$$

where the superscript T signifies transposition, and the superscript $*$ complex conjugation. The two-component, complex Dirac spinors $\psi_a, a=1,2$, have an inner product defined by

$$\bar\psi_a\chi_a\equiv\psi_a^{\dagger}\gamma_{ab}^0\chi_b \tag{A5}$$

with the adjoint spinor $\bar\psi_a$ defined as

$$\bar\psi_a=(\psi^{\dagger}\gamma^0)_a=\psi_b^{\dagger}\gamma_{ba}^0\ . \tag{A6}$$

The discrete symmetry operations of parity, charge conjugation, and time reversal are represented by the operators $\mathscr{P}$, $\mathscr{C}$, and $\mathscr{T}$, respectively. Under a parity transformation, the spinor field ψ transforms as

$$\mathscr{P}^{-1}\psi_a(x)\mathscr{P}=L(\mathscr{P})_{ab}\psi_b(x^P)\ , \tag{A7}$$

where

$$X^{P\mu}=P^{\mu}{}_{\nu}X^{\nu} \tag{A8}$$

with

$$P^{\mu}{}_{\nu}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{A9}$$

and $L(\mathscr{P})$ satisfies

$$L(\mathscr{P})\gamma^{\mu}L^{-1}(\mathscr{P})=P^{\mu}{}_{\nu}\gamma^{\nu}\,, \qquad L(\mathscr{P})^{\dagger}L(\mathscr{P})=1\,. \tag{A10}$$

In our choice of representation,

$$L(\mathscr{P})=\gamma^{0}\,. \tag{A11}$$

For charge conjugation,

$$C^{\dagger}\psi_{a}(x)\mathscr{C}=L(\mathscr{C})_{ab}\bar{\psi}_{b}(x)\,, \tag{A12}$$

where

$$L(\mathscr{C})^{-1}\gamma_{\mu}L(\mathscr{C})=-\gamma_{\mu}^{T}\,, \qquad L(\mathscr{C})^{\dagger}L(\mathscr{C})=1\,. \tag{A13}$$

In our choice of representation

$$L(\mathscr{C})=i\gamma^{1}\,. \tag{A14}$$

Under the antiunitary operation of time reversal, ψ transforms as

$$\mathscr{T}^{-1}\psi_{a}(x)\mathscr{T}=L(\mathscr{T})_{ab}\psi_{b}(x^{T})\,, \tag{A15}$$

where

$$x^{T\mu}=T^{\mu}{}_{\nu}x^{\nu}\,, \tag{A16}$$

with

$$T^{\mu}{}_{\nu}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \tag{A17}$$

and $L(\mathscr{T})$ satisfies

$$L(\mathscr{T})\gamma^{\mu}L(\mathscr{T})^{-1}=-T^{\mu}{}_{\nu}\gamma^{\nu *}\,, \qquad L(\mathscr{T})^{\dagger}L(\mathscr{T})=1\,. \tag{A18}$$

In our choice of representation

$$L(\mathscr{T})=\gamma^{0}\,. \tag{A19}$$

Thus the combined operation of parity and time reversal has a simple representation

$$(\mathscr{P}\mathscr{T})^{-1}\psi_{a}(x)(\mathscr{P}\mathscr{T})=L(\mathscr{P}\mathscr{T})_{ab}\psi_{b}(-x) \tag{A20}$$

with

$$L(\mathscr{P}\mathscr{T})=L(\mathscr{P})L(\mathscr{T})\,, \qquad L(\mathscr{P}\mathscr{T})^{-1}\gamma_{\mu}^{*}L(\mathscr{P}\mathscr{T})=\gamma_{\mu}\,. \tag{A21}$$

In our choice of representation, $L(\mathscr{P}\mathscr{T})$ is just the identity

$$L(\mathscr{P}\mathscr{T})=1\,. \tag{A22}$$

Thus

$$\begin{aligned}(\mathscr{P}\mathscr{T})^{-1}\psi(x)(\mathscr{P}\mathscr{T})&=\psi(-x)\,,\\ (\mathscr{P}\mathscr{T})^{-1}j^{\mu}(x)(\mathscr{P}\mathscr{T})&=j^{\mu}(-x)\,,\\ (\mathscr{P}\mathscr{T})\,|\,p\,\rangle&=|\,p\,\rangle\,,\\ (\mathscr{P}\mathscr{T})\,|\,0\rangle&=|\,0\rangle\end{aligned} \tag{A23}$$

and due to the antiunitary nature of $\mathscr{P}\mathscr{T}$

$$\langle\alpha\,|\,\beta\rangle^{*}=\langle\alpha^{PT}\,|\,\beta^{PT}\rangle\,, \tag{A24}$$

where $|\,\alpha^{PT}\rangle$ are the parity–time-reversal transformed states

$$|\,\alpha^{PT}\rangle=(\mathscr{P}\mathscr{T})\,|\,\alpha\rangle\,. \tag{A25}$$

[1]G. B. West, Phys. Rev. Lett. 46, 1365 (1981).

[2]J. M. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976).

[3]D. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

[4]A. B. Zamolodchikov and Al. B. Zamolodchikov, Phys. Lett. 72B, 481 (1978); M. Karowski and H. J. Thun, Nucl. Phys. B190, 61 (1981).

[5]C. P. Korthals Altes and E. de Rafael, Nucl. Phys. B106, 237 (1976).

[6]J. Schwinger, Phys. Rev. 128, 2425 (1962); in *Proceedings of the Seminar on Theoretical Physics, Trieste, 1962* (IAEA, Vienna, 1963), p. 89.

[7]G. B. West, Phys. Rev. Lett. 27, 762 (1971).

[8]F. Cooper and H. Pagels, Phys. Rev. D 2, 228 (1970).

[9]M. Karowski and P. Weisz, Nucl. Phys. B139, 455 (1978).

[10]L. S. Brown Nuovo Cimento 29, 617 (1963).