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A remnant of chiral symmetry on the lattice

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A new criterion for chiral symmetry in lattice theories of fermions is derived within a block-spin formalism. This "remnant" symmetry criterion properly incorporates the Adler Bell-Jackiw anomaly and avoids the fermion-doubling problem of other lattice fermion methods. Some obstacles to implementing this approach in the presence of fully dynamical gauge fields are discussed.

Since the introduction of lattice gauge theories, the implementation of lattice fermions has been recognized as somewhat problematic.[1] The most straightforward implementation results in an undesirable replication of fermionic species (the "doubling" problem), and methods[2] for eliminating the extra species have not proven entirely satisfactory. Recently, a connection between the fermion-doubling problem and the anomaly structure of the theory has been pointed out.[3] The regulation provided by the lattice allows no anomalies[4] in Ward identities and enforces this by realizing a naturally anomaly-cancelling spectrum. While the authors of Ref. 3 have emphasized the role of the color anomaly, a similar argument can be applied with perhaps greater generality to the anomalies of the *ungauged* fermion theory.[5] The conclusion is that some form of symmetry breaking must be present at the level of the lattice if the associated continuum spectrum is to have the desired anomaly structure. Otherwise, the lattice theory automatically leads to a continuum theory devoid of color or flavor anomalies, typically courtesy of unwanted anomaly-cancelling species.

Under these circumstances, how can a lattice theory serve to represent a continuum situation where the symmetry does not suffer explicit breaking? This is the case, for example, in strong-interaction dynamics, where our understanding of the physical spectrum depends on the breakdown of an approximately realized chiral symmetry. A tunable parameter has been introduced[1] to recover the symmetry limit in conjunction with spectrum calculations by strong-coupling or Monte Carlo techniques. The idea is that the spectrum may be tuned to exhibit some signal of the symmetry restoration, e.g., a vanishing Goldstone-boson mass. This state of affairs, while not unnatural, is slightly awkward in its requiring a proper and perhaps delicate parameter tuning to find the symmetry limit. In this article, we shall pursue a somewhat different approach, suggested by block-spin techniques. The strategy will be to exhibit a rigid criterion for what is meant by a chiral-symmetry limit within a framework where the symmetry is explicitly broken.

For the sake of simplicity, we shall introduce the basic idea in a familar theory of one Dirac fermion. Consider an action $A_I(\phi,\bar{\phi})$ possessing the invariance

$$A_I(e^{i\epsilon\gamma^5}\phi,\bar{\phi}e^{i\epsilon\gamma^5})=A_I(\phi,\bar{\phi}) \ . \tag{1}$$

$A_I(\phi,\bar{\phi})$ might be a continuum action but it can remain unspecified in what follows; for our purposes only the property (1) is of importance. We now define a new action $A(\psi,\bar{\psi})$ by means of the block-spin transformation[6]

$$e^{-A(\psi,\bar{\psi})}=\int_{\phi\bar{\phi}}\exp[-(\bar{\psi}_{n'}-\bar{\phi}_{n'})\alpha_{n'm'}(\psi_{m'}-\phi_{m'})-A_I(\phi,\bar{\phi})] \tag{2}$$

(we assume an implicit summation over Dirac indices and repeated primed spatial indices; our metric is Euclidean). The ϕ_n and $\bar{\phi}_n$ appearing in the first term of the exponent in Eq. (2) are block variables constructed in some well-prescribed way from those degrees of freedom of the original $A_I(\phi,\bar{\phi})$ in the vicinity of the new block lattice site n. The matrix α may have a nontrivial γ-matrix dependence but must in any case have a chirally noninvariant piece in order to ensure that the transformation (2) admits a nonsingular fixed-point action. In the following we shall consider only the simplest case where the matrix α is proportional to the unit matrix in Dirac space.

We would now like to determine what, if any, are the properties possessed by $A(\psi,\bar{\psi})$ by virtue of its having been constructed from an initially chirally invariant action $A_I(\phi,\bar{\phi})$. Under a global chiral transformation, we have

$$\exp[-A(e^{-i\epsilon\gamma^5}\psi,\bar{\psi}e^{-i\epsilon\gamma^5})]=\int_{\phi\bar{\phi}}\exp[-(\bar{\psi}e^{-i\epsilon\gamma^5}-\bar{\phi})\alpha(e^{-i\epsilon\gamma^5}\psi-\phi)-A_I(\phi,\bar{\phi})]$$
$$=\int_{\phi\bar{\phi}}\exp[-(\bar{\psi}-\bar{\phi})e^{-i\epsilon\gamma^5}\alpha e^{-i\epsilon\gamma^5}(\psi-\phi)-A_I(\phi,\bar{\phi})]\ , \tag{3}$$

where the second line follows from a change of variables $\phi\to e^{-i\epsilon\gamma^5}\phi$, $\bar{\phi}\to\bar{\phi}e^{-i\epsilon\gamma^5}$ together with the invariance (1) of $A_I(\phi,\bar{\phi})$. If $A_I(\phi,\bar{\phi})$ is quadratic in the fermion fields then so is $A(\psi,\bar{\psi})$, and we may write

$$A(\psi,\bar{\psi})\equiv\bar{\psi}h\psi=\bar{\psi}_{n'}h_{n'm'}\psi_{m'}\ . \tag{4}$$

Expanding Eq. (3) to first order in ϵ then results in

$$e^{-A(\psi,\bar{\psi})}[1+i\epsilon\bar{\psi}\{\gamma^5,h\}\psi]=\int_{\phi\bar{\phi}}[1+i\epsilon(\bar{\psi}-\bar{\phi})\{\gamma^5,\alpha\}(\psi-\phi)]\exp[-(\bar{\psi}-\bar{\phi})\alpha(\psi-\phi)-A_I(\phi,\bar{\phi})]$$
$$=\left[1-i\epsilon\frac{\partial}{\partial\psi}\alpha^{-1}\{\gamma^5,\alpha\}\alpha^{-1}\frac{\partial}{\partial\bar{\psi}}\right]\int_{\phi,\bar{\phi}}\exp[-(\bar{\psi}-\bar{\phi})\alpha(\psi-\phi)-A_I(\phi,\bar{\phi})]$$
$$=\left[1-i\epsilon\frac{\partial}{\partial\psi}\{\gamma^5,\alpha^{-1}\}\frac{\partial}{\partial\bar{\psi}}\right]e^{-A(\psi,\bar{\psi})}\ . \tag{5}$$

[An additional term $-i\epsilon\,\mathrm{Tr}\alpha^{-1}\{\gamma^5,\alpha\}$ has been eliminated from the right-hand side of (5) by $\mathrm{Tr}\gamma^5=0$.] The crucial element entering Eq. (5) is the possibility of reexpressing the ϕ dependence in terms of derivatives with respect to the new ψ variables[7] which then allows performing the $\phi,\bar{\phi}$ functional integration to recover $e^{-A(\psi,\bar{\psi})}$ on the right-hand side. By use of

$$h\psi e^{-\bar{\psi}h\psi}=-\frac{\partial}{\partial\bar{\psi}}e^{-\bar{\psi}h\psi}\ ,$$
$$\bar{\psi}he^{-\bar{\psi}h\psi}=\frac{\partial}{\partial\psi}e^{-\bar{\psi}h\psi}\ , \tag{6}$$

the terms linear in ϵ in Eq. (5) become

$$i\epsilon\bar{\psi}\{\gamma^5,h\}\psi e^{-\bar{\psi}h\psi}=i\epsilon\bar{\psi}h\{\gamma^5,\alpha^{-1}\}h\psi e^{-\bar{\psi}h\psi}\ . \tag{7}$$

The content of Eq. (7) is that the remnant of chiral symmetry which filters through the block-spin transformation (2) to $A(\psi,\bar{\psi})=\bar{\psi}h\psi$ is embodied in the relation

$$\{\gamma^5,h\}=h\{\gamma^5,\alpha^{-1}\}h=2h\gamma^5\alpha^{-1}h\ . \tag{8a}$$

In terms of the propagator $h^{-1}=\langle\psi\bar{\psi}\rangle$, this relation takes the form

$$\{\gamma^5,h^{-1}\}=2\gamma^5\alpha^{-1}\ . \tag{8b}$$

From the opening discussion, we recall that any lattice theory with an undoubled spectrum must have certain symmetries broken. The relation (8) provides the key to making precise what is meant by the symmetry limit of such symmetries. It need only be abstracted from the present derivation as follows: Any h, in particular a fixed-point h approached after many iterations of the block-spin transformation (2), is said to be chirally invariant if it satisfies (8). The remainder of this article is devoted to illustrating in an explicit calculation the utility of expressing lattice chiral symmetry in this manner.

The calculation we shall perform is an evaluation of the anomalous three-current expectation value $\langle J_\mu(x)J_\nu(0)\partial^\rho J^5_\rho(x)\rangle$ in the theory with action (4). In the continuum, this quantity is related to the invariant amplitude for $\pi^0\to2\gamma$ decay

$$T_{\mu\nu}(p,k)=\epsilon_{\mu\nu\alpha\beta}p^\alpha k^\beta T(k^2) \tag{9a}$$

(k is the π^0 momentum, p is a photon momentum) by

$$T_{\mu\nu}(p,k)=\frac{k^2-m_\pi{}^2}{f_\pi m_\pi{}^2}\int d^4x\int d^4z\, e^{ipx}e^{ikz}\langle J_\mu(x)J_\nu(0)\partial^\rho J^5_\rho(z)\rangle\ , \tag{9b}$$

where the π^0 field is taken to be $\phi(z)=(1/f_\pi m_\pi{}^2)\partial^\rho J_\rho^5(z)$. It follows from Eqs. (9a) and (9b) that $T(0)$ satisfies

$$\epsilon_{\mu\nu\alpha\beta}T(0)=\frac{1}{f_\pi}\int d^4x\int d^4z\, x_\alpha z_\beta\langle J_\mu(x)J_\nu(0)\partial^\rho J_\rho^5(z)\rangle \ . \tag{10}$$

To properly motivate the lattice discussion which follows, we shall briefly recall some essential aspects of the continuum short-distance analysis[8] of Eq. (10). If everything were sufficiently well defined, the algebraic identity

$$\begin{aligned}x_\alpha z_\beta\partial_z^\rho T_{\mu\nu\rho}(x,z)=& x_\alpha z_\nu(\partial_x^\rho+\partial_z^\rho)T_{\mu\rho\beta}(x,z)-x_\mu z_\nu\partial_x^\rho T_{\rho\alpha\beta}(x,z)+\partial_z^\rho x_\alpha z_\beta T_{\mu\nu\rho}(x,z)\\ &-(\partial_x^\rho+\partial_z^\rho)x_\alpha z_\nu T_{\mu\rho\beta}(x,z)+\partial_x^\rho x_\mu z_\nu T_{\rho\alpha\beta}(x,z)\end{aligned} \tag{11}$$

($\partial_z^\rho\equiv\partial/\partial z_\rho$) could be applied to $T_{\mu\nu\rho}(x,z)=\langle J_\mu(x)J_\nu(0)J_\rho(z)\rangle$ in the integrand of (10) to eliminate the divergence of the axial-vector current in favor of divergences of vector currents (which vanish) plus total divergences which would seem to yield only vanishing surface terms upon integration. This, then, would constitute a proof of the (false) Sutherland-Veltman theorem[9]: $T(0)=0$. As pointed out in Ref. 8, however, a careful definition of Eq. (10) requires short-distance cutoffs to exclude the singular points in the domain of integration which occur when any two currents coincide in space-time. The cutoff-dependent surface terms so generated have a finite limit as the short-distance cutoffs are taken to zero, giving the required nonvanishing of $T(0)$. The continuum perturbation-theory result, in substantial agreement with experiment, is[4]

$$T(0)=\frac{1}{4\pi^2 f_\pi} \ . \tag{12}$$

To perform a lattice analysis of (10), we must define the lattice vector and axial-vector currents $J_{n\mu}$ and $J_{n\mu}^5$. It is important that these currents transform properly under the lattice cubic symmetry group (discrete Lorentz symmetries) in order to ensure that only the pseudotensor structure of (10) survives to the continuum, where it is required by the usual Lorentz symmetries. Let us examine the vector current $J_{n\mu}$. It is associated with the link in the $\hat\mu$ direction from the site n so is conveniently interpreted as residing at $n+\hat\mu/2$. Under a parity transformation ($n\to -n$), we thus impose

$$J_{n\mu}\to -J_{-n-\hat\mu,\mu} \ .$$

Defining $\partial_\mu J_{n\mu}=J_{n\mu}-J_{n-\hat\mu,\mu}$, it then follows that

$$\partial_\mu J_{n\mu}\to -J_{-n-\hat\mu,\mu}+J_{-n,\mu}=\partial_\mu J_{-n,\mu} \ ,$$

as desired for a parity-even operator at n. Under the axis interchange symmetry $\hat\mu\leftrightarrow\hat\nu$, we must require

$$J_{n\mu}\to J_{n'\nu} \ ,$$

where n' is the vector n with μ and ν coordinates interchanged. The currents explicitly constructed below will be seen to have the above properties.

$\partial^\mu J_{n\mu}$ can be extracted from the formal prescription

$$A\to A+i\sum_{n'}\epsilon_{n'}\partial^\mu J_{n'\mu} \tag{13a}$$

under the infinitesimal local transformation

$$\psi_n\to\psi_n+i\epsilon_n\psi_n,\ \ \bar\psi_n\to\bar\psi_n-\bar\psi_n i\epsilon_n \ . \tag{13b}$$

Inserting (13b) into the action (4), we find that

$$\begin{aligned}\bar\psi h\psi\to\ &\bar\psi h\psi-i\epsilon_{n'}\bar\psi_{n'}h_{n'm'}\psi_{m'}\\ &+i\epsilon_{n'}\bar\psi_{m'}h_{m'n'}\psi_{n'} \ ,\end{aligned}$$

so (13a) implies

$$\partial^\mu J_{n\mu}=\bar\psi_{m'}h_{m'n}\psi_n-\bar\psi_n h_{nm'}\psi_{m'} \ . \tag{14}$$

To see that this behaves as expected for the total divergence of a conserved current, we note [by (4) and (6)] that

$$\partial^\mu J_{n\mu}e^{-A}=\left[\bar\psi_n\frac{\partial}{\partial\bar\psi_n}-\psi_n\frac{\partial}{\partial\psi_n}\right]e^{-A} \ . \tag{15a}$$

This means, after integration by parts, that

$$\sum_{n'}\langle R\,\partial^\mu J_{n'\mu}\rangle=\sum_{n'}\left\langle\left[\bar\psi_{n'}\frac{\partial}{\partial\bar\psi_{n'}}-\psi_{n'}\frac{\partial}{\partial\psi_{n'}}\right]R\right\rangle \tag{15b}$$

for any quantity R. The operator acting on R in this expression simply counts its total "baryonic" charge ($Q=\sum_{n'}J_{n'0}$), with each $\bar\psi$ or ψ in R counting $+1$ and -1, respectively. We thus recognize in (15b) the usual relation

$$\int d^4x \langle R(y)\partial_\mu J^\mu(x)\rangle$$
$$=\int d^4x \frac{\partial}{\partial x^\mu}\langle 0|TR(y)J^\mu(x)|0\rangle$$
$$=\int d^4x \{\langle 0|TR(y)\partial_\mu J^\mu(x)|0\rangle$$
$$+\delta(x^0-y^0)\langle 0|[J^0(x),R(y)]|0\rangle\}$$
$$=\langle 0|[Q,R(y)]|0\rangle .$$

Using translation invariance $h_{n-m}\equiv h_{nm}$, (14) may be reexpressed in the form

$$\partial^\mu J_{n\mu}=\bar\psi_{n+l'}h_{l'}\psi_n-\bar\psi_n h_{l'}\psi_{n-l'} , \quad (16)$$

which allows a most convenient construction of $J_{n\mu}$. For all l with $l_\mu \geq 0$, we let each of the

$$L=\frac{(|l_0|+|l_1|+|l_2|+|l_3|)!}{|l_0|!|l_1|!|l_2|!|l_3|!}$$

shortest-length paths from n to $n+l$ contribute $(1/L)\bar\psi_{n+m}h_l\psi_{n+m-l}$ to $J_{n\mu}$ for each link coming into $n+m$ from $n+m-\hat\mu$; for $l_\mu<0$, reflection symmetry then dictates a contribution $-(1/L)\bar\psi_{n+m}h_l\psi_{n+m-l}$ to $J_{n\mu}$ for each link going from $n+m$ to $n+m-\hat\mu$. $J_{n\mu}$ so defined satisfies (16) since each link gives a contribution

$$\frac{1}{L}(\bar\psi_{n+m}h_l\psi_{n+m-l}-\bar\psi_{n+m-\hat\mu}h_l\psi_{n+m-\hat\mu-l})$$

to $\partial_\mu J_{n\mu}$ and the contributions along each shortest-length path simply cancel in pairs except for the boundary terms

$$\frac{1}{L}(\bar\psi_{n+l}h_l\psi_n-\bar\psi_n h_l\psi_{n-l}) .$$

The sum over shortest-length paths, included to preserve the lattice rotational invariance, then cancels the $1/L$. Properties of $J_{n\mu}$ are most easily expressed in terms of the kernel K defined by

$$J_{n\mu}=\bar\psi_{n+l'}K_{\mu l'l''}\psi_{n+l''} . \quad (17)$$

Some of these properties which will later be useful are derived in the Appendix.

Extracting $\partial^\mu J^5_{n\mu}$ from (4) by the same canonical prescription requires a little more care than $\partial^\mu J_{n\mu}$ since

$$\psi\to e^{i\epsilon\gamma^5}\psi,\quad \bar\psi\to\bar\psi e^{i\epsilon\gamma^5} \quad (18)$$

is not assumed to be an exact global symmetry of (4). To proceed, we decompose

$$h=(h\gamma^5)\gamma^5=\tfrac{1}{2}[h,\gamma^5]\gamma^5+\tfrac{1}{2}\{h,\gamma^5\}\gamma^5$$

and use only the chirally invariant part $\frac{1}{2}[h,\gamma^5]\gamma^5$ to define $\partial^\mu J^5_{n\mu}$ via (13) and a local (18):

$$\partial^\mu J^5_{n\mu}=\tfrac{1}{2}\bar\psi_{m'}[h_{m'n},\gamma^5]\psi_n -\tfrac{1}{2}\bar\psi_n[h_{nm'},\gamma^5]\psi_{m'} . \quad (19)$$

Comparing (19) and (14), we see that $J^5_{n\mu}$ may be constructed just as was $J_{n\mu}$ above. The result is

$$J^5_{n\mu}=\bar\psi_{n+l'}K^5_{\mu l'l''}\psi_{n+l''} , \quad (20)$$

where K^5_μ is related to K_μ of (17) by

$$K^5_\mu=\tfrac{1}{2}[K_\mu,\gamma^5] . \quad (21)$$

Let us now assume that there is an explicit deviation from the chiral-symmetry limit (8),

$$\{h,\gamma^5\}=2h\gamma^5\alpha^{-1}h-2M^5 , \quad (22)$$

with the deviation parametrized by M^5, the analog of a continuum mass term. From Eq. (19), then, the divergence of the now only partially conserved axial-vector current is

$$\partial^\mu J^5_{n\mu}=\bar\psi_{m'}h_{m'n}\gamma^5\psi_n+\bar\psi_n\gamma^5 h_{nm'}\psi_{m'} -\tfrac{1}{2}\bar\psi\{\gamma^5,h\}\psi_n-\tfrac{1}{2}\bar\psi_n\{\gamma^5,h\}\psi$$
$$=\bar\psi_{m'}h_{m'n}\gamma^5\psi_n+\bar\psi_n\gamma^5 h_{nm'}\psi_{m'} -\bar\psi h\gamma^5\alpha^{-1}h\psi_n-\bar\psi_n h\gamma^5\alpha^{-1}h\psi+\phi_n , \quad (23)$$

where

$$\phi_n\equiv\bar\psi_{m'}M^5_{m'n}\psi_n+\bar\psi_n M^5_{nm'}\psi_{m'} \quad (24)$$

vanishes in the chiral-symmetry limit. The Fourier transform

$$\phi_p=\sum_n e^{ipn}\phi_n$$

satisfies

$$\phi_p=\int_{p'}[\bar\psi_{-p'}M^5(p')\psi_{p+p'} +\bar\psi_{p-p'}M^5(p')\psi_{p'}] , \quad (25)$$

where

$$M^5(p)=\sum_{l'}e^{ipl'}M^5_{l'} =h(p)\gamma^5\alpha^{-1}(p)h(p)-\tfrac{1}{2}\{h(p),\gamma^5\}$$

and

$$\int_{p'}\equiv(1/2\pi)\int_{-\pi}^{\pi}dp' .$$

To understand Eq. (25), we must digress momentarily to discuss the solution to (8b). A rather general solution with a symmetry-breaking mass term takes the form

$$h^{-1}(p)=\alpha^{-1}(p)+\sum_{\{j_\mu\}}\frac{\gamma^2(p+2\pi j)}{i(p+2\pi j)_\mu\gamma^\mu+m}\ . \tag{26}$$

$\gamma(p)$ is an arbitrary function of momentum and 2π periodicity in each momentum component is enforced by the sum over integers j_μ. As an example, the fixed-point h approached by iterating the block-spin transformation (2) on an initial undoubled action

$$A=4\left[\sum_n\bar\psi_n\psi_n-\kappa\sum_{n\mu}[\bar\psi_{n+\hat\mu}(1+\gamma^\mu)\psi_n+\bar\psi_{n-\hat\mu}(1-\gamma^\mu)\psi_n]\right] \tag{27}$$

has

$$\gamma(p)=\prod_{\nu=0}^{3}\left[\frac{\sin(p_\nu/2)}{p_\nu/2}\right] \tag{28}$$

when the block variables are constructed by averaging over hypercubes with sharp boundaries.[10] For a general γ normalized to $\gamma^2(0)=1$, Eq. (26) to leading order in m and p reduces to

$$h(p)\approx m+i\not p\ .$$

Consequently

$$M^5(p)\underset{p,m\ll 1}{\approx}-m\gamma^5$$

and up to corrections in higher orders of mass m and momenta p,p', the field ϕ of (24) is simply $-2m\bar\psi\gamma^5\psi$. Thus by the usual definition $\langle 0|\bar\psi\gamma_\mu\gamma^5\psi|\pi\rangle=2ik_\mu f_\pi$, implying $-2m\langle 0|\bar\psi\gamma^5\psi|\pi\rangle=2m_\pi{}^2f_\pi$, ϕ in the low-momentum limit is just $2m_\pi{}^2f_\pi$ times the π^0 field. Thus

$$2f_\pi\epsilon_{\mu\nu\alpha\beta}T(0)=\sum_{m'n'}m'_\alpha n'_\beta\langle J_{m'\mu}J_{0\nu}\phi_{n'}\rangle \tag{29}$$

is for our purposes the appropriate lattice version of Eq. (10). Before continuing, we should establish the relevance of (29) to the continuum result by showing that it is indeed dominated by its long-distance behavior. From (17) and (24), we note that the expectation value in (29) will contain three propagators h^{-1}. For m' and n' of $O(1/\text{mass})$, the three propagators each give a factor $(\text{mass})^3$, m' and n' each a factor 1/mass, and together with the explicit factor of mass from ϕ and the $(\text{mass})^{-8}$ phase-space factor, we find an overall contribution of $O(1)$. All other regions of the summation are easily seen to be suppressed by powers of mass. The region with m',n' both of $O(1)$, for example, contributes $O(\text{mass})$ due to the explicit factor of mass from ϕ. Equation (29) is thus dominated by $m',n'\sim O(1/\text{mass})$ and so should revert to the continuum result in the mass$\rightarrow$0 limit.

To proceed in analyzing (29), a lattice version of the continuum identity (11) is needed. A bit of care is in order since the lattice chain rule

$$\begin{aligned}\partial_\mu(f_ng_n)&=(\partial_\mu f_n)g_n+f_{n-\hat\mu}\partial_\mu g_n\\&=f_n\partial_\mu g_n+(\partial_\mu f_n)g_{n-\hat\mu}\end{aligned} \tag{30}$$

involves a shift in spatial argument. With a little perseverance, one arrives at

$$\begin{aligned}m_\alpha n_\beta\partial_n^\rho T_{\mu\nu\rho}(m,n)=&\ m_\alpha n_\nu\partial_{mn}^\rho T_{\mu\rho\beta}(m-\hat\nu,n-\hat\beta)-m_\mu n_\nu\partial_m^\rho T_{\rho\alpha\beta}(m+\hat\mu-\hat\nu,n-\hat\beta)\\&+\partial_n^\rho m_\alpha n_\beta T_{\mu\nu\rho}(m,n)+\partial_m^\rho m_\mu n_\nu T_{\rho\alpha\beta}(m+\hat\mu-\hat\nu,n-\hat\beta)\\&-[\partial_n^\rho m_\alpha n_\nu T_{\mu\rho\beta}(m+\hat\rho-\hat\nu,n+\hat\rho-\hat\beta)+\partial_m^\rho m_\alpha n_\nu T_{\mu\rho\beta}(m+\hat\rho-\hat\nu,n-\hat\beta)]\ ,\end{aligned} \tag{31}$$

where $\partial_{mn}^\rho T(m,n)\equiv T(m+\hat\rho,n+\hat\rho)-T(m,n)$ in the first term. The strategy from here in evaluating (29) will be to substitute for ϕ_n via (23), then reexpress the $\partial^\rho J_{n\rho}^5$ term via (31) and the remaining terms via (6). Typical manipulations involve identities such as those of (15). We shall be able to drop the total divergence terms generated from (31) since they produce only negligible surface terms as long as m' and n' are summed over regions of characteristic length much greater than the infrared cutoff 1/mass. We find, after integrating by parts,

$$2f_\pi\epsilon_{\mu\nu\alpha\beta}T(0)=\sum_{m'n'}\left\{m'_\alpha n'_\beta\left\langle\left[-\bar\psi_{n'}h\gamma^5\alpha^{-1}\frac{\partial}{\partial\bar\psi}-(\alpha^{-1}\gamma^5 h\psi_{n'})\frac{\partial}{\partial\psi}\right]J_{m'\mu}J_{0\nu}\right\rangle\right. \quad (32a)$$

$$+m'_\alpha n'_\beta\left\langle\left[\bar\psi_{n'}\gamma^5\frac{\partial}{\partial\bar\psi_{n'}}+(\gamma^5\psi_{n'})\frac{\partial}{\partial\psi_{n'}}\right]J_{m'\mu}J_{0\nu}\right\rangle \quad (32b)$$

$$-m'_\alpha n'_\nu\left\langle\left[\bar\psi_0\frac{\partial}{\partial\bar\psi_0}-\psi_0\frac{\partial}{\partial\psi_0}\right]J_{m'-\hat\nu,\mu}J^5_{n-\hat\beta,\beta}\right\rangle \quad (32c)$$

$$\left.-m'_\alpha n'_\nu\left\langle\left[\bar\psi_{m'+\hat\mu-\hat\nu}\frac{\partial}{\partial\bar\psi_{m'+\hat\mu-\hat\nu}}-\psi_{m'+\hat\mu-\hat\nu}\frac{\partial}{\partial\psi_{m'+\hat\mu-\hat\nu}}\right]J_{0\alpha}J^5_{n'-\hat\beta,\beta}\right\rangle\right\}, \quad (32d)$$

where (32c) and (32d) come from the $\partial^\rho J^5_{n\rho}$ part of (23), and (32a) and (32b) from the other two terms. Since the lattice regulates the short-distance singularities alluded to following Eq. (11), it should come as no surprise that (32c) and (32d) will prove to give vanishing contributions in the continuum limit. It turns out [see Eq. (36)] that (32b) does not contribute either, and only (32a) survives in the limit of interest.

To spare some algebraic tedium, we shall explicitly exhibit here the initial simplification of (32a) only; the rest of (32) is treated similarly and reappears in Eq. (35). Substituting (17) and taking the fermion derivatives in (32a) results in

$$\sum_{m'n'}-m'_\alpha n'_\beta\langle[\bar\psi_{n'}h\gamma^5\alpha^{-1}K_{\mu l'_1 l''_1}\psi_{m'+l''_1}+\bar\psi_{m'+l'_1}K_{\mu l'_1 l''_1}\alpha^{-1}\gamma^5 h\psi_{n'}]\bar\psi_{l'_2}K_{\nu l'_2 l''_2}\psi_{l''_2}$$
$$+\bar\psi_{m'+l'_1}K_{\mu l'_1 l''_1}\psi_{m+l''_1}[\bar\psi_{n'}h\gamma^5\alpha^{-1}K_{\nu l'_2 l''_2}\psi_{l''_2}+\bar\psi_{l'_2}K_{\nu l'_2 l''_2}\alpha^{-1}\gamma^5 h\psi_{n'}]\rangle\ . \quad (33)$$

Performing the functional integration in (33) then shows the two lines to be identical after a change of summation variables; they combine to produce

$$\sum_{m'n'}2m'_\alpha n'_\beta\mathrm{Tr}(K_{\mu l'_1 l''_1}h^{-1}_{m'+l''_1-l'_2}K_{\nu l'_2 l''_2}h^{-1}_{l''_2-n'}h_{n'-i'}\gamma^5\alpha^{-1}_{i'-m'-l'_1}$$
$$+K_{\mu l'_1 l''_1}\alpha^{-1}_{m'+l''_1-i'}\gamma^5 h_{i'-n'}h^{-1}_{n'-l'_2}K_{\nu l'_2 l''_2}h^{-1}_{l''_2-m'-l'_1})\ . \quad (34)$$

We now use the Fourier transforms

$$h^{\pm 1}_n=\int_p e^{ipn}h^{\pm 1}(p),\quad K_{\mu l_1 l_2}=\int_{p_1 p_2}e^{ip_1 l_1-ip_2 l_2}K_\mu(p_1,p_2)\ ,$$

together with the identity

$$\sum_{m'}m'_\alpha f(p')e^{ip'm'}=i\delta(p')\partial'_\alpha f(p')\quad(\partial'_\alpha\equiv\partial/\partial p'_\alpha)$$

to rewrite the four parts of (32) as

$$2f_\pi\epsilon_{\mu\nu\alpha\beta}T(0)=-2\partial'_\alpha\partial''_\beta\int_p\mathrm{Tr}\{K_\mu(p+p'',p+p')h^{-1}(p+p')K_\nu(p+p',p)h^{-1}(p)h(p+p'')\gamma^5\alpha^{-1}(p+p'')\}$$
$$-2\partial''_\alpha\partial'_\beta\int_p\mathrm{Tr}\{K_\mu(p+p'',p+p')\alpha^{-1}(p+p')\gamma^5 h(p+p')h^{-1}(p)K_\nu(p,p+p'')h^{-1}(p+p'')\} \quad (35a)$$

$$+2\partial'_\alpha\int_p\mathrm{Tr}\{\gamma^5[\partial_\beta K_\mu(p,p+p')]h^{-1}(p+p')K_\nu(p+p',p)h^{-1}(p)\} \quad (35b)$$

$$+\partial'_\nu\int_p\mathrm{Tr}\{[\partial_\alpha K_\mu(p+p',p)]h^{-1}(p)K^5_\beta(p,p'+p)h^{-1}(p+p')\}$$
$$+\partial'_\alpha\int_p\mathrm{Tr}\{[\partial_\nu K^5_\beta(p+p',p)]h^{-1}(p)K_\mu(p,p+p')h^{-1}(p+p')\} \quad (35c)$$

$$-\partial'_\nu\int_p\mathrm{Tr}\{[\partial_\mu K_\alpha(p+p',p)]h^{-1}(p)K^5_\beta(p,p+p')h^{-1}(p+p')$$
$$+[\partial_\mu K^5_\beta(p,p+p')]h^{-1}(p+p')K_\alpha(p+p',p)h^{-1}(p)\}\ . \quad (35d)$$

The terms in (35) are all understood to be evaluated at $p'=p''=0$.

In arriving at (35), we have not yet made use of its overall pseudotensor structure, i.e., the assured total antisymmetry in $\mu\nu\alpha\beta$ which allows us to drop any symmetric parts. It should first be noted that (35c) and (35d) are identical term by term after odd permutations of indices. Moreover, we recall that by construction $K_\beta^5=\frac{1}{2}[K_\beta,\gamma^5]$ but because of the γ^5, only the γ^5-odd parts of the two K_σ's and two h^{-1}'s in the trace need be retained (the trace of γ^5 and fewer than four γ matrices vanish) so we may write $K_\sigma^5=K_\sigma\gamma^5$ in (35c) and (35d). It thus emerges that (35b), (35c), and (35d) are all identical. Combined they give

$$6\partial'_\alpha\int_p \mathrm{Tr}\{\gamma^5[\partial_\beta K_\mu(p,p+p')]h^{-1}(p+p')K_\nu(p+p',p)h^{-1}(p)\}$$
$$=6\int_p \mathrm{Tr}\{\gamma^5[\partial_\beta\partial'_\alpha K_\mu(p,p+p')]h^{-1}(p+p')K_\nu(p+p',p)h^{-1}(p)\}$$
$$-6\int_p \mathrm{Tr}\{\gamma^5[\partial_\beta K_\mu(p,p)]h^{-1}(p)\partial'_\alpha[K_\nu(p,p+p')h^{-1}(p+p')]\}\ . \quad (36)$$

But by the results of the Appendix [Eq. (A6)],

$$K_\mu(p,p)=i\partial_\mu h(p),\ \ \partial'_\nu K_\mu(p,p+p')\,|_{p'=0}=\frac{i}{2}\partial_\mu\partial_\nu h(p)-\frac{\delta_{\mu\nu}}{2}h(p)\ , \quad (37)$$

the first and second terms on the right-hand side of (36) are symmetric in $\alpha\mu$ and $\beta\mu$, respectively, and so neither contributes to $\epsilon_{\mu\nu\alpha\beta}T(0)$.

This leaves (35a) as the only nonvanishing contribution. Applying (37) and keeping only the fully antisymmetric parts gives

$$2f_\pi\epsilon_{\mu\nu\alpha\beta}T(0)=2\int_p \mathrm{Tr}\{h_\mu(p)h_\alpha^{-1}(p)h_\nu(p)h^{-1}(p)[\partial_\beta(h(p)\gamma^5\alpha^{-1}(p))]\}$$
$$+2\int_p \mathrm{Tr}\{h_\mu(p)[\partial_\beta(\alpha^{-1}(p)\gamma^5h(p)]h^{-1}(p)h_\nu(p)h_\alpha^{-1}(p)\}\qquad [h_\mu^{\pm1}(p)\equiv\partial_\mu h^{\pm1}(p)]\ . \quad (38)$$

There is potentially a problem in taking the massless limit of Eq. (38) due to the singularity in the integrand at $p=0$. But for $m=0$, $h(p)\sim p\!\!\!/$ in the infrared, and we find that (38) behaves only as a harmless $\int d^4p/p^3$; the finite mass effects are thus small and the leading contribution to $T(0)$ may be safely calculated at $m=0$. Now we would also like to make use of the symmetry condition (8), but indiscriminate substitution of

$$2h\gamma^5\alpha^{-1}=\gamma^5+h\gamma^5h^{-1} \quad (39)$$

in (38) *would* produce a logarithmically divergent expression. To set up a more careful treatment, we first define the integrals in (38) to exclude the region $|p|<\delta$, for small but finite δ. Substitution of (8) may then be made in the remainder of the integration region and the resulting integral will turn out to be reducible to a $|p|=\delta$ surface term. Owing to the original convergence of (38), the necessarily nonsingular $\delta\to0$ limit of this surface term determines $T(0)$, up to negligible higher-order corrections in the mass.

We now proceed as outlined above, starting with the insertion of (39) into (38) supplied with $|p|=\delta$ cutoff. Using again the antisymmetry in $\mu\nu\alpha\beta$, along with the relation $h_\mu^{-1}=-h^{-1}h_\mu h^{-1}$, we obtain

$$2f_\pi\epsilon_{\mu\nu\alpha\beta}T(0)=\int_p \mathrm{Tr}\{h_\mu h_\alpha^{-1}h_\nu h^{-1}[h_\beta\gamma^5h^{-1}+h\gamma^5h_\beta^{-1}]\}+\int_p \mathrm{Tr}\{h_\mu[h_\beta^{-1}\gamma^5h+h^{-1}\gamma^5h_\beta]h^{-1}h_\nu h_\alpha^{-1}\}$$
$$=\int_p(-\mathrm{Tr}\gamma^5h_\mu^{-1}h_\alpha h_\nu^{-1}h_\beta+\mathrm{Tr}\gamma^5h_\beta^{-1}h_\mu h_\alpha^{-1}h_\nu+\mathrm{Tr}\gamma^5h_\nu h_\alpha^{-1}h_\mu h_\beta^{-1}-\mathrm{Tr}\gamma^5h_\beta h_\nu^{-1}h_\alpha h_\mu^{-1})$$
$$=4\int_p \mathrm{Tr}\{\gamma^5h_\mu^{-1}(p)h_\nu(p)h_\alpha^{-1}(p)h_\beta(p)\}\ . \quad (40)$$

The antisymmetric part of (40) is a total divergence, $\partial_\mu[h^{-1}h_\nu h_\alpha^{-1}h_\beta]$, whose integral may indeed be reexpressed as a $p=\delta$ surface term. To zeroth order in m, we take $h(p)=p\!\!\!/$ and find

$$2f_\pi\epsilon_{\mu\nu\alpha\beta}T(0)\approx4\int_{p=\delta}n_\mu\mathrm{Tr}\left[\gamma^5\frac{1}{p\!\!\!/}\gamma_\nu\left[\frac{\gamma_\alpha}{p^2}-2\frac{p\!\!\!/}{p^4}p_\alpha\right]\gamma_\beta\right]$$
$$=4\int_{p=\delta}4\epsilon_{\tau\nu\alpha\beta}\frac{n_\mu p^\tau}{p^4}=4\frac{1}{(2\pi)^4}2\pi^2p^3\epsilon_{\mu\nu\alpha\beta}\frac{p}{p^4}\bigg|_{p=\delta}=\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}+O(m)\ . \quad (41)$$

We thus conclude that $T(0)=1/4\pi^2 f_\pi$, in agreement with (12).

Given the evident utility[11] of the criterion (8) for expressing lattice chiral symmetry, the reader is undoubtedly wondering why we cannot now use our formalism in calculations of, say, the mass spectrum of QCD. The incorporation of dynamical gauge fields in our formalism involves taking a free-fermion relation like (8), making it locally gauge invariant, and then solving it for h. We have unfortunately not yet found either (8a) or (8b) to yield any tractable gauge-invariant solutions. For calculational purposes, we prefer an action, and hence an h, sufficiently local to justify retaining as few couplings as possible, ideally only nearest neighbor and maybe some next-nearest neighbor. But the symmetry criterion in terms of h, Eq. (8a), is difficult to solve directly because it is nonlinear. In terms of h^{-1}, on the other hand, the linear Eq. (8b) does yield solutions but they are nonlocal. We have not been able to exhibit an h^{-1} whose nonlocal strings of flux compactly invert to produce a (roughly) local h. Consequently, our remnant-symmetry formalism has not yet allowed calculations with gauge fields, even when the remnant symmetry is only an ungauged flavor symmetry. Finding a way to go ahead and actually gauge a symmetry present only in remnant form stands as a further challenge.

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APPENDIX

This appendix establishes the properties of K_μ quoted in Eq. (37).

With the definition (17) rewritten

$$J_{n\mu}=\sum_{ml}\bar{\psi}_{n+m}K_{\mu\,m,m-l}\psi_{n+m-l}\ , \tag{A1}$$

it follows from the construction of $J_{n\mu}$ described after (16) that

$$K_{\mu\,m,m-l}=L_{\mu\,ml}h_l\ , \tag{A2}$$

where $L_{\mu\,ml}$ is equal to $\mathrm{sgn}\,l_\mu$ times the fraction of shortest-length paths from 0 to l which pass through the link from $m-\hat{\mu}$ to m. By induction, it is easily shown that $L_{\mu\,ml}$ satisfies

$$\sum_m L_{\mu\,ml}=l_\mu\ , \tag{A3}$$

$$\sum_m m_\nu L_{\mu\,ml}=\frac{l_\mu(l_\nu+\delta_{\mu\nu})}{2}\ .$$

The Fourier transform

$$\begin{aligned}K_\mu(p,p+p')&=\sum_{l'l''}e^{-ipl'}e^{i(p+p')l''}K_{\mu l'l''}\\&=\sum_{l'm'}e^{-i(p+p')l'}h_{l'}e^{ip'm'}L_{\mu\,m'l'}\end{aligned} \tag{A4}$$

and its derivative with respect to p'_ν are only needed evaluated at $p'=0$. Thus we need only evaluate (A4) to leading order in p':

$$\begin{aligned}K_\mu(p,p+p')&\approx\sum_{l'm'}e^{-ipl'}h_{l'}(1-ip'l'+ip'm')L_{\mu\,m'l'}\\&=\sum_{l'}e^{-ipl'}h_{l'}l'_\mu\left[1-ip'l'+ip'_\alpha\frac{(l^\alpha+\delta^\alpha_\mu)}{2}\right]=i\partial_\mu\left[1+\frac{p'_\alpha\partial^\alpha}{2}+\frac{ip'_\mu}{2}\right]h(p)\end{aligned} \tag{A5}$$

to find

$$K_\mu(p,p)=i\partial_\mu h(p),\quad \partial'_\nu K_\mu(p,p+p')\,|_{p'=0}=\frac{i}{2}\partial_\mu\partial_\nu h(p)-\frac{\delta_{\mu\nu}}{2}h(p)\ . \tag{A6}$$

We notice that $\partial'_\nu K_\mu(p,p+p')\,|_{p'=0}$ is symmetric in μ and ν.

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[7]Our fermion derivatives always act from the left and satisfy $(\partial/\partial\psi_m)\,\psi_n=(\partial/\partial\bar{\psi}_m)\,\bar{\psi}_n=\delta_{mn}$. In the case of more than one flavor, the procedure would be complicated slightly by the necessity of adding multifermion terms (e.g., effective instanton interactions) to the action in order to split flavor-singlet from flavor-nonsinglet operators. For a purely quadratic action, the validity of the flavor-nonsinglet Ward identities would automatically imply the same for their flavor-singlet counterparts.

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[11]Our formalism can also be used, in the spirit following Eq. (15), to derive the usual soft-pion theorems for multioperator Green's functions. They differ from their continuum analogs by at most constant, non-pole terms which do not affect pion properties.