Matrix elements of the exchange operator for arbitrary-angular-momentum two-meson states

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A method allowing for the calculation of all matrix elements of the exchange operator between two mesons is presented. A close formula for such matrix elements is also given, together with graphical rules that allow easy numerical calculations.

I. INTRODUCTION

Recently,[1] it has been shown that a proper calculation of meson masses must take into account the effect of virtual channels into which a given meson can eventually decay, and which, therefore, contribute significantly to its mass. The spontaneous creation of a quark-antiquark pair, with the quantum numbers of the vacuum ($^3P_0$) is, together with the Pauli principle, responsible for such decays, as is depicted in Fig. 1.

Because quarks are fermions, any state made out of two quarks and two antiquarks must be properly antisymmetrized. If we assume that the quarks are labeled 1 and 3, and the antiquarks 2 and 4 (see Fig. 1), the antisymmetrization procedure can be written as an equation:

$$\psi_A(13,\overline{24})=A\phi_{\rm I}(1\bar{2})\psi_{\rm II}(3\bar{4}) ,$$
$$A=\frac{1}{N}(1-P^{13})(1-P^{24}) , \tag{1.1}$$

where $\phi_{\rm I}(1,\bar{2})$ and $\phi_{\rm II}(3,\bar{4})$ represent meson 1 and meson 2, respectively, and ψ_A represents the $q^2-\bar{q}^2$ properly antisymmetrized wave function. P^{13}, for instance, represents the exchange of quarks 1 and 3 according to Fig. 2. P^{24} does the same for the antiquarks. One antisymmetrizes separately for quarks and for the antiquarks, whence the form of A, the total antisymmetrizer. N is a normalizing constant.

It can be shown that A can be also written as

$$A=(1-P^{13})(1+P_t)/N , \tag{1.2}$$

where P_t induces the exchange of the two mesons $\phi_{\rm I}$ and $\phi_{\rm II}$ as a whole. It is apparent, therefore, that one only needs to be concerned with the calculation of the matrix elements for P^{13}, P_t being essentially 1 (boson statistics). The purpose of this paper is to calculate such matrix elements.

We are interested in the quantities

$$M(l_1',m_1',l_2',m_2',L_{12},M_{12};l_3',m_3',l_4',m_4',L_{34},M_{34})$$
$$=\langle \phi_{\rm I}'(l_3',m_3')\phi_{\rm II}'(l_4',m_4')g_{\rm I,II}'(L_{34},M_{34})\,|\,P^{13}\,|\,\phi_{\rm I}(l_1',m_1')\phi_{\rm II}(l_2',m_2')g_{\rm I,II}(L_{12},M_{12})\rangle . \tag{1.3}$$

The mesons ϕ are assumed to have arbitrary angular momentum. $\phi_{\rm I}$, for instance, is defined above to have angular momentum l_1', with z projection m_1'. The wave function $g_{\rm I,II}$ ($g_{\rm I,II}'$) describes the relative movement between $\phi_{\rm I}$ and $\phi_{\rm II}$ ($\phi_{\rm I}'$ and $\phi_{\rm II}'$), and is also assumed to have arbitrary angular momentum. From what has been said, it is obvious that the above matrix elements are going to "control" the relative strength of the coupling of a

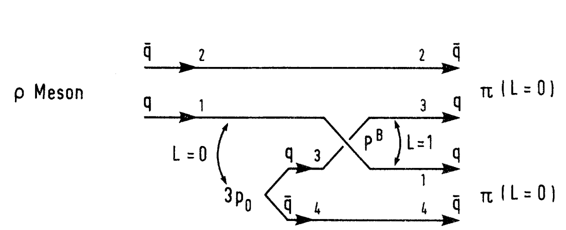

FIG. 1. $^3P_0$ mechanism to $\rho\rightarrow\pi\pi$.

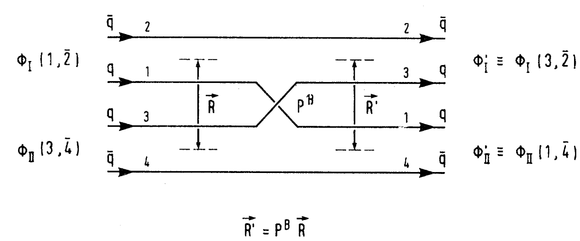

FIG. 2. The effect of P^{13}.

given meson to the two-meson channels into which it may decay. These quantities also play a role in the meson-meson scattering reactions.[2]

The rest of the paper is divided into two sections. In Sec. II, the general formalism is introduced. Section III deals with the relevant calculations and results. In the Appendices, we derive some formulas that play important roles in obtaining the final results.

II. BARGMAN HILBERT SPACES[3]

The Bargman Hilbert space is a Hilbert space $B_h^{(n)}$, where one defines the inner product of two elements $f, g \in B_h$ as

$$(f,g)=\int \bar{f}(z)g(z)d\mu_n(z)\ ,$$
$$d\mu_n(z)=\pi^{-n}e^{-\bar{z}\cdot z}\prod_{k=1}^{n} dx_k dy_k\ , \tag{2.1}$$
$$\bar{z}\cdot z=\sum_{k=1}^{n}\bar{z}_k z_k\ .$$

z is an n-fold complex variable; f and g are entire analytic functions defined on C^n.

One defines the usual inner product for usual Hilbert space $H^{(n)}$ as

$$(\psi_1,\psi_2)=\int \bar{\psi}(q)\psi(q)d^n q\ ,$$
$$\psi_1,\psi_2\in H^{(n)}\ . \tag{2.2}$$

There is an integral transform A that is a unitary map of $H^{(n)}$ into $B_h^{(n)}$, i.e.,

$$(\psi_1,\psi_2)=(f_1,f_2)\ ,$$
$$f_i(z)=A\psi_i(q)\Longleftrightarrow f_i(z)=\int A(z,q)\psi(q)dq\ . \tag{2.3}$$

A is given by (n dimensions)

$$A(z,q)=\pi^{-n/4}\exp[-\tfrac{1}{2}(z^2+q^2)+\sqrt{2}z\cdot q]\ .$$

Namely, it transforms the harmonic-oscillator wave functions (which constitute a complete set in $H^{(n)}$) into suitable defined monomials of the variable z, which constitute also a complete set in $B_h^{(n)}$.

For simplicity, let us consider the one-dimensional case $H^{(1)}$. The set of harmonic-oscillator wave functions is given by

$$\phi_m(q)=(2^m m!\sqrt{\pi})^{1/2}H_m(q)e^{-q^2/2}\ .$$

The $H_m(q)$ are the well-known Hermite polynomials. Now,

$$U_m(z)=\frac{z^m}{\sqrt{m!}}=A\phi_m(q)\ . \tag{2.5}$$

The functions $U_m(z)$ constitute a complete set in $B_h^{(n)}$. In other words, A can be decomposed as

$$A(z,q)=\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{m!}}\phi_m(q)\ . \tag{2.6}$$

Such decompositions can be achieved for arbitrary n. We will show such a decomposition for the three dimensional case in spherical coordinates.[4] We will consider the closely related kernel

$$G(\vec{s},\vec{\rho})=\exp\left[s^2-\frac{\rho^2}{z}+2i\vec{s}\cdot\vec{\rho}\right]\ . \tag{2.7}$$

$G(s,\rho)$ is obtained from $A(z,\rho)$ by the substitution $z\rightarrow i\sqrt{2}s$.

We are going to show that

$$G(\vec{s},\vec{\rho})=\sum_{nlm} i^l\left[\frac{2\pi^3}{n!\Gamma(n+l+\frac{3}{2})}\right]^{1/2}|s|^{2n+1}Y^*_{l,m}(\hat{s})\phi_{nlm}(\vec{\rho})\ . \tag{2.8}$$

The proof is as follows: First one writes down the general expression for the harmonic-oscillator wave function,

$$\phi_{nlm}(\rho)=\left[\frac{2(n!)}{\Gamma(n+l+\frac{3}{2})}\right]^{1/2}e^{-\rho^2/2}|\rho|^l L_n^{1+1/2}(\rho^2)Y_{l,m}(\vec{\rho})\ . \tag{2.9}$$

One sees that one needs a generating functional to handle the Laguerre polynomials $L_n^{l+1/2}(\rho^2)$:

$$e^z(xz)^{-(1/2)\alpha}J_\alpha(2\sqrt{xz})=\sum_{p=0}^{\infty}\frac{L_p^{(\alpha)}(x)}{\Gamma(p+\alpha+1)}z^p,\ \ \alpha>-1 \tag{2.10}$$

with J_α being the Bessel functions. If we substitute

$$x \to \rho^2, \quad z \to s^2, \quad p \to n, \quad \alpha = l + \tfrac{1}{2} ,$$

we have

$$e^{s^2}(s\rho)^{-\alpha} J_{l+1/2}(2s\rho) = \sum_{n=0}^{\infty} \frac{L_n^{l+1/2}(\rho^2) s^{2n}}{\Gamma(n+l+\frac{3}{2})} . \tag{2.11}$$

We consider the relation

$$J_{l+1/2}(2s\rho) = \left[\frac{4s\rho}{\pi} \right]^{1/2} j_l(s\rho) \tag{2.12}$$

with $J_l(s\rho)$ being spherical Bessel functions, to obtain

$$e^{s^2}(s\rho)^{-\alpha} J_{l+1/2}(2s\rho) = \sum_{n=0}^{\infty} \frac{L_n^{l+1/2}(\rho^2) s^{2n}}{\Gamma(n+l+\frac{3}{2})} . \tag{2.13}$$

Multiplying both terms by $4\pi i^l Y_{lm}(\hat{\rho}) Y_{lm}^*(\hat{s}) e^{-(1/2)\rho^2}$ and summing in both sides over l, m, we get

$$e^{s^2-(1/2)\rho^2} \sum_{l,m} 4\pi i^l j_l(2s\rho) Y_{l,m}(\hat{\rho}) Y_{l,m}^*(\hat{s})$$
$$= \sum_{nlm} \frac{2\sqrt{\pi^3}}{\Gamma(n+l+\frac{3}{2})} [\,|\rho|^l L_n^{l+1/2}(\rho^2) Y_{lm}(\hat{\rho}) e^{-(1/2)\rho^2}]\,|s|^{2n+l} Y_{lm}^*(\hat{s}) . \tag{2.14}$$

Noticing that

$$\sum_{l,m} 4\pi i^l j_l(2s\rho) Y_{l,m}(\hat{\rho}) Y_{l,m}^*(\hat{s}) = e^{2i\vec{s}\cdot\vec{\rho}} \tag{2.15}$$

and taking into consideration the normalization of the harmonic-oscillator wave function, one obtains the result of (2.8).

III. THE EXCHANGE-OPERATOR MATRIX ELEMENTS

To find the matrix elements $\langle p^{13} \rangle$, we proceed via two steps.

(1) First we evaluate

$$P(s_i, R, R') = \langle G(\vec{s}_3, \vec{\rho}_{12}) G(\vec{s}_4, \vec{\rho}_{34}) \,|\, P^{13} \,|\, G(\vec{s}_1, \vec{\rho}_{12}) G(\vec{s}_2, \vec{\rho}_{34}) \rangle \quad (i=1,4)$$

and show it to be equal to

$$(2\pi)^{3/2} G\left[\frac{(\vec{s}_2 - \vec{s}_1)}{\sqrt{2}}; \sqrt{2}\vec{R} \right] G\left[\frac{(\vec{s}_3 - \vec{s}_4)}{\sqrt{2}}; \sqrt{2}\vec{R}' \right] e^{(\vec{s}_1 + \vec{s}_2)\cdot(\vec{s}_3 + \vec{s}_4)}$$

with $\vec{R}$ being the vector distance between the two mesons and R' representing $P^{13}\vec{R}$. By the symbol $\langle\ \rangle$ we mean integration on the quark (antiquark) individual coordinates. $\vec{\rho}_{12}$, for instance, stands for the difference $\vec{r}_1 - \vec{r}_2$, with $\vec{r}_1$, $\vec{r}_2$ being the positions of quark 1 and antiquark $\bar{2}$.

(2) We perform for the above equality a power-series expansion of both terms, and equate powers of s. This will imply relations among the coefficients of such expansions, which in turn will yield the sought matrix elements for P^{13}.

A. Evaluation of $P(\vec{s}_i, \vec{R}, \vec{R}')$

To start with, let us introduce the kinematical variables relevant for the process we are interested in. In Fig. 3 we depict them. The effect of the operator P^{13} can be summarized as follows:

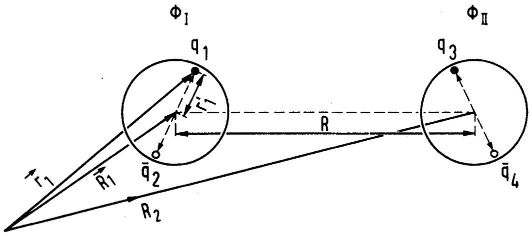

FIG. 3. Some relevant kinematical variables.

$$\vec{r}_1 \rightarrow \vec{r}_3,\ \ \vec{r}_3 \rightarrow \vec{r}_1,\ \ \vec{r}_2 \leftrightarrow \vec{r}_2,\ \ \vec{r}_4 \leftrightarrow \vec{r}_4\ . \tag{3.1}$$

In the calculations which follow, we will make use of the variable $\vec{R}\,'$, that is the vector image of $\vec{R}$ under the transformation P^{13}:

$$\vec{R}\,' = P^{13}\vec{R}\ . \tag{3.2}$$

Using the definition of $\vec{R}$, and acting upon this vector with P^{13}, we get

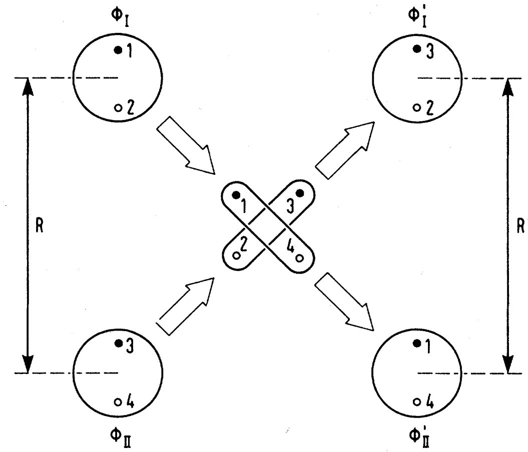

FIG. 4. The overlap "interaction."

$$\vec{R}_{\mathrm{I,II}} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2 - \vec{r}_3 - \vec{r}_4) \underset{P^{13}}{\longrightarrow} \tfrac{1}{2}(\vec{r}_3 + \vec{r}_2 - \vec{r}_1 - \vec{r}_4) = \vec{R}\,'_{\mathrm{I,II}}\ . \tag{3.3}$$

Next we introduce the "center-of-mass extracted" vector variables r'_1 that are defined by

$$\vec{r}\,'_i = \vec{r}_i - \vec{R}_1\ \ (i=1,2),\ \ \vec{r}\,'_j = \vec{r}_j - \vec{R}_2\ \ (j=3,4)\ . \tag{3.4}$$

In terms of these variables, $\vec{R}\,'$ can be very simply written as

$$\vec{R}'_{\mathrm{I,II}} = \vec{r}\,'_3 - \vec{r}\,'_1 = \vec{r}\,'_2 - \vec{r}\,'_4\ . \tag{3.5}$$

For the sake of clarity, we sketch in Fig. 4 what happens when we exchange quarks 1 and 3.

The integration volume is

$$dv = d^3\vec{r}_1 d^3\vec{r}_2 d^3\vec{r}_3 d^3\vec{r}_4 = |J|\, d^3\vec{R}_T d^3\vec{R}\, d^3\vec{r}_1 d^3\vec{r}_2 \tag{3.6}$$

with J being a suitable Jacobian, and $\vec{R}_T = \vec{R}_1 + \vec{R}_2$. Trivial insertions of unity, like for instance $1 = d^3\vec{R}\,'\delta^3(\vec{R}\,' - (\vec{r}\,'_2 - \vec{r}\,'_4))$, will lead us to

$$dv \cong d^3\vec{R}_T d^3\vec{R}\, d^3\vec{R}\,'\delta^3(\vec{R}\,' - (\vec{r}\,'_2 - \vec{r}\,'_4)) d^3\vec{r}\,'_1 d^3\vec{r}\,'_2 \delta^3(\vec{r}\,'_1 + \vec{r}\,'_2) d^3\vec{r}\,'_3 d^3\vec{r}\,'_4 \delta^3(\vec{r}\,'_3 + \vec{r}\,'_4)\ . \tag{3.7}$$

It will become apparent that none of the algebraic manipulations we are about to execute excite the degree of freedom $\vec{R}_T$, which means that we need only to be concerned with the actual differential volume,

$$dv' \cong d^3\vec{R}\, d^3\vec{R}\,' \prod_{i=1}^{4} d^3\vec{r}\,'_i \delta^3(\vec{r}\,'_1 + \vec{r}\,'_2)\delta^3(\vec{r}\,'_2 + \vec{r}\,'_4)\delta^3(\vec{R}\,' - (\vec{r}\,'_2 - \vec{r}\,'_4))\ . \tag{3.8}$$

A further modification is required in order to account for the unusual normalization used where defining the harmonic wave functions for the mesons. As an example, the ground-state wave function for a meson reads

$$\phi(\vec{\rho}_{12}) = \frac{1}{N}\exp\left[-\frac{(\vec{r}\,'_1 - \vec{r}\,'_2)^2}{2R_0{}^2}\right],\ \ N = \pi^{3/4}R_0{}^{3/2} \tag{3.9}$$

and the integration volume, $dv = 2^3 d^3\vec{r}\,'_1 d^3\vec{r}\,'_2 \delta^3(\vec{r}\,'_1 + \vec{r}\,'_2)$, such that $\langle \phi(\vec{\rho}_{12}) \,|\, \phi(\vec{\rho}_{12}) \rangle = 1$. With the above functions, one writes dv' as

$$dv' = d^3\vec{R}\,' d^3\vec{R}\, dv'',\ \ dv'' = 2^6 \prod_{i=1}^{4} d^3\vec{r}\,'_1 \delta^3(\vec{R}\,' - (\vec{r}\,'_2 - \vec{r}\,'_4)) d^3(\vec{r}\,'_1 + \vec{r}\,'_2) d^3(\vec{r}\,'_3 + \vec{r}\,'_4)\ . \tag{3.10}$$

The symbol $\langle\ \rangle$ in the definition of $P(s_i,R,R')$ means that the integration $\int dv''$ is performed.

It is perhaps clarifying to show a special case,

$$\langle \phi_{0,0,0}(\vec{\rho}_{12})\phi_{0,0,0}(\vec{\rho}_{34}) \,|\, P^{13} \,|\, \phi_{0,0,0}(\vec{\rho}_{12})\phi_{0,0,0}(\vec{\rho}_{34}) \rangle = P(0,\vec{R},\vec{R}') \ . \tag{3.11}$$

It is easy, albeit cumbersome, to show that $P(0,\vec{R},\vec{R}')$, with the integration measure (3.10), reads

$$\frac{2^{3/4}}{\pi^{3/2}} e^{-R^2} \frac{2^{3/4}}{\pi^{3/2}} e^{-R'^2} = \phi_{0,0,0}(\sqrt{2}R)\phi_{0,0,0}(\sqrt{2}R'), \quad R_0 = 1 \ . \tag{3.12}$$

Because the algebraic calculations leading to the results (3.12) are the same as those used in deriving the general result, we refer the reader to Sec. III B.

It is convenient, at this stage, to refer to two properties possessed by the generating functional $G(\vec{s},\vec{\rho})$:

(i) $G^*(\vec{s},\vec{\rho}) = G(-\vec{s},\vec{\rho})$.

(ii) If

$$G(\vec{s},\vec{\rho}) = \sum_{nlm} A_{nlm}\phi_{nlm}(\vec{\rho}) \,|\, \vec{s} \,|^{2n+l} Y^*_{lm}(\hat{s}) \ , \tag{3.13a}$$

then

$$G\left[\frac{\vec{s}}{\lambda}, \lambda\vec{\rho}\right] = \sum_{nlm} \lambda^{-3/4} A_{nlm}\phi_{nlm}(\lambda\vec{\rho}) \left[\frac{|\vec{s}|}{\lambda}\right]^{2n+l} Y^*_{lm}(\hat{s}) \ . \tag{3.13b}$$

Consider now,

$$\langle G(\vec{s}_3,\vec{\rho}_{12})G(\vec{s}_4,\vec{\rho}_{34}) \,|\, P^{13} \,|\, G(\vec{s}_1,\vec{\rho}_{12})G(\vec{s}_2,\vec{\rho}_{34}) \rangle \ .$$

The ket expression corresponds to (for notational convenience we suppress the arrow superscripts)

$$\exp[s_1{}^2 - \tfrac{1}{2}(r_1' - r_2')^2 + 2is_1\cdot(r_1' - r_2') + s_2{}^2 - \tfrac{1}{2}(r_3' - r_4')^2 + 2is_3\cdot(r_3' - r_4')] \ , \tag{3.14}$$

$\rho_{12} = r_1 - r_2 = r_1' - r_2'$, $\rho_{34} = r_3 - r_4 = r_3' - r_4'$. Using the fact that $P^{13}R_{\mathrm{I,II}} = R'_{\mathrm{I,II}} = R' = r_3' - r_1' = r_2' - r_4'$, we have (when $r_1 \leftrightarrow r_3$)

$$\exp\{ s_1{}^2 + s_2{}^2 - \tfrac{1}{2}[(r_1' - r_2')^2 + (r_3' - r_4')^2] - (R - R')^2 + 2is_1(r_3 - r_2) + 2is_2(r_1 - r_4) \} \tag{3.15}$$

so that the total product $G \times G \times P^{13} \times G \times G$ reads

$$\exp\{ s_1{}^2 + s_2{}^2 + s_3{}^2 + s_4{}^2 - [(r_1' - r_2')^2 + (r_3' - r_4')^2] - (R - R')^2 + 2is_1(r_3 - r_2) + 2is_3(r_1 - r_2) + 2is_2(r_1 - r_4) + 2is_4(r_3 - r_4) \} = \exp(z) \ . \tag{3.16}$$

Using

$$r_3 - r_2 = r_3' - r_2' + R_2 - R_1 = r_3' - r_2' - R, \quad r_1 - r_2 = r_1' - r_2', \text{etc.} \ , \tag{3.17}$$

the imaginary part of z becomes

$$2i[s_1(r_3' - r_2' - R) + s_3(r_1' - r_2') + s_2(r_1' - r_4' + R) + s_4(r_3' - r_4')] \ . \tag{3.18}$$

(a). Integration in r_3', r_1' uses up the δ functions $\delta^3(r_1' + r_2')\delta^3(r_3' + r_4')$ and yields

$$2i[s_1(-r_4' - r_2' - R) + s_3(-2r_2') + s_2(-r_2' - r_4' + R) + s_4(-2r_4')] \ . \tag{3.19}$$

(b). Integration in $r_4'\delta^3(R' - (r_2' - r_4'))$ gives

$$2i\{ s_1(-2r_2' + R' - R) + s_3(-2r_2') + s_2(-2r_2' + R' - R) + s_4[-2(r_2' - R')] \} \tag{3.20}$$

which can be written as

$$2i[s_1(-2A - R) + s_3(-2A - R') + s_2(-2A + R) + s_4(-2A + R')] \tag{3.21}$$

with $A = r_2' - R'/2$.

So much for the imaginary part of z. The real part of z going through steps (a) and (b) yields

$$s_1{}^2+s_2{}^2+s_3{}^2+s_4{}^2-8A^2-R^2-R'^2 . \tag{3.22}$$

We get for $\langle GG \mid P^{13} \mid GG \rangle$ the expression

$$2^6 \int \exp[s_1{}^2+s_2{}^2+s_3{}^2+s_4{}^2-8A^2-4iA(s_1+s_2+s_3+s_4)-2i(s_1-s_2)\cdot R-2i(s_3-s_4)\cdot R']d^3r_2' . \tag{3.23}$$

Rearranging terms,

$$\langle GG \mid P^{13} \mid GG \rangle = 2^6 \int \exp\left[\frac{(s_1+s_2)^2}{2}+\frac{(s_3+s_4)^2}{2}-8A^2-4iA(s_1+s_2+s_3+s_4) + \frac{(s_1-s_2)^2}{2}-2i(s_1-s_2)R+\frac{(s_3-s_4)^2}{2}-2i(s_3-s_4)R'\right]d^3r_2' ,$$

$$A=r_2'-\frac{R'}{2} . \tag{3.24}$$

Further rearrangement yields

$$\langle GG \mid P^{13} \mid GG \rangle = 2^6 \int \exp[-8A^2+\tfrac{1}{2}(s_1+s_2+s_3+s_4)^2-4iA(s_1+s_2+s_3+s_4)]d^3r_2' \times\exp[-(s_1+s_2)(s_3+s_4)]\exp\left[\frac{(s_1-s_2)^2}{2}-2i(s_1-s_2)R-R^2\right] \times\exp\left[\frac{(s_3-s_4)^2}{2}-2i(s_3-s_4)R'-R'^2\right] . \tag{3.25}$$

The integration can be easily performed $r_2' \rightarrow A$, and we obtain

$$\langle GG \mid P^{13} \mid GG \rangle = (2\pi)^{3/2} e^{-(s_1+s_2)(s_3+s_4)} G\left[\frac{(s_1-s_2)}{\sqrt{2}},\sqrt{2}R\right] G\left[\frac{(s_3-s_4)}{\sqrt{2}},\sqrt{2}R'\right] . \tag{3.26}$$

The expression $\langle GG \mid P^{13} \mid GG \rangle$ means $\int dv'' G^\dagger G^\dagger P^{13} GG$, therefore using (3.13a), we finally obtain

$$\langle G(s_3,\rho_{12})G(s_4,\rho_{34}) \mid P^{13} \mid G(s_1,\rho_{12})G(s_2,\rho_{34})\rangle = (2\pi)^{3/2} G\left[\frac{s_2-s_1}{\sqrt{2}},\sqrt{2}R\right] G\left[\frac{s_3-s_4}{\sqrt{2}},\sqrt{2}R'\right] e^{(s_1+s_2)(s_3+s_4)} . \tag{3.27}$$

B. Power-series expansions of the G's

We start with $e^{(s_1+s_2)(s_3+s_4)}$:

$$e^{(s_1+s_2)(s_3+s_4)} = e^{s_1s_3+s_2s_3+s_1s_4+s_2s_4} . \tag{3.28}$$

Using the formulas

$$e^{s_is_j} = 4\pi j_l(s_is_j)Y^*_{lm}(\hat{s}_i)Y_{lm}(\hat{s}_j), \quad Y_{l_im_i}(\hat{s})Y_{l_jm_j}(\hat{s}) = \begin{bmatrix} l_i & l_j & L_{ij} \\ m_i & m_j & M_{ij} \end{bmatrix}_y Y_{L_{ij},M_{ij}}(\hat{s}) \tag{3.29}$$

with

$$\begin{bmatrix} l_i & l_j & L_{ij} \\ m_i & m_j & M_{ij} \end{bmatrix}_y = \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L_{ij}+1)}\right]^{1/2} \begin{bmatrix} l_1 & l_2 & L_{12} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} \tag{3.30}$$

and with

$$\begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix}, \quad \begin{bmatrix} l_1 & l_2 & L_{12} \\ 0 & 0 & 0 \end{bmatrix}$$

being appropriate Clebsch-Gordan coefficients for SO(3), we get

$$e^{(s_1+s_2)(s_3+s_4)} = (4\pi)^4 j_{l_a}(s_1s_3)j_{l_b}(s_2s_3)j_{l_c}(s_1s_4)j_{l_d}(s_2s_4)$$
$$\times \begin{bmatrix} l_a & l_c & l_{ac} \\ m_a & m_c & m_{ac} \end{bmatrix}_y \begin{bmatrix} l_b & l_d & l_{bd} \\ m_b & m_d & m_{bd} \end{bmatrix}_y \begin{bmatrix} l_a & l_b & l_{ab} \\ m_a & m_b & m_{ab} \end{bmatrix}_y \begin{bmatrix} l_c & l_d & l_{cd} \\ m_c & m_d & m_{cd} \end{bmatrix}_y$$
$$\times Y^*_{l_{ac},m_{ac}}(\hat{s}_1)Y^*_{l_{bd},m_{bd}}(\hat{s}_2)Y_{l_{ab},m_{ab}}(\hat{s}_3)Y_{l_{cd},m_{cd}}(\hat{s}_4) \ . \tag{3.31}$$

(Summing over the repeated indices is assumed from now on.) $G((s_2-s_1)/\sqrt{2},\sqrt{2}R)$ gives [see (3.13b)]

$$G\left[\frac{s_2-s_1}{\sqrt{2}},\sqrt{2}R\right] = \sum_{nlm} i^l \left[\frac{\pi^3}{\sqrt{2}n!\Gamma(n+l+\frac{3}{2})}\right]^{1/2} \phi_{nlm}(\sqrt{2}R)\frac{|s_2-s_1|^{2n+l}}{(\sqrt{2})^{2n+l}}Y^*_{lm}(\hat{s}_{2-1}) \ ,$$

where $\hat{s}_{2-1}=(\vec{s}_2-\vec{s}_1)/|\vec{s}_2-\vec{s}_1|$. There is an analogous expression for $G((s_3-s_4)/\sqrt{2},\sqrt{2}R')$. Next we use the result (see Appendix A)

$$Y_{L_{12}M_{12}}(\rho_2-\rho_1)|\rho_2-\rho_1|^{L_{12}} = (-1)^{l_1}\left[\frac{2\pi^{3/2}\Gamma(L_{12}+\frac{3}{2})}{\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})}\right]^{1/2} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} |\rho|^{l_1}|\rho|^{l_2}Y_{l_1m_1}(\hat{\rho}_1)Y_{l_2m_2}(\hat{\rho}_2)$$
$$\times (C_{l_1}^{L_{12}})^{1/2}\delta(L_{12}-l_1-l_2) \ , \tag{3.32}$$

where $C_{l_1}^{L_0}$ represents $L_0!/l_1!(L_0-l_1)!$.

Substituting all the above results into

$$G\left[\frac{s_2-s_1}{\sqrt{2}},\sqrt{2}R\right]G\left[\frac{s_3-s_4}{\sqrt{2}},\sqrt{2}R'\right]e^{(s_1+s_2)(s_3+s_4)}$$

yields trivially [we use the notational convention that $\Gamma(l+\frac{3}{2})\equiv\Gamma_1(l)$]

$$(2\pi)^{10}\sum i^{L_{12}+L_{34}}\left[\frac{\Gamma_1(L_{12})\Gamma_1(L_{34})}{N_1!\Gamma_1(L_{12}+N_1)N_2!\Gamma_1(N_2+L_{34})\Gamma_1(l_1)\Gamma_1(l_2)\Gamma_1(l_3)\Gamma_1(l_4)}\right]^{1/4}$$
$$\times(-1)^{l_1+l_4}\frac{\phi_{N_1L_{12}M_{12}}(\sqrt{2}R)\phi^*_{N_2L_{34}M_{34}}(\sqrt{2}R')}{(\sqrt{2})^{2N_1+2N_2+L_1+L_2}}(C_{l_1}^{L_{12}}C_{l_3}^{L_{34}})^{1/2}$$
$$\times\begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix}\begin{bmatrix} l_3 & l_4 & L_{34} \\ m_3 & m_4 & M_{34} \end{bmatrix}\delta(L_{12}-l_1-l_2)\delta(L_{34}-l_3-l_4)$$
$$\times|s_1|^{l_1}|s_2|^{l_2}|s_3|^{l_3}|s_4|^{l_4}\begin{bmatrix} l_1 & l_{ac} & L_{1ac} \\ m_1 & m_{ac} & M_{1ac} \end{bmatrix}_y\begin{bmatrix} l_2 & l_{bd} & L_{2bd} \\ m_2 & m_{bd} & M_{2bd} \end{bmatrix}_y$$
$$\times\begin{bmatrix} l_3 & l_{ab} & L_{3ab} \\ m_3 & m_{ab} & M_{3ab} \end{bmatrix}_y\begin{bmatrix} l_4 & l_{cd} & L_{4cd} \\ m_4 & m_{cd} & M_{4cd} \end{bmatrix}_y\begin{bmatrix} l_a & l_c & l_{ac} \\ m_a & m_c & m_{ac} \end{bmatrix}_y\begin{bmatrix} l_b & l_d & l_{bd} \\ m_b & m_d & m_{bd} \end{bmatrix}_y$$
$$\times\begin{bmatrix} l_a & l_b & l_{ab} \\ m_a & m_b & m_{ab} \end{bmatrix}_y\begin{bmatrix} l_c & l_d & l_{cd} \\ m_c & m_d & m_{cd} \end{bmatrix}_y|s_1-s_2|^{2N_1}|s_3-s_4|^{2N_2}j_{l_a}(s_1s_3)j_{l_d}(s_2s_3)j_{l_c}(s_1s_4)j_{l_d}(s_2s_4)$$
$$\times Y^*_{L_{1ac},M_{1ac}}(\hat{s}_1)Y^*_{L_{2bd},M_{2bd}}(\hat{s}_2)Y_{L_{3ab},M_{3ab}}(\hat{s}_3)Y_{L_{4cd},M_{4cd}}(\hat{s}_4) \ . \tag{3.33}$$

If, on the other hand, we develop the expression $\langle GG \mid P^{13} \mid GG \rangle$, we obtain

$$\sum i^{l_1'+l_2'-l_3'-l_4'} \left[\frac{(2\pi^3)^4}{\Gamma_1(l_1')\Gamma_1(l_2')\Gamma_1(l_3')\Gamma_1(l_4')} \right]^{1/2} |s_1|^{2n_1+l_1'} |s_2|^{2n_2+l_2'} |s_3|^{2n_3+l_3'} |s_4|^{2n_4+l_4'}$$

$$\times Y^*_{l_1',m_1'}(\hat{s}_1) Y^*_{l_2',m_2'}(\hat{s}_2) Y_{l_3',m_3'}(\hat{s}_3) Y_{l_4',m_4'}(\hat{s}_4)$$

$$\times \langle \phi_{n_3,l_3',m_3'}(\rho_{12}) \phi_{n_4,l_4',m_4'}(\rho_{34}) \mid P \mid \phi_{n_1,l_1',m_1'}(\rho_{12}) \phi_{n_2,l_2',m_2'}(\rho_{34}) \rangle \ . \quad (3.34)$$

We recall that the

$$G \left[\frac{s_3-s_4}{\sqrt{2}}, \sqrt{2}R \right] G \left[\frac{s_2-s_1}{\sqrt{2}}, \sqrt{2}R' \right] e^{(s_1+s_2)(s_3+s_4)}$$

s-angular dependence is of the form $Y^*_{L_{1ac},M_{1ac}}(\hat{s}_1) \cdots Y_{L_{4cd},M_{4cd}}(\hat{s}_4)$. The variables s_i are arbitrary; hence,

$$l_1' = L_{1ac}, \quad l_2' = L_{2bd}, \quad l_3' = L_{3ab}, \quad l_4' = L_{4cd} \ . \quad (3.35)$$

Returning to Eq. (3.33) we see that we can set $N_1, N_2 = 0$. It suffices to use the remaining integrations in R, R' to project, with the help of $g_{\rm I,II}(R) = \phi_{0LM}(\sqrt{2}R)$; $g'_{\rm I,II}(R') - \phi_{0L'M'}(\sqrt{2}R')$, the $N_{1,2}=0$ part. A particular set of values L, M; L', M' can also be chosen this way. We use the result (see Appendix B)

$$\left[\frac{\Gamma_1(L_0)}{2\pi^{3/2}\Gamma_1(l_1)\Gamma_1(l_2)} \right]^{1/2} \begin{bmatrix} l_1 & l_2 & \big| L_0 \\ m_1 & m_2 & \big| M_0 \end{bmatrix}_y = (C_{l_1}^{L_0})^{1/2} \begin{bmatrix} l_1 & l_2 & \big| L_0 \\ m_1 & m_2 & \big| M_0 \end{bmatrix} \quad (3.36)$$

to simplify expression (3.33). We get

$$2^6\pi^4 \sum i^{L_{12}+L_{34}} (-1)^{l_1+l_4} \left[\frac{\Gamma_1{}^2(l_a)\Gamma_1{}^2(l_b)\Gamma_1{}^2(l_c)\Gamma_1{}^2(l_d)}{\Gamma_1(L_{1ac})\Gamma_1(L_{2bd})\Gamma_1(L_{3ab})\Gamma_1(L_{4cd})} \right]^{1/2}$$

$$\times \left[C_{l_1}^{L_{12}} C_{l_3}^{L_{34}} C_{l_1}^{L_{1ac}} C_{l_2}^{L_{2bd}} C_{l_3}^{L_{3ab}} C_{l_4}^{L_{4cd}} C_{l_a}^{l_{ac}} C_{l_b}^{l_{bd}} C_{l_a}^{l_{ab}} C_{l_c}^{l_{cd}} \right]^{1/2}$$

$$\times (l_1 l_2 \mid L_{12})(l_3 l_4 \mid L_{34})(l_1 l_{ac} \mid L_{1ac})(l_2 l_{bd} \mid L_{2bd})(l_3 l_{ab} \mid L_{3ab})(l_4 l_{cd} \mid L_{4cd})$$

$$\times (l_a l_c \mid l_{ac})(l_b l_d \mid l_{bd})(l_a l_b \mid l_{ab})(l_c l_d \mid l_{cd})$$

$$\times \frac{\phi_{L_{12}M_{12}}(\sqrt{2}R)\phi^*_{L_{34}M_{34}}(\sqrt{2}R') \, |s|^{l_1} |s_2|^{l_2} |s_3|^{l_3} |s_4|^{l_4}}{(\sqrt{2})^{L_{12}+L_{34}}}$$

$$\times j_{l_a}(s_1 s_3) j_{l_b}(s_2 s_3) j_{l_c}(s_1 s_4) j_{l_d}(s_2 s_4)$$

$$\times Y^*_{L_{1ac}M_{1ac}}(\hat{s}_1) Y^*_{L_{2bd}M_{2bd}}(\hat{s}_2) Y_{L_{3ab}M_{3ab}}(\hat{s}_3) Y_{L_{4cd}M_{4cd}}(\hat{s}_4) \delta(L_{12}-l_1-l_2)\delta(L_{34}-l_3-l_4) \ . \quad (3.37)$$

(The symbols $(l_i l_j \mid L_{ij})$ stand for

$$\begin{bmatrix} l_i & l_j & \Big| L_{ij} \\ m_i & m_j & \Big| M_{ij} \end{bmatrix} .$$

We must still equate the magnitude of the s's. For that we use the following expansion of the Bessel functions:

$$j_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!(2n+k+1)!!}x^{2k+n} . \tag{3.38}$$

There the s dependence of $j_{l_a}(s_1 s_3)j_{l_b}(s_2 s_3)j_{l_c}(s_1 s_4)j_{l_d}(s_2 s_4)$ looks like

$$|s_1|^{2(k_1+k_3)+l_a+l_c}|s_2|^{2(k_2+k_4)+l_b+l_d}|s_3|^{2(k_1+k_2)+l_a+l_b}|s_4|^{2(k_3+k_4)+l_c+l_d} . \tag{3.39}$$

Notice that in the expansion of $\langle GG \mid P^{13} \mid GG \rangle$, we arrived at an $|s|$ dependence of the form

$$|s_1|^{2n_1+L_{1ac}}|s_2|^{2n_2+L_{2bd}}|s_3|^{2n_3+L_{3ab}}|s_4|^{L_{4cd}+2n_4} .$$

$L_{1ac}, L_{2bd}, L_{3ab}, L_{4cd}$, we have seen, can be fixed, so that we are left with a remaining sum in n_1, n_2, n_3, n_4. We can set them to zero, and look in the other expansion [the one of $G((s_2-s_1)/\sqrt{2},\dots)$ $G((s_3-s_4)/\sqrt{2},\dots)e^{\cdots}$] for the *matching* powers. We have, therefore,

$$\begin{aligned} &L_{3ab}=l_a+l_b+l_3+2(k_1+k_2),\quad L_{2bd}=l_b+l_d+l_2+2(k_2+k_4),\\ &L_{1ac}=l_1+l_a+l_c+2(k_1+k_3),\quad L_{4cd}=l_c+l_d+l_4+2(k_3+k_4) , \end{aligned} \tag{3.40}$$

but on the other hand, in the expansion of $G((s_2-s_1)/\sqrt{2},\dots)$ we had Clebsch-Gordan coefficients like $(l_1,l_{ac} \mid L_{1ac})$ $(l_2,l_{bd} \mid L_{2bd})$ $(l_3,l_{ab} \mid L_{3ab})$ $(l_4,l_{cd} \mid L_{4cd})$. They imply

$$L_{1ac}\le l_1+l_{ac},\quad L_{2bd}\le l_2+l_{bd},\quad L_{3ab}\le l_3+l_{ab},\quad L_{4cd}\le l_4+l_{cd} ; \tag{3.41}$$

the k_i's are positive quantities. *Then they must be zero.*

Returning to $j_{l_a}(s_1 s_3)j_{l_b}(s_2 s_3)j_{l_c}(s_1 s_4)j_{l_d}(s_2 s_4)$, we notice that

$$(2n+1)!!=\Gamma_1(n)2^n/\Gamma_1(0) . \tag{3.42}$$

[Observe, that according to our notation, $\Gamma_1(0)=\Gamma(\frac{3}{2})$.] Therefore

$$j_{l_a}(s_1 s_3)\cdots j_{l_d}(s_2 s_4)\,|_{\{k_i=0\}}\,|s_1|^{l_1}|s_2|^{l_2}|s_3|^{l_3}|s_4|^{l_4}$$

$$=[\Gamma(0)]^4\frac{|s_1|^{L_{1ac}}|s_2|^{L_{2bd}}|s_3|^{L_{3ab}}|s_4|^{L_{4cd}}}{(\sqrt{2})^{2(l_a+l_b+l_c+l_d)}}\frac{1}{\Gamma_i(l_a)\Gamma_1(l_b)\Gamma_1(l_c)\Gamma_1(l_d)} . \tag{3.43}$$

This can be inserted in expression (3.37), to yield the final result $[\Gamma_1(0)=\Gamma(\frac{3}{2})=\sqrt{\pi}/2]$

$$(-1)^{l_1+l_3}\left[\prod C_{l_\beta}^{L_\alpha}\right]^{1/2}\left[\prod \begin{bmatrix} l_i & l_j & \Big| L_{ij} \\ m_i & m_j & \Big| M_{ij} \end{bmatrix}\right]\phi_{L_{12}M_{12}}(\sqrt{2}R)\phi^*_{L_{34}M_{34}}(\sqrt{2}R')\frac{1}{(\sqrt{2})^{L_{12}+L_{34}+2(l_a+l_b+l_c+l_d)}}$$

$$=\langle \phi_{L_{3ab}M_{3ab}}(\vec{\rho}_{12}),\phi_{L_{4cd}M_{4cd}}(\vec{\rho}_{34}) \mid P^{13} \mid \phi_{L_{1ac}M_{1ac}}(\vec{\rho}_{12}),\phi_{L_{2bd}M_{2bd}}(\vec{\rho}_{34})\rangle . \tag{3.44}$$

The products of ten Clebsch-Gordan and ten combinatorials are abbreviated by

$$\prod \begin{bmatrix} l_i & l_j & \Big| L_{ij} \\ m_i & m_j & \Big| M_{ij} \end{bmatrix} \text{ and } \prod C_{l_\beta}^{L_\alpha} ,$$

respectively.

C. Results and graphical rules

We have

$$
\begin{aligned}
\langle \phi_{L_{3ab}M_{3ab}}\phi_{L_{4cd}M_{4cd}} \,|\, P^{13} \,|\, \phi_{L_{1ac}M_{1ac}}\phi_{L_{2bd}M_{2bd}} \rangle
&= \sum_{\{l,m\}} (-1)^{l_1+l_3} \phi_{L_{12}M_{12}}(\sqrt{2}R)\phi^{*}_{L_{34}M_{34}}(\sqrt{2}R') \\
&\times \left[C_{l_1}^{L_{12}} C_{l_3}^{L_{34}} C_{l_1}^{L_{1ac}} C_{l_3}^{L_{3ab}} C_{l_2}^{L_{2bd}} C_{l_4}^{L_{4cd}} C_{l_a}^{l_{ac}} C_{l_b}^{l_{bd}} C_{l_a}^{l_{ab}} C_{l_c}^{l_{cd}} \right]^{1/2} \\
&\times \left[\begin{array}{cc|c} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{array}\right]
\left[\begin{array}{cc|c} l_3 & l_4 & L_{34} \\ m_3 & m_4 & M_{34} \end{array}\right]
\left[\begin{array}{cc|c} l_1 & l_{ac} & L_{1ac} \\ m_1 & m_{ac} & M_{1ac} \end{array}\right]
\left[\begin{array}{cc|c} l_2 & l_{bd} & L_{2bd} \\ m_2 & m_{bd} & M_{2bd} \end{array}\right] \\
&\times \left[\begin{array}{cc|c} l_3 & l_{ab} & L_{3ab} \\ m_3 & m_{ab} & M_{3ab} \end{array}\right]
\left[\begin{array}{cc|c} l_4 & l_{cd} & L_{4cd} \\ m_4 & m_{cd} & M_{4cd} \end{array}\right]
\left[\begin{array}{cc|c} l_a & l_c & l_{ac} \\ m_a & m_c & m_{ac} \end{array}\right]
\left[\begin{array}{cc|c} l_b & l_d & l_{bd} \\ m_b & m_d & M_{bd} \end{array}\right] \\
&\times \left[\begin{array}{cc|c} l_a & l_b & l_{ab} \\ m_a & m_b & M_{ab} \end{array}\right]
\left[\begin{array}{cc|c} l_c & l_d & L_{cd} \\ m_c & m_d & M_{cd} \end{array}\right]
\frac{\delta(L_{12}-l_1-l_2)\delta(L_{34}-l_3-l_4)}{(\sqrt{2})^{2(l_a+l_b+l_c+l_d)+L_{12}+L_{34}}} \\
&\times \delta(l_{ab}-l_a-l_b)\delta(l_{cd}-l_c-l_d)\delta(l_{ac}-l_a-l_c)\delta(l_{bd}-l_b-l_d) \\
&\times \delta(L_{3ab}-l_3-l_{ab})\delta(L_{2bd}-l_2-l_{bd})\delta(L_{1ac}-l_1-l_{ac})\delta(L_{4cd}-l_4-l_{cd}) \,, \qquad (3.45)
\end{aligned}
$$

where one sums over all contributing l's and m's (L_{12}, L_{34} are fixed by external conditions).

One can deduce from the above expression a simple graphical rule.

(1) Draw straight lines representing the mesons, and also two straight lines to represent the relative angular momentum between them (dashed lines); see Fig. 5.

(2) Connect all lines except for the connection (L_{12},L_{34}) as is shown in Fig. 6. Note that the lines that connect any meson with a relative angular momentum line are different from the others (wiggly).

(3) Identity "propagators" (Fig. 7).

(4) Identity "vertices" (Fig. 8).

(5) The value of a particular diagram is obtained by multiplying all the vertices and propagators. Sum over all possible intermediate "states." Note that for moderately low l's, only a few diagrams contribute.

IV. CONCLUSIONS

The matrix elements of the exchange operator between two mesons for arbitrary angular momenta have been derived. A straightforward extension to include radial excitations is possible. By explicit calculation we have seen that the exchange operator for pure angular momentum states (no radial excitations) does not excite intermediate radial excitations. This is a considerable simplification. If we allow some of the mesons

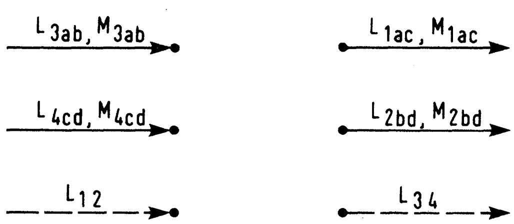

FIG. 5. Rule 1.

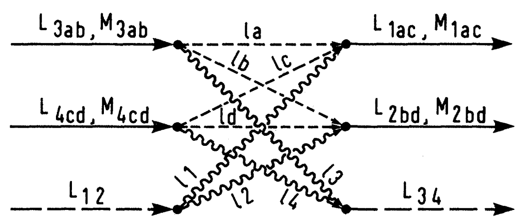

FIG. 6. Rule 2.

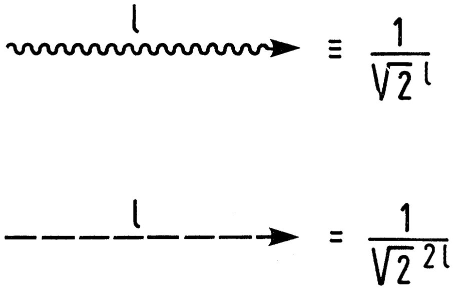

FIG. 7. Rule 3.

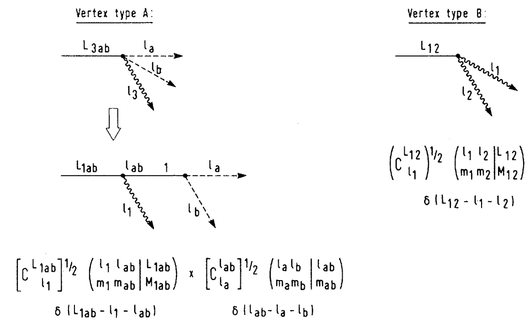

FIG. 8. Rule 4.

ϕ and/or the relative wave functions between them to have radial excitations, we are forced to sum over a family of internal radial excitations and the number of diagrams which contribute for a given process is comparatively much larger.

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APPENDIX A

We start by considering the functional generator $G(s,\vec{\rho})$. There it is a straightforward piece of algebra to show that

$$G(\vec{s},\vec{\rho}_1)G(\vec{s},\vec{\rho}_2)=G\left[\sqrt{2}\vec{s},\frac{\vec{\rho}_1+\vec{\rho}_2}{\sqrt{2}}\right]e^{-(\vec{\rho}_1-\vec{\rho}_2)^2/4}\ . \tag{A1}$$

Now we use property (3.13-ii) to expand $G(\sqrt{2}s,(\vec{\rho}_1+\vec{\rho}_2)/\sqrt{2})$. Expanding also each of the $G(\vec{s},\vec{\rho}_i)$ $(i=1,2)$ we have

$$\sum_{\substack{n_1l_1m_1\\ n_2l_2m_2}} i^{l_1+l_2}\left[\frac{4\pi^6}{n_1!n_2!\Gamma(n_1+l_1+\frac{3}{2})\Gamma(n_2+l_2+\frac{3}{2})}\right]^{1/2}|\vec{s}\,|^{2(n_1+n_2)+l_1+l_2}$$

$$\times Y^*_{l_1m_1}(\hat{s})Y_{l_2m_2}(\hat{s})\phi_{2n_1l_1m_1}(\vec{\rho}_1)\phi_{2n_2l_2m_2}(\vec{\rho}_2)$$

$$=\sum_{NLM} i^L\left[\frac{4\sqrt{2}\pi^3}{N!\Gamma(N+l+\frac{3}{2})}\right]^{1/2}Y^*_{LM}(\hat{s})\phi_{2NLM}\left[\frac{\vec{\rho}_1+\vec{\rho}_2}{\sqrt{2}}\right]|\vec{s}\,|^{2N+L}(\sqrt{2})^{2N+L_{12}}e^{-(\vec{\rho}_1-\vec{\rho}_2)^2/4}\ . \tag{A2}$$

Integration in $Y_{LM}(\hat{s})$, and using the fact that

$$Y_{l_1m_1}(\hat{s})Y_{l_2m_2}(\hat{s})=\left[\begin{matrix}l_1 & l_2\\ m_1 & m_2\end{matrix}\middle|\begin{matrix}L_{12}\\ M_{12}\end{matrix}\right]_y Y_{L_{12},M_{12}}(\hat{s})\ ,$$

and also—see Appendix B—that

$$\left[\begin{matrix}l_1 & l_2\\ m_1 & m_2\end{matrix}\middle|\begin{matrix}L_{12}\\ M_{12}\end{matrix}\right]_y$$

is expressible in Clebsch-Gordan coefficients, we are led into

$$\sum_{n_1 l_1 m_1} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} \left[\frac{\pi^{3/2} C_{l_1}^{L_0}}{2n_1! n_2! \Gamma(L_{12}+\frac{3}{2})} \right]^{1/2} \phi_{2n_1 l_1 m_1}(\vec{\rho}_1)\phi_{2n_2 l_2 m_2}(\vec{\rho}_2) \,|\, s \,|^{\,2(n_1+n_2)+l_1+l_2}$$

$$= \sum_{N L_0 M_0} \left[\frac{\sqrt{2}}{N!\Gamma(N+L_{12}+\frac{3}{2})} \right]^{1/2} \phi_{2NL_{12}M_{12}} \left[\frac{\vec{\rho}_1+\vec{\rho}_2}{\sqrt{2}} \right] |\, s \,|^{\,2N+L_{12}} e^{-(\vec{\rho}_1-\vec{\rho}_2)^2/4} (\sqrt{2})^{2N+L_{12}} \,. \quad \text{(A3)}$$

The integration in $Y_{LM}(\hat{s})$ fixed for us on L_{12}. Now if we set $N=0$, then

$$|\,\vec{s}\,|^{\,2(n_1+n_2)+l_1+l_2} \equiv |\,\vec{s}\,|^{\,L_{12}} \,. \quad \text{(A4)}$$

The Clebsch-Gordan boundary condition $L_{12} \le l_1+l_2$ implies that $n_1=n_2=0$ and $L_{12}=l_1+l_2$.

The harmonic-oscillator wave function $\phi_{2NL_{12}M_{12}}[(\vec{\rho}_1+\vec{\rho}_2)/\sqrt{2}]$ contains a term that goes like exp $[(\vec{\rho}_1+\vec{\rho}_2)^2/4]$. This term multiplied by exp $[(\vec{\rho}_1-\vec{\rho}_2)^2/4]$ yields exp $(-\frac{1}{2}\vec{\rho}_1^{\,2}-\frac{1}{2}\vec{\rho}_2^{\,2})$, which cancels against similar exponentials (coming from ϕ_{l_1},ϕ_{l_2}) on the left-hand side of expression (A3). We get, therefore,

$$\sum_{\substack{l_1 m_1 \\ l_2 m_2}} \delta_{l_1+l_2,L_{12}} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} \left[\frac{2\pi^{3/2}\Gamma(L_{12}+\frac{3}{2})}{\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})} C_{l_1}^{L_{12}} \right]^{1/2}$$

$$\times |\,\vec{\rho}_1\,|^{\,l_1} |\,\vec{\rho}_2\,|^{\,l_2} Y_{l_1 m_1}(\vec{\rho}_1) Y_{l_2 m_2}(\vec{\rho}_2) = (\vec{\rho}_1+\vec{\rho}_2)^{L_0} Y_{L_{12}M_{12}}(\hat{\rho}_{1+2}) \,, \quad \text{(A5)}$$

where $\hat{\rho}_{1+2}=(\vec{\rho}_1+\vec{\rho}_2)/|\,\vec{\rho}_1+\vec{\rho}_2\,|$.

APPENDIX B

We are going to prove that

$$\begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix}_y = \left[\frac{\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})}{2\pi^{3/2}\Gamma(L_{12}+\frac{3}{2})} \right]^{1/2} (C_{l_1}^{L_0})^{1/2} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} . \quad \text{(B1)}$$

By definition

$$\begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix}_y = \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} \begin{bmatrix} l_1 & l_2 & L_{12} \\ 0 & 0 & 0 \end{bmatrix} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_0+1)} \right]^{1/2} . \quad \text{(B2)}$$

We must prove that

$$\begin{bmatrix} l_1 & l_2 & L_{12} \\ 0 & 0 & 0 \end{bmatrix} = C_{l_1}^{L_{12}} / (C_{2l_1}^{2L_{12}})^{1/2} \quad \text{(B3a)}$$

and that

$$\left[\frac{\Gamma(L_{12}+\frac{3}{2})}{\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})} \right]^{1/2} = \left[C_{2l_1+2}^{2L_{12}+1} / (2l_2+1) C_{l_1+1}^{L_{12}} \right]^{1/2} \frac{\sqrt{2}}{\pi^{1/4}} \,. \quad \text{(B3b)}$$

Substitution of (B3a) and (B3b) in (B1) leads us to

$$\left[\frac{\Gamma(L_{12}+\frac{3}{2})}{\Gamma(l_1+\frac{3}{2})\Gamma(l_2+\frac{3}{2})} \right]^{1/2} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix}_y = \left[\frac{l_2}{l_1+1} \frac{1}{C_{l_1+1}^{L_{12}}} \right]^{1/2} \frac{1}{\sqrt{2}\pi^{3/4}} C_{l_1}^{L_{12}} \begin{bmatrix} l_1 & l_2 & L_{12} \\ m_1 & m_2 & M_{12} \end{bmatrix} . \quad \text{(B4)}$$

Noticing that

$$\frac{l_2}{l_1+1}\frac{1}{C_{l_1+1}^{L_{12}}}=\frac{1}{C_{l_1}^{L_{12}}}\,, \tag{B5}$$

we arrive at (B1).

Proof of (B3a). From the definition,[5] we have

$$\begin{bmatrix} l_1 & l_2 & L_{12} \\ 0 & 0 & 0 \end{bmatrix} = \left[\frac{(l_1+l_2)!^2(2l_1)!(2l_2)!}{[2(l_1+l_2)]!(l_1!)^2(l_2!)^2}\right]^{1/2} = C_{l_1}^{L_{12}}/(C_{2l_1}^{2L_{12}})^{1/2}\,. \tag{B6}$$

Proof of (B3b). From the definition, we have

$$\Gamma(N+\tfrac{3}{2})=\frac{(2N+1)!!}{2^N}\Gamma(\tfrac{3}{2})\,. \tag{B7}$$

Then it is a simple algebraic manipulation to show that

$$\Gamma(N+M+\tfrac{3}{2})=2^{1-2M}\frac{(2N+2M+1)!(N+1)!}{(2N+2)!(N+M)!}\Gamma(N+\tfrac{3}{2})\,. \tag{B8}$$

In the case $N=0$,

$$\Gamma(N+\tfrac{3}{2})=2^{-2N}\frac{(2N+1)!}{N!}\Gamma(\tfrac{3}{2})\,. \tag{B9}$$

Then,

$$\frac{\Gamma(N+M+\tfrac{3}{2})}{\Gamma(N+\tfrac{3}{2})\Gamma(M+\tfrac{3}{2})}=\frac{2(M)!(2N+2M+1)!(N+1)!}{(2M+1)!(2N+2)!(N+M)!}\frac{1}{\Gamma(\tfrac{3}{2})}\,. \tag{B10}$$

Multiplying the right-hand side by $(N-1)!/(M-1)!$, we obtain, after trivial manipulations,

$$\frac{\Gamma(N+M+\tfrac{3}{2})}{\Gamma(N+\tfrac{3}{2})\Gamma(M+\tfrac{3}{2})}=\frac{C_{2N+2}^{2N+2M+1}}{\Gamma(\tfrac{3}{2})(2M+1)C_{2N+2}^{2N+2M+1}}\,,\quad \Gamma(\tfrac{3}{2})=\frac{\sqrt{\pi}}{2}\,. \tag{B11}$$

Equation (B11) for the case $N=l_1$, $M=l_2$ yields (B3b).

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