Absence of stationary solutions to Einstein-Yang-Mills equations

R. Weder
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20-726, Mexico 20, D.F.
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We prove the absence of nonsingular, localized, stationary solutions of the coupled Einstein-Yang-Mills equations, with compact gauge group.

INTRODUCTION

Since the early days of general relativity many investigations have been dedicated to the study of the possibility of the existence of nonsingular, localized solutions to the Einstein equations. The importance of this question for the self-consistence of general relativity was first discovered by Hilbert,[1] Levi-Civita,[2] and G. Darmois. Serini[3] and Racine[4] proved that every static, localized, nonsingular solution is locally Minkowski space-time. This research culminated in the work by Lichnerowicz,[5–7] Einstein,[8] and Einstein and Pauli,[9] where it was proven that every nonsingular, localized, stationary solution to the Einstein equations is locally Minkowski space-time. It is also known that if one considers general nonsingular, localized solutions this statement is not true.[4]

The same question has been investigated by Thiry[10] in the case of Einstein-Maxwell equations. He proved that every stationary, localized, nonsingular solution to these equations is trivial. Of course if the assumption of localization is removed there are nonsingular solutions, namely static Wheeler geons.[11–12]

It was not many years after Yang and Mills[13] introduced non-Abelian gauge fields that similar global questions were investigated for Yang-Mills equations. The first result is due to Deser[14] and Coleman,[15] who demonstrated the absence of static finite-energy solutions going to zero fast enough at spatial infinity. Then Coleman[16] demonstrated the absence of finite-energy solutions going sufficiently fast to zero at infinity uniformly in time. Finally Weder[17] proved that the local energies of every finite-energy solution to pure Yang-Mills equations decay in time, i.e., all the energy is radiated out to spatial infinity. This result excludes the possibility of the existence of any classical lump for pure Yang-Mills equations, and is the analog of the Einstein-Lichnerowicz-Pauli-Thiry theorems in the gravitational and Abelian cases.

The natural continuation of this research is to consider both gravitation and non-Abelian gauge fields, that is to say to investigate classical lumps in the case of Einstein-Yang-Mills (EYM) equations. In what follows we give an elementary proof of the absence of localized nonsingular stationary solutions with space-oriented space sections. Recently Deser[18] conjectured this result in the particular case of static solutions. Our elementary proof is different from the ones suggested in Ref. 18, and from Thiry's proof.[10]

ABSENCE OF SOLUTIONS

Let V_4 denote a four-dimensional Riemannian manifold of class C^2 with a metric of normal hyperbolic type. Consider a principal fiber bundle with base manifold V_4, and with gauge group a compact Lie group G. Denote by A a Lie-algebra-valued connection one-form, and let F be the curvature two-form. In a local coordinate system we have

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g^{abc}A^b_\mu A^c_\nu \ , \qquad (2.1)$$

where g^{abc} are the structure constants of the Lie algebra of G, and $\mu,\nu=0,1,2,3$ are space-time indices. The energy-momentum tensor of the Yang-Mills fields is given by

$$T_{\mu\nu}=-F^a_{\mu\gamma}F^{a\gamma}_{\nu}+\tfrac{1}{4}g_{\mu\nu}F^a_{\gamma\delta}F^{a\gamma\delta} \ . \qquad (2.2)$$

The EYM equations are

$$G_{\mu\nu}=T_{\mu\nu} \ , \qquad (2.3)$$

$$D_\mu F^{\mu\nu}=(\nabla_\mu F^{\mu\nu}+[A_\mu,F^{\mu\nu}])=0 \ , \qquad (2.4)$$

where ∇_μ stands for the covariant derivative on V_4, and [,] denotes the Lie brackets on the Lie algebra of G.

Let us consider now a stationary space-time V_4

(Ref. 19). These spaces are homeomorphic to the product (in a topological sense) $R\times V_3$, where V_3 is a three-dimensional manifold (the space manifold). Moreover there exist systems of adapted local coordinates such that the potentials $g_{\mu\nu}$ are independent of the time variable x_0, and where x_i is any system of local coordinates of V_3. Moreover in an adapted system $\eta^2=g_{00}>0$. The manifold $x_0=$constant (denoted W_3) of an adapted system of coordinates will be called the space sections of the system. The W_3 are clearly homeomorphic to V_3. From now on we will always work in an adapted system of local coordinates.

As usual we consider in W_3 the metric given by

$$g_{ij}^3=g_{ij}-\frac{g_{0i}g_{0j}}{g_{00}} . \tag{2.5}$$

Since the metric is normally hyperbolic g_{ij}^3 defines a negative-definite metric. W_3 and V_3 are Riemannian manifolds with the metric given by (2.5). We denote by ∇_i^3 the associated covariant derivative. We will always assume that V_3 and W_3 are complete manifolds.

DEFINITION I

We say that the space sections W_3 are space oriented if the tangent plane T_x at each point of W_3 is space oriented, that is to say, if for every $x\in W_3$ and every tangent vector $dx^i/d\mu$ we have

$$g_{ij}\frac{dx^i}{d\mu}\frac{dx^j}{d\mu}<0 . \tag{2.6}$$

The assumption that the space sections are space oriented means that the metric of "three-dimensional space" is negative definite. This is true in particular if V_4 is static, i.e., if the space is stationary and moreover $g_{0i}=0$, because in this case it follows from (2.5) that $g_{ij}=g_{ij}^3$ and the later metric is negative definite.

By a trivial solution to the EYM equations we mean a solution with $F_{\mu\nu}=0$, and V_4 locally Minkowski space-time.

We first consider the case where the space sections are compact.

Theorem I. The only nonsingularly, stationary solution to the EYM equations with compact, space-oriented, space sections is the trivial one.

Proof. We denote by $R_{\mu\nu}$ the Ricci tensor on V_4. It follows from a straightforward computation that we omit[19] that

$$\eta R_0^0=-\nabla_i^3 q^i , \tag{2.7}$$

where

$$\eta=(g_{00})^{1/2} ,$$

$$q_i=\frac{\partial}{\partial x^i}\eta+\frac{\eta}{2}g_{0k}g^{kj}\left(\partial_j\frac{g_{0i}}{\eta^2}-\partial_i\frac{g_{0j}}{\eta^2}\right) \tag{2.8}$$

and $q^i=g^{ij}q_j$. Then

$$\int_{W_3}\eta R_0^0 dV_3=-\int_{W_3}\nabla_i^3 q^i dV_3=0 . \tag{2.9}$$

Since the trace of $T_{\mu\nu}$ is zero, (2.3) is equivalent to

$$R_{\mu\nu}=T_{\mu\nu} . \tag{2.10}$$

Then we have from (2.9)

$$\int_{W_3}\eta T_0^0 dV_3=\int_{W_3}\eta R_0^0 dV_3=0 . \tag{2.11}$$

From (2.2) we obtain

$$T_0^0=\tfrac{1}{4}g^{ij}g^{kp}F_{ik}^a F_{jp}^a+\tfrac{1}{2}g^{0i}F_{0i}^a g^{0j}F_{0j}^a \, . \\ -\tfrac{1}{2}g^{00}g^{ij}F_{0i}^a F_{0j}^a . \tag{2.12}$$

Since the space sections are space orientated the metric defined by g^{ij} is negative definite, and $g^{00}>0$. Then the first and last terms on the right of (2.12) are non-negative. The second is obviously non-negative. Hence we have $T_0^0\geq 0$ with equality if and only if $F_{\mu\nu}^a\equiv 0$. But since $\eta>0$ it follows from (2.11) that $T_0^0\equiv 0$, and then $F_{\mu\nu}^a\equiv 0$. We are now in the case of Einstein equations and it follows from the Lichnerowicz theorem[5] that V_4 is locally Minkowski space-time. Q.E.D. Once this result has been established, the extension to noncompact space sections is easy. There are many ways to define a localized solution. In general they have a similar intuitive meaning, but many differ from the technical point of view. Since we prefer a precise concept of localization we will consider solutions with space sections asymptotically Euclidean in the sense of Lichnerowicz; we refer to Ref. 19 for details. In particular this means that for some $R>0$ and every $x\in W_3$, such that $r=d(a,x)\geq R$,

$$|g_{\mu\nu}-\eta_{\mu\nu}|\leq\frac{C}{r},\quad |\partial_k g_{0i}|\leq\frac{C}{r^2},\quad |\partial_k g_{00}|\leq\frac{C}{r^{2+\epsilon}} , \tag{2.13}$$

where C,ϵ are positive constants, a is a fixed, but arbitrary, point of W_3, and $r=d(a,x)$ is the geodesic distance from a to x, and $\eta_{\mu\nu}=(+1,-1,-1,-1)$.

Theorem II. The only nonsingular stationary solution to the EYM equations, with space-oriented space sections, that is localized in the sense of

Lichnerowicz is the trivial solution.

Proof. We follow the proof of Theorem I:

$$\int_{W_3} \eta T^0_0 dV_3 = \int_{W_3} \eta R^0_0 dV_3 = -\lim_{R\to\infty} \int_{B_R} \nabla^3_i q^i dV_3 = \lim_{R\to\infty} \int_{\partial B_R} q^i ds^i = 0 . \qquad (2.14)$$

As before, this implies $F^a_{\mu\nu} \equiv 0$, and the result follows from the pure gravitational case.[7] Q.E.D. Clearly the conditions (2.13) are stronger than what is really needed, however we have chosen simplicity rather than generality.

Finally we consider in the case of static V_4 a theorem where we have a different type of assumption in the behavior of the metric at spatial infinity. We recall[19] that V_3 is said to have a domain at infinity if for some fixed $a \in V_3$, and every $r > 0$ there exists $x \in V_3$ such that $d(a,x) > r$.

Theorem III. The only nonsingular static solution to EYM equations with space manifold V_3 having a domain at infinity and such that g_{00} tends to 1 uniformly by values larger than or equal to 1 in the domain at infinity is the trivial solution.

Proof. In an orthonormal Cartan moving frame we have[19]

$$R_{\underline{00}} = \frac{1}{\eta}(-\Delta^3 \eta) . \qquad (2.15)$$

Then from (2.3)

$$\frac{1}{\eta}(-\Delta^3 \eta) = R_{\underline{00}} = T_{\underline{00}} = \frac{1}{\eta^2} T_{00} , \qquad (2.16)$$

since $T_{00} \geq 0, -\Delta^3 \eta \geq 0$, and since η tends to 1 uniformly by values larger than or equal to 1, it follows from the maximum principle that $\eta \equiv 1$. But then from (2.16) it follows that $T_{00} \equiv 0$, and then $F^a_{\mu\nu} \equiv 0$. Hence V_4 is locally Minkowski spacetime by the Racine theorem.[4] Q.E.D. Notice that in our proof we only used the facts that $T^\mu_\mu = 0$, $T^0_0 \geq 0$, in Theorem I, and II, and $T_{00} \geq 0$, in Theorem III. Then our results hold in the more general case where we consider Einstein equation with any energy-momentum tensor having these properties.

Our theorems do not cover the case where g_{00} tends to 1 by values smaller than or equal to 1, at slower rate than $1/r^{1+\epsilon}$. An interesting open question is to extend our results to that case.

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