Evolution of radiating anisotropic spheres in general relativity

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A method used to study the evolution of radiating fluid spheres is extended to the case of anisotropic spheres. Explicit forms of the equations are written down for two models. One of the models is numerically integrated to display the difference between the isotropic and the anisotropic models for different degrees of anisotropy.

I. INTRODUCTION

In a recent paper[1] (hereafter referred to as HJR) a general method was developed by means of which one can construct different families of radiating fluid spheres from the exact static solution of the Einstein equations for a spherically symmetric distribution of matter. It is the purpose of this paper to extend the procedure mentioned above to the case of anisotropic matter.

Anisotropy could be introduced by the existence of a solid core, by the presence of a type-P superfluid, or by other physical phenomena. In this paper we do not discuss the mechanisms for inducing anisotropy. Rather we concentrate on the following two questions:

(a) What is the extent to which isotropic models differ from anisotropic ones?

(b) What is the evolution of the anisotropy in the process of contraction (or expansion) and radiation?

We recall that anisotropic matter has already been considered,[2-4] and it has been shown that some properties of anisotropic spheres may differ drastically from the properties of isotropic spheres. For example, the maximal red-shift at the surface may be larger than the value of the corresponding isotropic case, the stability under adiabatic contractions depend on the degree and the kind of anisotropy, etc.

This paper is organized as follows: In Sec. II we include the field equations, the general conditions, as well as the conventions used.

In Sec. III we describe the method to obtain the models following closely the program sketched in HJR. The surface equations and the equation of state for the tangential pressure are discussed in Sec. IV. Two examples are worked out explicitly in Secs. V and VI. Finally, the results are discussed in the last section. Some details of intermediate calculations are included in Appendices A and B.

II. THE FIELD EQUATIONS AND CONVENTIONS

As in HJR, our starting point is Bondi's approach to studying the evolution of gravitating spheres,[5] the only difference is that we shall consider anisotropic matter instead of perfect fluids.

Thus, let us consider a nonstatic distribution of matter which is spherically symmetric: In radiation coordinates[6] the metric takes the form

$$ds^2 = e^{2\beta}\left(\frac{V}{r}du^2 + 2\,du\,dr\right) - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,,$$

where $\beta \to 0$ as $r \to \infty$. Both β and V are functions of u and r. Here $u \equiv x^0$ is the timelike coordinate, $r \equiv x^1$ is a null coordinate, and $\theta, \phi \equiv x^2, x^3$ are the usual angle coordinates. In these coordinates the components of the energy-momentum tensor are distinguished by a bar and differentiation with respect to u and r denoted by suffixes 0 and 1, respectively.

Thus the Einstein equations are

$$-8\pi\overline{T}_{00} = -\frac{V_0 - 2\beta_0 V}{r^2} - \frac{V}{r^3}(e^{2\beta} - V_1 + 2\beta_1 V)\,, \tag{1}$$

$$-8\pi\overline{T}_{01} = -\frac{1}{r^2}(e^{2\beta} - V_1 + 2\beta_1 V)\,, \tag{2}$$

$$-8\pi\overline{T}_{11} = -4\beta_1/r\,, \tag{3}$$

$$-8\pi\overline{T}^2_2 = -8\pi\overline{T}^3_3 = -e^{-2\beta}\{2\beta_{01} - \tfrac{1}{2}r^{-2}[rV_{11} - 2\beta_1 V + 2r(\beta_{11}V + \beta_1 V_1)]\}\,. \tag{4}$$

Following Bondi, local Minkowski coordinates (t, x, y, z) are introduced by

$$dt = e^{\beta}[(V/r)^{1/2}du + (r/V)^{1/2}dr]\,, \qquad dx = e^{\beta}(r/V)^{1/2}dr\,, \quad dy = r\,d\theta\,, \quad dz = r\sin\theta\,d\phi\,. \tag{5}$$

Denoting the Minkowski components of the energy-momentum tensor by a caret,

$$\overline{T}_{00} = \hat{T}_{00}\left(\frac{e^{2\beta}V}{r}\right),$$

$$\overline{T}_{01} = (\hat{T}_{00} + \hat{T}_{01})e^{2\beta}\,,$$

$$\overline{T}_{11} = e^{2\beta}\left(\frac{r}{V}\right)(\hat{T}_{00} + \hat{T}_{11} + 2\hat{T}_{01})\,,$$

$$\overline{T}^{2}_{2} = \overline{T}^{3}_{3} = \hat{T}^{3}_{3} = \hat{T}^{2}_{2}\,.$$

Next we assume that for an observer moving relative to these coordinates with velocity ω in the radial direction, the space contains (a) anisotropic matter of density $\hat{\rho}$, radial pressure $\hat{P}$, and tangential pressure $\hat{P}_{\perp}$, (b) isotropic radiation of energy density $3\hat{\sigma}$, and (c) unpolarized radiation of energy density $\hat{\epsilon}$ traveling in the radial direction.

For this moving observer, the covariant energy tensor is

$$\begin{bmatrix} \hat{\rho} + 3\hat{\sigma} + \hat{\epsilon} & -\hat{\epsilon} & 0 & 0 \\ -\hat{\epsilon} & \hat{P} + \hat{\sigma} + \hat{\epsilon} & 0 & 0 \\ 0 & 0 & \hat{P}_{\perp} + \hat{\sigma} & 0 \\ 0 & 0 & 0 & \hat{P}_{\perp} + \hat{\sigma} \end{bmatrix}.$$

A Lorentz transformation readily shows that

$$\overline{T}_{00} = e^{2\beta}\frac{V}{r}\left(\frac{\rho + P\omega^2}{1 - \omega^2} + \epsilon\right), \tag{6}$$

$$\overline{T}_{01} = e^{2\beta}\frac{1}{1+\omega}(\rho - P\omega)\,, \tag{7}$$

$$\overline{T}_{11} = e^{2\beta}\frac{r}{V}\,\frac{1-\omega}{1+\omega}(\rho + P)\,, \tag{8}$$

$$\overline{T}^{2}_{2} = \overline{T}^{3}_{3} = -P_{\perp}\,, \tag{9}$$

where

$$\rho = \hat{\rho} + 3\hat{\sigma}\,, \quad P = \hat{P} + \hat{\sigma}\,,$$

$$P_{\perp} = \hat{P}_{\perp} + \hat{\sigma}\,, \quad \epsilon = \hat{\epsilon}\frac{1+\omega}{1-\omega}\,.$$

Note also that from (5) the velocity of matter in the radiative coordinate is given by

$$\frac{dr}{du} = \frac{V}{r}\frac{\omega}{1-\omega}\,. \tag{10}$$

Outside the matter, Eqs. (1)–(9) show that

$$\beta = 0\,, \quad V = r - 2\tilde{m}(u)\,, \quad \epsilon = -\frac{\tilde{m}_0}{4\pi r(r - 2\tilde{m})}\,, \tag{11}$$

where $\tilde{m}$ is a function of integration depending on u. This function is the same as the "mass aspect" defined in Ref. 6. In the static case it coincides with the Schwarzschild mass.

Inside the matter, the function $\tilde{m}(u)$ is generalized to $\tilde{m}(u,r)$ by putting everywhere

$$V = e^{2\beta}[r - 2\tilde{m}(u,r)]\,. \tag{12}$$

Substituting (12) into (1)–(4) and using (6)–(9), one obtains

$$\frac{\rho + P\omega^2}{1-\omega^2} + \epsilon = \frac{r}{V}e^{-2\beta}\overline{T}_{00} = \frac{1}{4\pi r(r-2\tilde{m})}\left(-\tilde{m}_0 e^{-2\beta} + \frac{r - 2\tilde{m}}{r}\tilde{m}_1\right), \tag{13}$$

$$\frac{\rho - P\omega}{1+\omega} = e^{-2\beta}\overline{T}_{01} = \frac{\tilde{m}_1}{4\pi r^2}\,, \tag{14}$$

$$\frac{1-\omega}{1+\omega}(\rho + P) = \frac{V}{r}e^{-2\beta}\overline{T}_{11} = \frac{r - 2\tilde{m}}{2\pi r^2}\beta_1\,, \tag{15}$$

$$P_{\perp} = -\overline{T}^{2}_{2} = -\frac{\beta_{01}e^{-2\beta}}{4} + \frac{1}{8\pi}\left(1 - \frac{2\tilde{m}}{r}\right)\left(2\beta_{11} + 4\beta_1^2 - \frac{\beta_1}{r}\right) + \frac{3\beta_1(1 - 2\tilde{m}_1) - \tilde{m}_{11}}{8\pi r}\,. \tag{16}$$

Now, unlike the isotropic case ($P_{\perp} \equiv P$), it is not sufficient to give $\beta(u,r)$ and $\tilde{m}(u,r)$ to calculate ω, P, $P_{\perp}$, ρ, and ϵ.

Indeed, an equation of state relating the tangential pressure with the other dynamical quantities should be given. We shall discuss this point with more detail in Sec. IV.

The choice of functions $\beta(u,r)$ and $\tilde{m}(u,r)$ will be restricted by the conditions $\rho \geq 0$, $-1 < \omega < 1$, $\tilde{m} < \frac{1}{2}r$, $\beta_1 > 0$ and as a boundary condition at the outer surface [say $r = a(u)$] of matter we have $P = 0$.

Two remarks are in order at this point.

(a) If one allows discontinuities of the radiation flux across the surface, then $P|_{r=a}$ is not necessarily zero.

(b) In order to consider models which could be of physical interest we do not impose regularity conditions such as $\beta_1 = O(r)$, $\tilde{m} = O(r^3)$ at $r = 0$. Also, we observe that negative pressures could appear at some stages of the evolution [even in the case of a perfect fluid ($P \equiv P_{\perp}$)], so we do not require $P \geq 0$.

Finally, since $\beta = 0$ for $r > a$, and β should be a continuous function across $r = a(u)$ we impose $\beta = 0$ at $r = a(u) - 0$. The same is not true for $\tilde{m}_1$ since there may be a discontinuity of density, so $\tilde{m}_1 \neq 0$ at $r = a(u) - 0$.

The next section is devoted to presenting a general procedure to construct models satisfying the condition just outlined.

III. THE MODELS

Let us start by noting that because of (14) and (15)

$$\tilde{m}=\int_0^r 4\pi r^2 \frac{\rho - P\omega}{1+\omega}dr\,, \tag{17}$$

$$\beta=\int_a^r \frac{2\pi r^2}{r-2\tilde{m}}\,\frac{1-\omega}{1+\omega}\,(\rho+P)dr\,. \tag{18}$$

Next, let us define the two auxiliary functions:

$$\tilde{\rho}\equiv\frac{\rho-\omega P}{1+\omega}\,, \tag{19}$$

$$\tilde{P}\equiv\frac{P-\omega\rho}{1+\omega}\,. \tag{20}$$

It can be seen at once that $\tilde{\rho}=\rho$ and $\tilde{P}=P$ at $r=0$ because $\omega=0$ at $r=0$. Also in the static case $\tilde{P}=P$, $\tilde{\rho}=\rho$. Finally, note that because of (2) and (20)

$$\tilde{m}=\int_0^r 4\pi r^2\tilde{\rho}\,dr\,, \tag{17$'$}$$

$$\beta=\int_a^r \frac{2\pi r^2}{r-2\tilde{m}}(\tilde{\rho}+\tilde{P})dr\,. \tag{18$'$}$$

Thus $\tilde{m}$ and β are expressed in terms of $\tilde{\rho}$ and $\tilde{P}$ in the nonstatic case in the same way they are in terms of ρ and P in the static case. These considerations, which are the same as for the perfect-fluid case, suggest the following procedures to obtain models of radiating contracting spheres:

(1) Take a static interior solution of the Einstein equations for anisotropic matter with spherical symmetry and with given

$$\rho_{st}=\rho(r)\,,\quad P_{st}=P(r)\,.$$

(2) Assume that the r dependence of $\tilde{P}$ and $\tilde{\rho}$ is the same as of P_{st} and ρ_{st}, but being careful with the boundary condition, which now reads, because of (20),

$$\tilde{P}_a=-\,\omega_a\tilde{\rho}_a\,.$$

From now on the suffix a indicates that the quantity is evaluated at the surface.

(3) With the r dependence of $\tilde{\rho}$ and $\tilde{P}$ and using (17′) and (18′) one gets $\tilde{m}$ and β up to three functions of u, which will be specified below.

(4) For these three functions one has two differential equations, one of which is (10) evaluated at $r=a$ and the other is

$$[T^{\mu}_{1;\,\mu}]_a=0\,.$$

Another u-dependent equation can be obtained evaluating (11) at $r=a+0$. Thus,

$$E(u)\equiv(4\pi r^2\epsilon]_{r=a+0}=\left[\frac{\tilde{m}_0}{(r-2\tilde{m})}\right]_{r=a+0}\,.$$

Thus, one has three differential equations for five unknown functions of u, which are the three functions appearing in the definition of β and $\tilde{m}$, E, and the tangential pressure evaluated at $r=a$.

(5) Given one of the functions, and specifying the equation of state relating the tangential pressure with the other dynamical variables, the system may be integrated for any particular initial data.

(6) Feeding back the result of integration in the expressions for β and $\tilde{m}$, these two functions are completely determined.

(7) Using (13)–(16) and the equation of state for the tangential pressure, ρ, $P_\perp$, P, ω, and ϵ may be found.

In the above we have outlined the general program for obtaining models. In the next section we shall investigate in detail the differential equations mentioned in point (4) and the equation of state for the tangential pressure.

IV. THE EQUATIONS AT THE SURFACE AND THE EQUATION OF STATE

A. The equation of state

The chief question when introducing anisotropy is how to chose the equation of state relating the tangential pressure with the other dynamical variables. The ideal approach would be to infer such an equation on physical grounds; unfortunately this is an extremely difficult task. So we shall follow the procedure introduced in Ref. 4. Namely, for the static case the following equation of state is proposed (for mathematical simplicity):

$$P_\perp-P=C\,(\rho+P)\frac{[m(r)+4\pi r^3P]}{(r-2m)}\,, \tag{21}$$

where C is a constant measuring the anisotropy. Now, we shall generalize Eq (2) for the radiating case as follows:

$$P_\perp-P=\frac{C(\tilde{P}+\tilde{\rho})}{(r-2\tilde{m})}(4\pi r^3\tilde{P}+\tilde{m})\,. \tag{21$'$}$$

Another possible gneralization could be

$$P_\perp-P=C\frac{(P+\rho)}{(r-2\tilde{m})}(4\pi r^3P+\tilde{m})\,.$$

However, it can be shown that such a possibility is incompatible with the boundary condition $P_a=0$.

B. The equations at the surface

Two of the equations at the surface are the same as in the isotropic case (actually they are the same for any model with spherically symmetric distribution of matter). Thus from (10) and (11) we get

$$\dot{a}=(1-2m/a)\frac{\omega_a}{1-\omega_a}\quad\text{with } \dot{a}\equiv\frac{da}{du}, \tag{22}$$

where $m\equiv\tilde{m}_a$ is the total mass. Scaling the radius a, the total mass m, and the timelike coordinate u by the initial mass $m(u=0)\equiv m(0)$,

$$A\equiv a/m(0),\quad M\equiv m/m(0),\quad u/m(0)\rightarrow u\,, \tag{23}$$

and defining

$$F\equiv 1-2M/A\,, \tag{24}$$

$$\Omega\equiv\frac{1}{1-\omega_a}, \tag{25}$$

Equation (22) can be written as

$$\dot{A}=F(\Omega-1)\,. \tag{26}$$

The second equation relates the total mass-loss rate with the energy flux through the surface. This can be obtained by evaluating Eq. (11) for $r=a+0$ and takes the form

$$\dot{M}=-FE\,. \tag{27}$$

Using (24), (26), and (27),

$$\frac{\dot{F}}{F}=\frac{2E+(1-F)(\Omega-1)}{A}. \tag{28}$$

The third equation at the surface will be obtained from the conservation equation $T^{\mu}_{1;\mu}=0$, evaluated at the surface.

After a lengthy and tedious calculation we get

$$\frac{\dot{F}}{F}+\frac{\dot{\Omega}}{\Omega}-\frac{\dot{\tilde{\rho}}_a}{\tilde{\rho}_a}+\frac{\Omega^2RF}{\tilde{\rho}_a}+(\Omega-1)\left[4\pi a\tilde{\rho}_a(2+h)-\frac{4\pi a\tilde{\rho}_a h}{\Omega}-\frac{h(1-F)}{2a}-\frac{(1+F)}{a}+\frac{\Omega F}{\tilde{\rho}_a}\tilde{\rho}_{1a}\right]=0 \tag{29}$$

with

$$\tilde{R}\equiv\left[\frac{\partial\tilde{P}}{\partial r}+\frac{h(\tilde{P}+\tilde{\rho})}{(1-2\tilde{m}/r)}\left(4\pi r\tilde{P}+\frac{\tilde{m}}{r^2}\right)\right]_a$$

and

$$h=1-2C\,.$$

If the effective density $\tilde{\rho}$ is separable, i.e., $\tilde{\rho}=f(u)g(r)$, then it can be shown (see Ref. 1) that Eq. (29) becomes

$$\frac{\dot{F}}{F}+(1-F)\frac{\dot{\Omega}}{\Omega}=G(F,\Omega,A,h) \tag{30}$$

with

$$\frac{G(F,\Omega,A,h)}{m(0)}=-\frac{\Omega^2F(1-F)\tilde{R}}{\tilde{\rho}_a}+(F-1)(\Omega-1)\left[4\pi a\tilde{\rho}_a(2+h)-\frac{4\pi ha\tilde{\rho}_a}{\Omega}+k(a)F-\frac{h(1-F)}{2a}-\frac{(1+F)}{a}+\frac{\Omega F}{\tilde{\rho}_a}\tilde{\rho}_{1a}\right], \tag{31}$$

where

$$k(a)=\frac{d}{da}\ln\left[\frac{1}{a}\int_0^a dr\,r^2g(r)/g(a)\right].$$

C. Bouncing at the surface

As in the isotropic case it is possible to define a criterion to predict the bounce at the surface without integrating the equations at the surface. With this aim observe that the occurrence of a bounce is related to the occurrence of a minimum of the object's radius A during the evolution.

According to Eq. (26), this requires, as expected, $\Omega=1$, and we have

$$\ddot{A}=F\dot{\Omega}$$

which together with Eqs. (28) and (30) gives

$$\ddot{A}=\frac{F}{1-F}\left(G-\frac{2E}{A}\right). \tag{32}$$

All quantities are to be evaluated at the extremal point. It is obvious from (32) that a necessary condition for a bounce to occur is $G>0$, more specifically, a sufficient condition is

$$G>\frac{2E}{A}. \tag{33}$$

Observe also that $G>0$ is equivalent to $-R>0$. The difference between the isotropic and anisotropic cases depends on the presence of h in the expression for G.

In the following sections we shall exhibit two models. The first is a nonstatic radiating generalization of the Schwarzschild anisotropic model.[4] In this model $\tilde{R}=0$ so that the bouncing of the surface is impossible. In the second model (a nonstatic radiating generalization of the anisotropic Tolman IV solution[4]) $\tilde{R}$ is not identically zero allowing the occurrence of bounce.

V. THE ANISOTROPIC SCHWARZSCHILD-TYPE MODEL

Let us now illustrate the method presented above with a very simple model inspired by the anisotropic Schwarzschild solution.[4]

With this aim, and following the prescription of Sec. III, we take

$$\tilde{\rho}=\begin{cases} f(u), & r\leq a(u)\\ 0, & r>a(u)\end{cases} \tag{34}$$

and

$$\tilde{P}=f(u)\frac{[(1-\frac{8}{3}\pi f r^2)^{h/2}(1-3\omega_a)-(1-\omega_a)(1-\frac{8}{3}\pi f a^2)^{h/2}]}{[3(1-\omega_a)(1-\frac{8}{3}\pi f a^2)^{h/2}-(1-\frac{8}{3}\pi f r^2)^{h/2}(1-3\omega_a)]}\,. \tag{35}$$

Using (17′) and (18′),

$$\beta=\begin{cases} \frac{1}{2h}\ln\left[(1-\omega_a)\left(\frac{3(1-\frac{8}{3}\pi f a^2)^{h/2}}{2(1-\frac{8}{3}\pi f r^2)^{h/2}}-\frac{1}{2}\right)+\omega_a\right], & r<a\\ 0, & r>a\end{cases} \tag{36}$$

$$\tilde{m}=\begin{cases} \frac{4}{3}\pi f r^3, & r\leq a(u)\\ \frac{4}{3}\pi f a^3, & r>a\,.\end{cases} \tag{37}$$

And from (31) and the definition of $\tilde{R}$,

$$\tilde{R}\equiv 0\,,$$

$$G=-\frac{(1-F)^2(\Omega-1)}{2A\Omega}[2(2+h)\Omega-3h]\,.$$

Thus, the equations at the surface for this model are

$$\dot{A}=F(\Omega-1)\,,$$

$$\frac{\dot{F}}{F}=\frac{2E+(1-F)(\Omega-1)}{A}\,,$$

$$\frac{\dot{F}}{F}+(1-F)\frac{\dot{\Omega}}{\Omega}=G\,.$$

We still need to specify one function of u and the initial data. Following HJR we chose FE to be a Gaussian, so that the total radiated mass is one-tenth of the initial mass. As for the initial data the following cases will be described:

(a) $A|_{u=0}=5,\ \ \Omega|_{u=0}=1,\ \ F|_{u=0}=0.6$,

(b) $A|_{u=0}=5,\ \ \Omega|_{u=0}=0.83,\ \ F|_{u=0}=0.6$,

(c) $A|_{u=0}=3.3,\ \ \Omega|_{u=0}=1,\ \ F|_{u=0}=0.4$.

Figure 1 shows the evolution of the radius A for the three different values of h. In all cases (including the isotropic case) the final situation is a constant density sphere, the final density depending on h. Once equations at the surface are integrated, we are able to determine the functions ρ, ϵ, P, $P_\perp$, and ω for any piece of the material, following the algorithm indicated in Sec. III. (Some details of the calculations are included in Appendix A.) Figures 2 and 3 show the profile of the density for different values of h (or C) and for $r/a=1,\ \frac{1}{2}$.

Of special interest in the context of this work is Fig. 4 which shows the profile of the ratio $\eta=(P_\perp-P)/P$ for $r/a=\frac{1}{2}$. It is interesting to observe that during the maximum intensity of the pulse of radiation, η oscillates rapidly through zero, and the material is subject to large stresses. However, at the surface the tangential pressure increases during that time as shown in Fig. 5. Figure 6 shows the pulse of radiation.

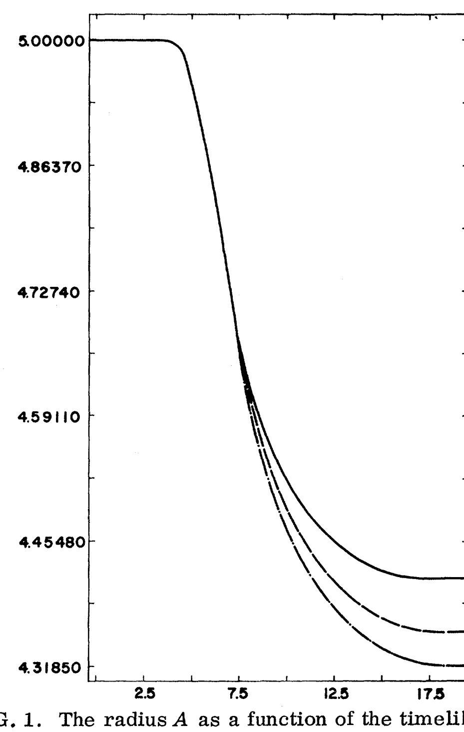

FIG. 1. The radius A as a function of the timelike coordinate for the initial value $A|_{u=0}=5$, $\Omega|_{u=0}=1$, $F|_{u=0}=0.6$, and different values of h. The solid line represents $h=0.33$, the dashed line $h=1$, and the dot-dashed line $h=1.33$.

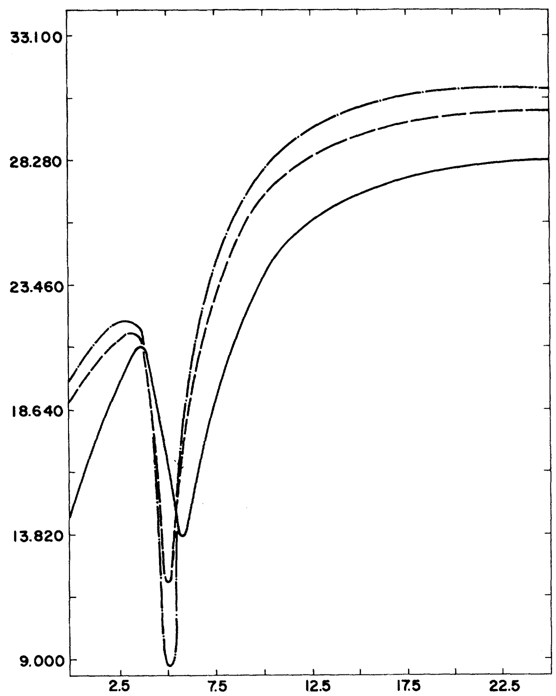

FIG. 2. $\bar{\rho} \equiv \rho m^2(0)$ as a function of the timelike coordinate for the initial data $A|_{u=0}=5$, $\Omega|_{u=0}=0.83$, $F|_{u=0}=0.6$, $r/a=0.5$, and different values of h. The solid line represents $h=0.33$, the dashed line $h=1$, and the dot-dashed line $h=1.33$.

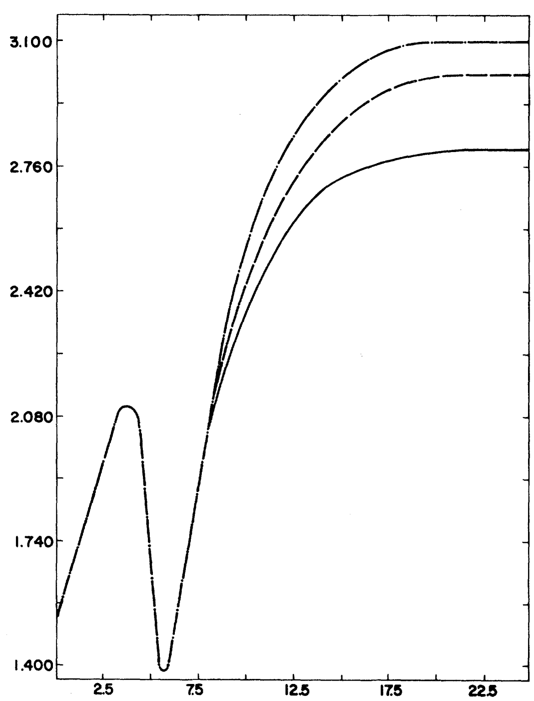

FIG. 3. Same as Fig. 2, but for $r/a=1$.

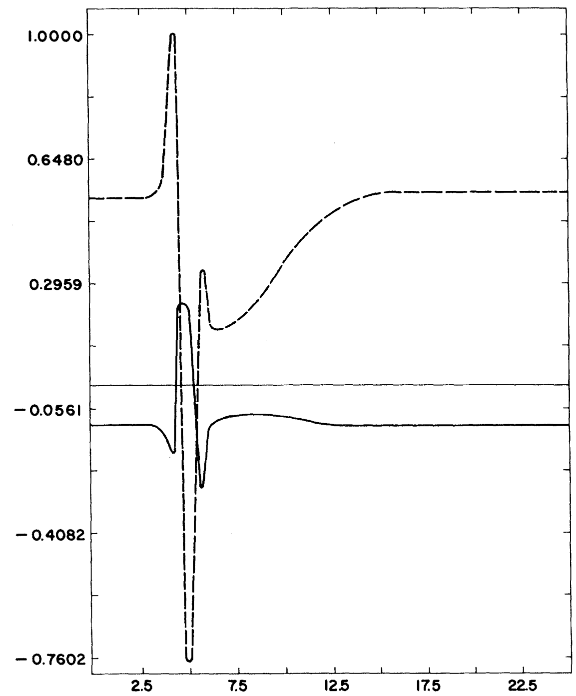

FIG. 4. $(P_\perp - P)/P$ as a function of the timelike coordinate for $r/a=0.5$, and different values of h. The solid line represents $h=0.33$ and the dashed line $h=1.33$. $\Omega|_{u=0}=1$, $A|_{u=0}=3.3$, $F|_{u=0}=0.4$.

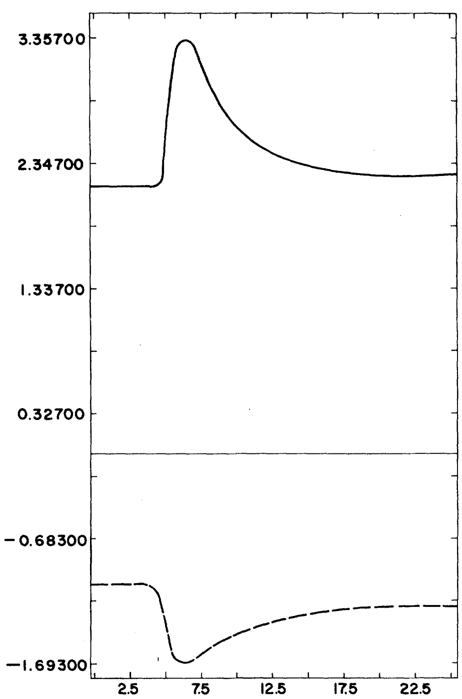

FIG. 5. $\bar{P}_\perp \equiv m^2(0)\bar{P}_\perp$ as a function of the timelike coordinate for the initial data $F|_{u=0}=0.6$, $A|_{u=0}=5$, $\Omega|_{u=0}=1$ for different values of h and $r/a=1$. The solid line represents $h=0.33$ and the dashed line $h=1.33$.

VI. ANISOTROPIC TOLMAN VI-TYPE MODEL

As a second example we shall give all the formulas for a model obtained from the anisotropic-static Tolman VI solution.[4]

Following the scheme of Sec. III, let us take

$$\tilde{\rho}=\frac{3g(u)}{r^2}, \tag{38}$$

$$\tilde{P}=\frac{g(u)}{hr^2}\left[\frac{I-\alpha(4-3h)^{1/2}Kr^{(4-3h)^{1/2}}}{1-\alpha(4-3h)^{1/2}r^{(4-3h)^{1/2}}}\right] \tag{39}$$

with

$$K=8-3h+4(4-3h)^{1/2},$$

$$I=8-3h-4(4-3h)^{1/2},$$

$$\alpha=\frac{I+3h\omega_a}{(4-3h)^{1/2}(K+3h\omega_a)a^{(4-3h)^{1/2}}}.$$

Substituting (38) and (39) into (17′) and (18′),

$$\tilde{m}=12\pi g r, \tag{40}$$

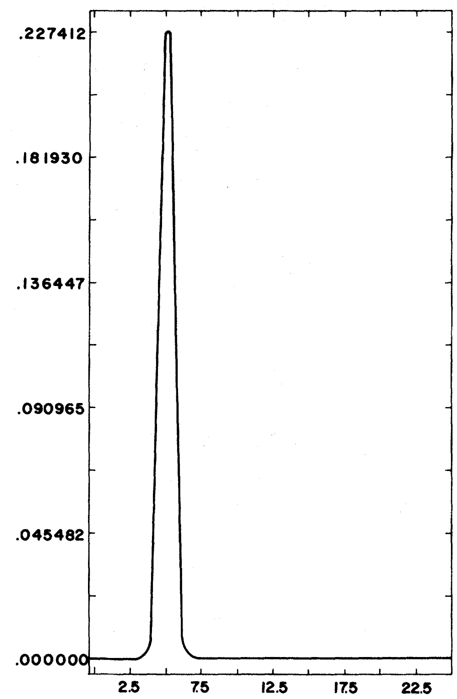

FIG. 6. E as a function of the timelike coordinate.

$$\beta=\frac{8\pi g}{(1-24\pi g)^h}\ln\left[\frac{[1-\alpha(4-3h)^{1/2}r^{(4-3h)^{1/2}}]^2}{[1-\alpha(4-3h)^{1/2}a^{(4-3h)^{1/2}}]^2}\left(\frac{r}{a}\right)^{2-(4-3h)^{1/2}}\right]. \tag{41}$$

Next, using (31) and the definition of $\tilde{R}$ one gets

$$\tilde{R}=\frac{(1-F)}{64\pi a^3\Omega^2 F}\left\{16(\Omega-1)F\Omega-\frac{F}{3h}[I\Omega+3(\Omega-1)h][K\Omega+3(\Omega-1)h]+4(1-F)h\right\}, \tag{42}$$

$$\frac{G}{m(0)}=\tilde{R}+(F-1)(\Omega-1)\left[\frac{(1-F)}{2a}(2+h)-\frac{(1-F)}{2a\Omega}+\frac{2}{a}F-\frac{h(1-F)}{2a}-\frac{(1+F)}{a}-\frac{2\Omega F}{a}\right]. \tag{43}$$

The surface equations may now be integrated as in the preceding example, in order to obtain ρ, P, $P_\perp$, and ϵ (some intermediate calculations are given in Appendix B). As we have indicated it is possible that in this case the sphere will bounce.

VII. CONCLUSIONS

We have seen so far that the method described in HJR for isotropic matter is very easily extended to the anisotropic matter.

The main results to appear are the peculair evolution of the anisotropy during the process of contraction and radiation (Figs. 2–4) and the influence of the anisotropy on the possibility of the bounce at the surface, as indicated in Eq. (33). Figures 1–6 for the Schwarzschild model show that the main features of collapse are qualitatively similar for the isotropic and for the anisotropic cases. The most obvious difference is the wild swings in the ratio $(P_\perp - P)/P$ when the pulse of radiation passes by.

This might have important consequences for an actual collapsing object; it quite possibly will lead to instability in the purely radial nature and to the onset of convective-type instabilities. Obviously a study of the stability of a collapsing spherically symmetric, nonisotropic object would be worthwhile.

It is a very difficult task to evaluate the extent to which the results exhibited here are model independent. However, two remarks may be made at this point.

(a) Since anisotropic and isotropic models are continuously connected through the parameter C (or h) it seems plausible to think that, at least, for small anisotropies the conclusions remain valid.

(b) The identification of the r dependence of $\tilde{P}$ and $\tilde{\rho}$ with the corresponding dependence of the static case may be expected to work for small velocities, since it is true in the limit $\omega \to 0$.

Obviously, it would be of great interest to see the effect on collapse of anisotropy based on a physically derived and realistic relationship between the tangential pressure and the radial pressure. Ideally, a nonconstant effective densitylike model should be examined to permit the possibility of a bounce, and the stability of the collapsing object with respect to nonradial motions should be examined.

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APPENDIX A

Once the surface equations are integrated for the anisotropic Schwarzschild-type case, it is useful to introduce the following nondimensional auxiliary quantities:

$$Y \equiv e^{2\beta} = \left\{\frac{1}{\Omega}\left[\frac{3}{2}\left(\frac{F}{Z}\right)^{h/2}\right] + \Omega - \frac{3}{2}\right\}^{1/h}, \tag{A1}$$

$$Z \equiv \left[1 - (1-F)\frac{r^2}{a^2}\right].$$

It is now easy to obtain

$$\beta_1 m(0) = \frac{3(1-F)}{4A\Omega}\frac{F^{h/2}}{Y^h Z^{(h/2)+1}}\left(\frac{r}{a}\right), \tag{A2}$$

$$\beta_{11} m(0)^2 = \frac{3(1-F)F^{h/2}}{4A^2\Omega Y^{2h} Z^{h+2}} \times\left\{Y^h Z^{(h/2)+1} - (1-Z)\left[\frac{3hF^{h/2}}{2\Omega} - Z^{h/2}(h+2)Y^h\right]\right\}, \tag{A3}$$

$$\beta_{01} m^2(0) = S\dot{F} + T\dot{\Omega} + V\dot{A}, \tag{A4}$$

$$S = -\frac{3F^{h/2}}{4\Omega Y^4 A Z^{(h/2)+2}}\left[1 + \frac{h(2-F)(3-2\Omega)}{4F\Omega Y^h}\right]\left(\frac{r}{a}\right),$$

$$T = -\frac{3F^{h/2}(1-F)}{4Z^{h/2+1}\Omega^2 Y^h A}\left\{1 + \frac{3}{2\Omega Y^h}\left[1 - \left(\frac{F}{Z}\right)^{h/2}\right]\right\}\left(\frac{r}{a}\right),$$

$$V = -\frac{3F^{h/2}(1-F)}{2A^2\Omega Y^h Z^{(h/2)+2}}\left[1 + \frac{h(Z-F)(3-2\Omega)}{4F\Omega Y^h}\right]\left(\frac{r}{a}\right) + \frac{3h(1-F)^2 F^{h-1}}{4\Omega Y^{2h} A^2 Z^{h+1}}\left[\frac{3}{2\Omega} - \frac{Y^h}{(F/Z)^{h/2}}\right]\left(\frac{r}{a}\right), \tag{A5}$$

$$\tilde{m}_1 = \frac{3}{2}\left(\frac{r}{a}\right)^2(1-F),$$

$$\frac{\tilde{m}_{11}}{r} m^2(0) = \frac{3(1-F)}{A^2}. \tag{A6}$$

Feeding back (A1)–(A6) into (16) gives P_1, then using (21′) we get P. The other field equations give

$$\omega = 1 - \frac{3(1-F)(F/Z)^{h/2}}{Y^h\Omega[3(1-F) + 8A^2\overline{P}]},$$

$$\rho = \frac{3(1-F)}{8\pi A^2}(1+\omega) + \overline{P}\omega,$$

and

$$\overline{\epsilon} = \frac{1}{8\pi Z Y^h A}\left[\dot{F} + \frac{2(1-F)\dot{A}}{A}\right]\left(\frac{r}{a}\right) + \frac{3(1-F)}{8\pi A^2} - \left[\frac{\overline{\rho} + \overline{P}\omega^2}{(1-\omega^2)}\right],$$

$$\overline{\rho} = \frac{3(1-F)(1+\omega)}{8\pi A^2} + \overline{P}\omega,$$

where

$$\overline{P} = m^2(0)P, \quad \overline{\rho} = \rho m^2(0), \quad \overline{\epsilon} \equiv \epsilon m^2(0).$$

APPENDIX B

In the anisotropic Tolman VI case we arrive at the following expressions:

$$\tilde{m}_1 = \frac{(1-F)}{2}, \quad \tilde{m}_{11} = 0,$$

$$\beta = \frac{(1-F)}{3Fh}\ln\left\{\left[\frac{[K\Omega + 3(\Omega-1)h][I\Omega + 3(\Omega-1)h](r/a)^{(4-3h)^{1/2}}}{[K\Omega + 3(\Omega-1)h] - [I\Omega + 3(\Omega-1)h]}\right]^2\left(\frac{r}{a}\right)^{2-(4-3h)^{1/2}}\right\},$$

$$Y \equiv e^{2\beta} = \left\{\left[\frac{[K\Omega + 3(\Omega-1)h] - [I\Omega + 3(\Omega-1)h](r/a)^{(4-3h)^{1/2}}}{[K\Omega + 3(\Omega-1)h][I\Omega + 3(\Omega-1)h]}\right]^2\left(\frac{r}{a}\right)^{2-(4-3h)^{1/2}}\right\}^{\frac{2(1-F)}{3Fh}},$$

$$\beta_1 m(0) = \frac{(1-F)}{3F(r/a)Ah}\left\{\frac{[K\Omega + 3(\Omega-1)h][2-(4-3h)^{1/2}] - [2+(4-3h)^{1/2}][I\Omega + 3(\Omega-1)h](r/a)^{(4-3h)^{1/2}}}{[K\Omega + 3(\Omega-1)h] - [I\Omega + 3(\Omega-1)h](r/a)^{(4-3h)^{1/2}}}\right\}.$$

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