PHYSICAL REVIEW D VOLUME 25, NUMBER 10 15 MAY 1982

Fluid with heat flux in a conformally flat space-time

S. R. Maiti
Department of Physics, Presidency College, Calcutta-700073, India
(Received 3 April 1981)

This paper presents a solution of Einstein's equation in a conformally flat and spherically symmetric space-time with perfect fluid and heat flux as sources when vorticity and shear vanish. With a certain choice of parameters the solution may represent an earlier stage of the Robertson-Walker model. Though acceleration remains, a singularity cannot be avoided in some cases. The horizon of the model cannot be definitely known, but for a particular case it can be assigned. Other behaviors of the solution are similar to those of the Robertson-Walker metric.

I. INTRODUCTION

In the literature there exists a large number of studies of the solutions of Einstein's equation with perfect-fluid source assuming a conformally flat space-time. In such a situation one has either spatial constancy of energy density (ρ) and vanishing of shear ($\sigma_{\alpha\beta}$) and vorticity (ω^α) of the fluid or a vanishing of the sum of pressure (p) and density. The second case reduces to the de Sitter universe with velocity vector undetermined or an altogether empty Minkowski space. In the first case, if expansion (θ) vanishes the Schwarzschild interior metric follows.[1] If $\theta \neq 0$, either we have the isotropic cosmological solution or a family of nonhomogeneous cosmological solutions.[2,3] In this paper we intend to study a more general situation, where the fluid is not in thermal equilibrium so that there is a heat flux in addition to perfect fluid. The assumption of conformally flat space-time now no longer leads vorticity and shear to vanish. However, for mathematical simplicity we assume that they vanish and we assume the space-time to be spherically symmetric, and thereby arrive at a class of solutions that may represent some early stages of the previously discussed models. As expected, when the fluid thermalizes our solution goes over to all the above solutions.

II. DERIVATION OF THE SOLUTION

We now take the energy-momentum tensor as

$$T_{\mu\nu}=(p+\rho)u_\mu u_\nu - p g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \,, \tag{1}$$

where u^α and q^α are the velocity and the heat-flux vectors with $u^\alpha u_\alpha = 1$ and $q_\alpha u^\alpha = 0$. We recall a few general equations from Ellis[4]:

$$h^\mu_\alpha h^\nu_\beta E^{\alpha\beta}{}_{;\nu} - \eta^{\mu\alpha\beta\nu} u_\alpha \sigma^\gamma_\beta H_{\nu\gamma} + 3H^\mu_\alpha \omega^\alpha = \tfrac{1}{3} h^{\mu\alpha}\rho_{,\alpha} - \tfrac{3}{2}\omega^\mu_\alpha q^\alpha + \tfrac{1}{2}\sigma^\mu_\alpha q^\alpha - \tfrac{1}{3}\theta q^\mu \,, \tag{2}$$

$$h^\alpha_\mu h^\beta_\nu (\sigma_{\alpha\beta})^{\cdot} - h^\alpha_\mu h^\beta_\nu \dot{u}_{(\alpha;\beta)} - \dot{u}_\mu \dot{u}_\nu + \omega_\mu \omega_\nu + \sigma_{\mu\alpha}\sigma^\alpha_\nu + \tfrac{2}{3}\theta\sigma_{\mu\nu} + h_{\mu\nu}(-\tfrac{1}{3}\omega^2 - \tfrac{2}{3}\sigma^2 + \tfrac{1}{3}\dot{u}^\alpha{}_{;\alpha}) + E_{\mu\nu} = 0 \,, \tag{3}$$

$$h^\mu_\alpha(\omega^{\alpha\beta}{}_{;\beta} + \sigma^{\alpha\beta}{}_{;\beta} + \tfrac{2}{3}\theta^{;\alpha}) + (\omega^\mu_\alpha + \sigma^\mu_\alpha)\dot{u}^\alpha = q^\mu \,, \tag{4}$$

$$H_{\alpha\beta} = 2\dot{u}_{(\alpha}\omega_{\beta)} - h^\mu_\alpha h^\nu_\beta(\omega^{\lambda;\gamma}_{(\mu} + \sigma^{\lambda;\gamma}_{(\mu})\eta_{\nu)\delta\lambda\gamma} u^\delta \,. \tag{5}$$

Here $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are electric and magnetic components derived from the Weyl tensor $C_{\alpha\beta\lambda\mu}$. In conformally flat space-time $C_{\alpha\beta\lambda\mu}=0$, $E_{\alpha\beta}=H_{\alpha\beta}=0$ (and conversely if $E_{\alpha\beta}=H_{\alpha\beta}=0$, the space-time is conformally flat) and with vanishing of shear and vorticity, Eq. (5) is trivially satisfied and Eqs. (2), (3), and (4) reduce to

$$h^{\mu\alpha}\rho_{,\alpha} = \theta q^\mu \,, \tag{6}$$

$$\dot{u}_{(\alpha;\beta)} + u_\alpha \dot{u}_\beta - \tfrac{1}{3}h_{\alpha\beta}\dot{u}^\mu{}_{;\mu} = 0 \,, \tag{7}$$

$$\tfrac{2}{3}\theta_{,\alpha}h^{\mu\alpha} = q^\mu \,. \tag{8}$$

From Eqs. (6) and (8)

$$h^{\mu\alpha}(\rho - \tfrac{1}{3}\theta^2)_{,\alpha} = 0 \,. \tag{9}$$

If we now use a comoving coordinate system, i.e., $u^\alpha = g_{00}{}^{-1/2}\delta^\alpha_0$, then from Eq. (9) $\rho - \frac{1}{3}\theta^2$ is a function of t alone.

Again the three-space Ricci tensor $R^*_{\alpha\beta}$ may be written for this case[4]

$$R^*_{lm} = \dot{u}_{(l;m)} + \dot{u}_l \dot{u}_m + \tfrac{1}{3}h_{lm}(2\rho - \tfrac{2}{3}\theta^2 - u^\alpha{}_{;\alpha}) \,. \tag{10}$$

Using Eq. (7) we have in our coordinate system

$$R^*_{lm} = \tfrac{2}{3} g_{lm}(\rho - \tfrac{1}{3}\theta^2) \,. \tag{11}$$

Hence the three-space is a space of constant curvature. So the line element can be written as (cf. Eisenhart[5])

$$ds^2 = g_{00}dt^2 - \frac{R^2}{(1+kr^2/4)^2}(dx^2+dy^2+dz^2) \,. \tag{12}$$

Here R is a function of t alone and $k=0$, $+1$, or

-1 for a Euclidean space, a spherical space, or a pseudospherical space. Using polar coordinates and assuming g_{00} to be a function of r and t we can solve Einstein's equation using Dingle's formula[6] for $T^1_1 = T^2_2$ and find

$$g_{00} = \left(A + \frac{B}{1+kr^2/4}\right)^2 , \qquad (13)$$

where A and B are functions of t alone. Hence the metric (12) is completely determined as

$$ds^2 = \left(A + \frac{B}{1+kr^2/4}\right)^2 dt^2 - \frac{R^2}{(1+kr^2/4)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) . \qquad (14)$$

The nonvanishing components of T^μ_ν being $T^1_1 = T^2_2 = T^3_3 = -p$ and $T^0_0 = \rho$ and $T^1_0 = q^1 u_0$, one can have the only nonvanishing radial component of q^μ is q^1. So we can calculate density, pressure, and heat flux as

$$\rho = \frac{3k}{R^2} + 3\left(\frac{\dot R}{R}\right)^2 \left(A + \frac{B}{1+kr^2/4}\right)^{-2} , \qquad (15)$$

$$p = -\frac{k}{R^2} - \frac{Bk}{R^2}\,\frac{1-kr^2/4}{1+kr^2/4}\left(A + \frac{B}{1+kr^2/4}\right)^{-1} - \left[2\frac{\ddot R}{R} + \left(\frac{\dot R}{R}\right)^2\right]\left(A + \frac{B}{1+kr^2/4}\right)^{-2} + 2\left(\frac{\dot R}{R}\right)\left(\dot A + \frac{\dot B}{1+kr^2/4}\right)\left(A + \frac{B}{1+kr^2/4}\right)^{-3} , \qquad (16)$$

$$q^1 = \frac{Bk\dot R r}{R^3}\left(A + \frac{B}{1+kr^2/4}\right)^{-2} , \qquad (17)$$

or

$$q = (-q^\alpha q_\alpha)^{1/2} = \frac{Bk\dot R r}{R^2(1+kr^2/4)}\left(A + \frac{B}{1+kr^2/4}\right)^{-2} . \qquad (17a)$$

The expansion and acceleration are

$$\theta = 3\frac{\dot R}{R}\left(A + \frac{B}{1+kr^2/4}\right)^{-1} , \qquad (18)$$

$$\dot u^\alpha = -\frac{1}{2}\left(A + \frac{B}{1+kr^2/4}\right)^{-1}\frac{Bkr}{R^2}\,\delta^\alpha_1 . \qquad (19)$$

From (15) and (16) we have

$$\rho + 3p = -6\frac{\ddot R}{R}\left(A + \frac{B}{1+kr^2/4}\right)^{-2} + 6\frac{\dot R}{R}\left(\dot A + \frac{\dot B}{1+kr^2/4}\right)\left(A + \frac{B}{1+kr^2/4}\right)^{-3} - 3\frac{Bk}{R^2}\,\frac{1-kr^2/4}{1+kr^2/4}\left(A + \frac{B}{1+kr^2/4}\right)^{-1} . \qquad (20)$$

If the heat flux is absent, either $B = 0$ or $\dot R = 0$. For $B = 0$, with a transformation of t such that $A = 1$, the solution reduces to the Robertson-Walker metric and in that case acceleration vanishes. For $\dot R = 0$, i.e., $\theta = 0$, the Schwarzschild interior metric follows with A and B as constants.[1] For $k = 0$ the solution reduces to the Friedmann universe having no heat flux and no acceleration.

III. DISCUSSION OF THE RESULT

Like the Robertson-Walker metric, the space sections orthogonal to the fluid velocity are flat, spherical, or hyperbolic accordingly as $k = 0$, 1, or -1. The t = constant hypersurfaces of the space-time admit the same six-parameter group of motions as such surfaces do in Robertson-Walker space-time. The entire space-time admits the group of rotations. However, the space-time as a whole is not homogeneous. One may talk of volume and topology of the space sections in much the same way as in the case of homogeneous space.

The solution (14) for $A \neq 0$ may be transformed (by a transformation of t such that $A = 1$) as

$$ds^2 = \left(1 + \frac{B}{1+kr^2/4}\right)^2 dt^2 - \frac{R^2}{(1+kr^2/4)^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) . \qquad (21)$$

The solution has a singularity at $R = 0$, the pressure and density being infinite. To see whether the singular state is actually attained at a finite time, we examine Eq. (20). For $\rho + 3p > 0$ and $\dot R = 0$, $\ddot R$ is negative for B positive with $k = 1$ and for B negative with $k = -1$ and the singularity state is attained. Again for B negative there is a singularity for $1 + B/(1+kr^2/4) = 0$. It is to be noted that the $R = 0$ singularity is a collapse of the entire space and not a localized singularity at different epochs as in the case for B negative. The singularity can be avoided altogether for B positive with $k = -1$.

The functions B and R remain arbitrary, the only constraints on them that can be reasonably set are the physical requirements $\rho > 0$, $p \geqslant 0$, and $\rho \geqslant p$. However, if B is a decreasing function with increasing R, then Eqs. (17a) and (19) suggest that heat flux and acceleration vanish more rapidly than pressure and density. So the solution (21) with such B may be treated as an early stage of the Robertson-Walker universe.

As g_{00} contains r and t, the horizon of the metric (21) cannot be understood clearly. If, however, B is constant, we can consider light rays to be traveling radially leaving r_1 at t_1 and reaching r_2 at t_2, then

$$\int_{r_1}^{r_2} \frac{dr}{1+kr^2/4+B} \int_{t_1}^{t_2} \frac{dt}{R} . \qquad (22)$$

For $R=0$ singularity models, $R\to 0$ at finite past, we consider that instant to be $t=0$. Then $t\to 0$, $R\sim t^n$ and from (20) for $\rho+3p>0$, $\ddot{R}/R<0$ and hence $n(n-1)<0$, i.e., $0<n<1$. With this restriction on n, integral $\int_0 R^{-1}dt$ converges and one can have a particle horizon. For models where the $R=0$ singularity is not attained, there exists a minimum for R at a finite time. We assume that time to be $t=0$. Then we may consider as $t\to 0$, $R\sim t^n$. For $\rho+3p>0$, n should be positive, otherwise $\rho+3p$ will diverge. For $n>0$, $\int_0 R^{-1}dt$ converges and there is a particle horizon.

Note added in proof. The result (13) is obtained from (12) using Dingle's formula for $T_2^1=T_3^1=T_3^2=0$ and so the assumption of spherical symmetry need not be required.

ACKNOWLEDGMENT

The author's grateful thanks are due Professor A. K. Raychaudhuri of Presidency College for his help and guidance.

[1]A. K. Raychaudhuri and S. R. Maiti, J. Math. Phys. 20, 245 (1979).
[2]M. Trumper, Z. Astro. Phys. 66, 215 (1967).
[3]L. C. Shepley and T. Taub, Commun. Math. Phys. 5, 237 (1967).
[4]G. F. R. Ellis, in *General Relativity and Cosmology, Corso, Varena, Italy, 1968,* edited by R. K. Sachs (Academic, New York, 1971).
[5]L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949).
[6]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Oxford, 1934).