Lepton-number conservation and the double-β decay of ^{128}Te and ^{130}Te

W. C. Haxton
*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
and Department of Physics, Purdue University, West Lafayette, Indiana 47907*

G. J. Stephenson, Jr. and D. Strottman
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
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We consider the classic argument of Pontecorvo that the ratio of $\beta\beta$-decay half-lives for ^{130}Te and ^{128}Te may indicate violation of lepton number. We believe that the theoretical support for the assumption underlying this argument, equality of the nuclear matrix elements mediating the 2ν $\beta\beta$ decay of these isotopes, is weak due to a significant disparity between the calculated and geochemical absolute rates. If one ignores this inconsistency and attributes the breaking of the γ_5 invariance of the weak leptonic current to a Majorana mass, faithful treatment of the radial dependence of the 0ν-$\beta\beta$-decay operators yields $\langle m^{\rm Maj}\rangle_\nu \cong 10$ eV.

Current theoretical attempts to unify the electroweak and strong interactions may succeed only if certain symmetries of the Glashow-Weinberg-Salam model, such as exactly conserved baryon and lepton numbers and massless neutrinos, are abandoned. Sensitive experimental tests of such symmetries are thus being pursued with great urgency. One powerful probe of lepton-number conservation, of the mass and charge-conjugation properties of the neutrino, and of possible right-handed admixtures in the weak leptonic current is nuclear double-β decay, $(A,Z-2)\rightarrow(A,Z)$.[1] This process can be observed in a number of even-even nuclei where, due to the pairing interaction, the competing decay $(A,Z-2)\rightarrow(A,Z-1)$ is energetically inaccessible. The key issue is whether $\beta\beta$ decay proceeds, as in the standard model, entirely by those second-order weak processes in which two electrons and two neutrinos are produced [see Fig. 1(a)], or whether additional mechanisms involving violation of lepton number also contribute. In particular, if the Majorana mass of the electron neutrino is nonzero, the neutrinoless $\beta\beta$ decay shown in Fig. 1(b) will occur.[2] It has also been suggested that a first-order process of a superweak interaction may produce a neutrinoless final state[3,4] and Georgi, Glashow, and Nussinov have recently considered a $\beta\beta$-decay mechanism in which the outgoing electrons are accompanied by a light scalar boson.[5]

Currently, the issue of lepton-number conservation in $\beta\beta$ decay must be argued in the context of limited experimental information. Laboratory bounds on the neutrinoless decay of Fig. 1(b) have been used to restrict the Majorana mass of the electron neutrino to $\langle m^{\rm Maj}\rangle_\nu \lesssim 15$ eV, assuming that there is no explicit right-handed coupling of the Majorana neutrino.[6] Somewhat less stringent limits can be obtained by attributing total geochemical $\beta\beta$-decay rates to neutrinoless decay.[7]

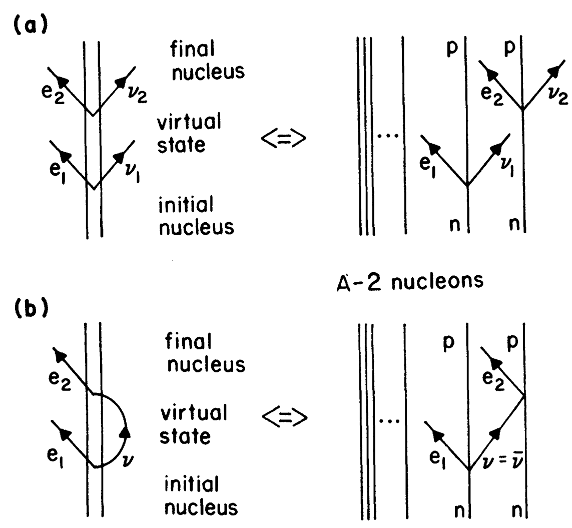

FIG. 1. Two-nucleon mechanisms for two-neutrino (a) and no-neutrino (b) $\beta\beta$ decay.

However, in both these approaches the limits on lepton-number violation can only be determined to within the accuracy permitted by nuclear-matrix-element calculations.

It is clearly of great importance to have a test of lepton-number violation which is less sensitive to structure uncertainties. One possibility, originally suggested by Pontecorvo,[3] is the ratio of the total $\beta\beta$-decay rates for ^{128}Te and ^{130}Te. The Pontecorvo argument has been used by Vergados[7] and by Bryman and Picciotto[8] to derive a nonzero value for a lepton-number-violating coupling in a phenomenological $\beta\beta$-decay Lagrangian. More recently the Osaka group has interpreted this coupling in terms of a neutrino mass, concluding that the electron neutrino may be a Majorana particle with $\langle m^{\rm Maj}\rangle_\nu \cong 30$ eV,[9] while Minkowski has found the much different value $\langle m^{\rm Maj}\rangle_\nu \cong 1$ eV.[10] It is thus unclear whether the mass determined in this manner is consistent[5] with the lepton-number-violating mechanism of Fig. 1(b), given the laboratory 0ν-$\beta\beta$-decay mass limit $\langle m^{\rm Maj}\rangle_\nu \lesssim 15$ eV.

In this paper we examine the tellurium decays and the putative evidence for lepton-number violation. We conclude that (1) the qualitative argument that the Te $\beta\beta$ decays indicate lepton-number violation is significantly weakened if the 2ν phase space is treated properly; (2) the use of nuclear-structure calculations to support the underlying assumption of the Pontecorvo argument, equality of the ^{128}Te and ^{130}Te matrix elements, disregards an existing discrepancy between theory and experiment in the absolute rates; and (3) as demonstrated by the significant discrepancy in the results of Refs. 9 and 10, a faithful treatment of the radial behavior of 0ν matrix elements is necessary for the determination of $\langle m^{\rm Maj}\rangle_\nu$ from the Te ratio. We present the mass estimate resulting from such a treatment.

The $\beta\beta$-decay amplitude for $J^\pi=0^+\to 0^+$ nuclear transitions is given, in the standard model, by the two-nucleon mechanism of Fig. 1(a). [A single-hadron mechanism where the $\Delta(1232)$ decays to a neutron and four leptons has been discussed.[11] However, the nuclear operator for this process is a spin vector and cannot contribute to $\Delta J=0$ transitions.] If we evaluate each nucleon β decay in the allowed approximation, the nuclear $\beta\beta$-decay amplitudes in time-dependent perturbation theory are

$$\sum_M \frac{\left\langle F\left|\sum_{i=1} O(i)\right|M\right\rangle\left\langle M\left|\sum_{i=1} O(i)\right|I\right\rangle}{E_M^N+E^L-E_I^N}\,, \qquad (1)$$

where the sum extends over a complete set of states M of the intermediate nucleus $(A,Z-1)$. Nuclear energies are denoted by E^N, and E^L is the sum of the lepton energies emitted in the first β decay. The $O(i)$ represent either the Fermi or Gamow-Teller operators, $\tau_+(i)$ or $\sigma(i)\tau_+(i)$.

The principal approximation we make is to complete the sum in Eq. (1) by closure after replacing E_M^N by an average value, either $\langle E_{\rm F}^N\rangle$ or $\langle E_{\rm GT}^N\rangle$. Our method of calculating these values is discussed in Ref. 6 and, more extensively, in Ref. 12. The resulting 2ν-$\beta\beta$-decay rate then can be written

$$\omega_{2\nu}=f_{\rm GT}\,|M_{\rm GT}|^2+f_{\rm F}\,|M_{\rm F}|^2 +f_{\rm F-GT}{\rm Re}(M_{\rm F}M_{\rm GT}^*)\,,$$

where

$$M_{\rm F}=\left\langle F\left|\left|\tfrac{1}{2}\sum_{ij}\tau_+(i)\tau_+(j)\right|\right|I\right\rangle,$$

$$M_{\rm GT}=\left\langle F\left|\left|\tfrac{1}{2}\sum_{ij}\vec\sigma(i)\cdot\vec\sigma(j)\tau_+(i)\tau_+(j)\right|\right|I\right\rangle,$$

$$f_i=\frac{G^4}{8\pi^7}\int_{m_e}^{T_0+m_e}\mathscr{F}(Z,\epsilon_1)\,|\vec{\rm k}_1|\,\epsilon_1 d\epsilon_1\int_{m_e}^{T_0+2m_e-\epsilon_1}\mathscr{F}(Z,\epsilon_2)\,|\vec{\rm k}_2|\,\epsilon_2 d\epsilon_2 \times\int_0^{T_0-\epsilon_1-\epsilon_2+2m_e}\nu_1^2\nu_2^2 d\nu_1 A_i,\quad i=\text{GT, F or F}-\text{GT}$$

with

$$A_{\rm GT}=F_A^4(K_{\rm GT}^2+L_{\rm GT}^2+K_{\rm GT}L_{\rm GT})/3,\quad A_{\rm F}=F_1^4(K_{\rm F}^2+L_{\rm F}^2-K_{\rm F}L_{\rm F})\,,$$

$$A_{\rm F-GT}=-F_1^2F_A^2(K_{\rm GT}L_{\rm F}+K_{\rm F}L_{\rm GT})\,,$$

$$K=\frac{1}{\langle E^N\rangle+\epsilon_1+\nu_1-E_I}+\frac{1}{\langle E^N\rangle+\epsilon_2+\nu_2-E_I}\,, \qquad (2)$$

$$L=\frac{1}{\langle E^N\rangle+\epsilon_1+\nu_2-E_I}+\frac{1}{\langle E^N\rangle+\epsilon_2+\nu_1-E_I}\,,$$

with F_A and F_1 the axial-vector and vector couplings, and where K_{GT} is obtained by assigning the appropriate subscript to $\langle E^N \rangle$, etc. In the above $(\epsilon, \vec{k})$ denotes an electron four-momentum, $T_0 = E_I - E_F - 2m_e$ is the total kinetic energy of the outgoing leptons, ν_1 is the first neutrino's energy, and $\nu_2 = E_I - E_F - \epsilon_1 - \epsilon_2 - \nu_1$. The factors $\mathscr{F}$ accounting for Coulomb distortion of the outgoing electrons in the field of the daughter nucleus of charge Z are taken from Ref. 13.

In early treatments of $\beta\beta$ decay by Primakoff and Rosen[1] and by Konopinski,[1] an approximation to this formula was often used. The Fermi matrix element vanishes in the limit of exact isospin; if Coulomb effects are included, we find typically $|M_F| < 0.02 << |M_{GT}|$. Thus M_F can be ignored for all but extremely suppressed $\beta\beta$-decay transitions. In addition, usually $\langle E^N \rangle - E_I >> \epsilon + \nu$, so we can approximate $\epsilon + \nu \cong T_0/2 + m_e$ without changing the energy denominators too much. Finally, a nonrelativistic point Coulomb correction is used in describing the distortion of the outgoing electrons,

$$\mathscr{F}(Z,\epsilon) = \frac{\epsilon}{|\vec{k}|}\mathscr{F}^{PR}(Z),$$
$$\mathscr{F}^{PR}(Z) = \frac{2\pi\alpha Z}{1-\exp(-2\pi\alpha Z)} . \tag{3}$$

Although this approximation fails badly for heavy nuclei, its use in the older papers of Ref. 1 was not inappropriate: estimates of $\beta\beta$-decay rates were then typically assigned uncertainties of 2 orders of magnitude. However, in the context of modern efforts to perform accurate calculations of $\beta\beta$-decay matrix elements, this approximation may lead to serious underestimation of rates. We shall return to this point in our discussion of the Te ratio.

These approximations permit an exact evaluation of the phase-space integrals of Eq. (2), yielding

$$\omega_{2\nu}^{PR} = f_{GT}^{PR} |M_{GT}|^2$$

with

$$f_{GT}^{PR} = \frac{16 G^4 F_A{}^4}{\pi^7}[\mathscr{F}^{PR}(Z)]^2 \frac{1}{(\langle E_N \rangle + \frac{1}{2}T_0 + m_e - E_I)^2} \frac{m_e{}^{11}}{8!}\tilde{T}_0{}^7 \left[1 + \frac{\tilde{T}_0}{2} + \frac{\tilde{T}_0{}^2}{9} + \frac{\tilde{T}_0{}^3}{90} + \frac{\tilde{T}_0{}^4}{1980}\right] \tag{4}$$

and with $\tilde{T}_0 = T_0/m_e$. This result agrees with that of Konopinski.[1]

We now turn to the evaluation of the lepton-number-violating two-nucleon process of Fig. 1(b). We use a phenomenological Lagrangian in which the weak leptonic current

$$\bar{\psi}_e(x)\gamma_\mu \left[(1-\gamma_5) + \eta(1+\gamma_5)\psi_{\nu_e}(x)\right] \tag{5}$$

couples to the usual hadronic current. The γ_5 invariance is broken by an explicit right-handed coupling η of the electron to the Majorana neutrino field ψ_ν and by the Majorana mass $\langle m^{\rm Maj} \rangle_\nu$.[14] We again restrict our consideration to $\Delta J = 0$ normal-parity transitions, we make the allowed approximation, and we complete the sum over intermediate nuclear states by closure. (We use $\langle E^N \rangle \cong \langle E_{GT}^N \rangle$ as the 0ν rate is rather insensitive to the value of $\langle E^N \rangle$.) In addition, we specialize to neutrino mass components $m_i^{\rm Maj} << \langle p_\nu \rangle$, with $\langle p_\nu \rangle$ the average momentum of the exchanged neutrino. (Physically this restricts us to $m_i^{\rm Maj} << 1/R_0$, where R_0 is the nuclear radius, and thus to $m_i^{\rm Maj}$, not more than a few tens of MeV.) We also make one further approximation. The matrix elements mediating 0ν decay have a radial dependence which varies as $[g(r_{ij}\xi_1) + g(r_{ij}\xi_2)]/2r_{ij}$,

$$g(x) = \frac{2}{\pi}\left\{\sin x \,\mathrm{Ci}(x) + \cos x \left[\frac{\pi}{2} - \mathrm{Si}(x)\right]\right\} \tag{6}$$

with $\xi_{1,2} = \langle E_N \rangle + \epsilon_{1,2} - E_I$ and r_{ij} the separation between nucleons i and j. For a heavy nucleus, $g(x)$ may vary from 1.0 to 0.4 as r_{ij} goes from 0 to $R_0 \cong 1.2A^{1/3}$. Thus it has been customary to approximate

$$\frac{g(r_{ij}\xi_1) + g(r_{ij}\xi_2)}{2r_{ij}} \cong \frac{1}{r_{ij}} \cong \frac{1}{R_0} ,$$

which of course is only a very rough guess of the effect of this radial dependence on the nuclear matrix elements. We instead make the approximation

$$g(r_{ij}\xi_{1,2}) \cong g(r_{ij}\xi_0) ,$$

with

$$\xi_0 = \langle E_N \rangle + \frac{T_0}{2} + m_e - E_I \cong \xi_{1,2} ,$$

and explicitly evaluate matrix elements with such radial dependences. As $\langle E_N \rangle - E_I >> \epsilon$, we are making a small change in an argument of a slowly varying function, and thus our approximation should be quite reliable.

The resulting rate for the 0ν two-nucleon process of Fig. 1(b) then becomes, for the $\langle m^{\rm Maj}\rangle_\nu \neq 0$ and $\eta = 0$ terms,

$$\omega_{0\nu,\eta=0} = \left[\frac{\langle m^{\rm Maj}\rangle_\nu}{m_e}\right]^2 f_{xx} |X|^2$$

with

$$f_{xx} = \frac{G^4 m_e^2}{8\pi^5}\int_{m_e}^{T_0+m_e} \mathscr{F}(Z,\epsilon_1)\mathscr{F}(Z,\epsilon_2) \times \epsilon_1\epsilon_2 |\vec{k}_1| \, |\vec{k}_2| d\epsilon_1 ,$$

where

$$X = F_A{}^2 M_2'' - F_1{}^2 M_1'' ,$$

$$M_1'' = \left\langle F \left| \tfrac{1}{2}\sum_{ij} \frac{g(r_{ij}\xi_0)}{r_{ij}} \tau_+(i)\tau_+(j) \right| I \right\rangle , \qquad (7)$$

$$M_2'' = \left\langle F \left| \tfrac{1}{2}\sum_{ij} \frac{g(r_{ij}\xi^0)}{r_{ij}} \vec{\sigma}(i)\cdot\vec{\sigma}(j) \times \tau_+(i)\tau_+(j) \right| I \right\rangle .$$

This agrees, in the proper limit, with the result of Doi *et al.* (The result for $\eta \neq 0$ is given in Ref. 12. We concentrate on the $\langle m^{\rm Maj}\rangle_\nu$ contribution because of the keen interest in Majorana mass limits, but caution the reader that the $\eta\langle m^{\rm Maj}\rangle_\nu$ interference terms occur in the general expression.)

We now consider the additional approximations which are often made and which, in contrast to those discussed above, may introduce appreciable uncertainties in estimates of lepton-number violations. If we take $\mathscr{F}(Z,\epsilon) = \mathscr{F}^{\rm PR}(Z)$, the phase-space integrals can be done exactly, yielding

$$\omega^{\rm GW}_{0\nu,\eta=0} = f^{\rm GW}_{xx}\left[\frac{\langle m^{\rm Maj}\rangle_\nu}{m_e}\right]^2 |X|^2 \qquad (8)$$

with

$$f^{\rm GW}_{xx} = \frac{G^4}{16\pi^5}\left[\mathscr{F}^{\rm PR}(Z)\right]^2 m_e{}^7 \times \left[\frac{\tilde{T}_0{}^5}{15} + \frac{2\tilde{T}_0{}^4}{3} + \frac{8\tilde{T}_0{}^3}{3} + 4\tilde{T}_0{}^2 + 2\tilde{T}_0\right] .$$

(We use the superscript GW as the first to derive the 0ν rate in this manner appear to be Grueling and Whitten.[15] Their $\eta = 0$ result, however, is a factor of 2 too large becasue of their omission of the statistical factor for identical particles in the final state.) In addition, the radial dependence of the matrix elements is approximated as

$$\frac{g(r_{ij}\xi_0)}{r_{ij}} \cong \frac{1}{r_{ij}} \cong \frac{1}{R_0} . \qquad (9)$$

As M_1'' then becomes proportional to $M_{\rm F}$, the isospin-forbidden matrix element, it is ignored, leading to the simple result

$$\omega^{\rm Approx}_{0\nu} = f^{\rm GW}_{xx}\frac{F_A{}^4}{R_0{}^2}\left[\frac{\langle m^{\rm Maj}\rangle_\nu}{m_e}\right]^2 |M_{\rm GT}|^2 . \qquad (10)$$

We now consider the Te decay ratio. This ratio is of interest because of the differing nuclear energy releases in ^{128}Te and ^{130}Te, $\tilde{T}_0 = 1.701$ and 4.957, respectively. As the leading-order contribution to $\omega^{\rm GW}_{0\nu,\eta=0}$ varies as $\tilde{T}_0{}^5$, while that for $\omega^{\rm PR}_{2\nu}$ varies as $\tilde{T}_0{}^{11}$, the ratio of these decay rates may depend sensitively on the mechanism for $\beta\beta$ decay. In particular, the ^{128}Te decay rate could be affected appreciably by 0ν decay proceeding via small Majorana mass terms $\langle m^{\rm Maj}\rangle_\nu / m_e$.

The half-life for the decay of ^{130}Te has been determined from the concentrations of ^{130}Xe in telluride ore samples by Inghram and Reynolds,[16] by Takaoka and Ogata,[17] by Kirsten *et al.*,[18] by Alexander *et al.*,[19] and by Srinivasan *et al.*[20] The more recent geochemical measurements[18–20] yield values in the range $(2.03-3.09)\times 10^{21}$ yr. The half-life ratio $\tau^{128}_{1/2}/\tau^{130}_{1/2} = (1.59\pm 0.05)\times 10^3$ has been determined by Hennecke *et al.*[21] from the ratios of ^{130}Xe/^{132}Xe and ^{128}Xe/^{132}Xe in old ore; an earlier, smaller value for this ratio is given in Ref. 17, indicating that background ^{128}Xe may have contributed in that measurement.

If we assume that the nuclear matrix elements mediating 2ν $\beta\beta$ decay are equal for ^{128}Te and ^{130}Te, then Eq. (2) yields $(\tau^{128}/\tau^{130})_{2\nu} = 5.07\times 10^3$. The corresponding ratio employing the approximate result of Eq. (4) is $(\tau^{128}/\tau^{130})^{\rm approx}_{2\nu} = 5.69\times 10^3$. Likewise, the ratios for 0ν $\beta\beta$ decay are, with the same assumption, $(\tau^{128}/\tau^{130})_{0\nu,\eta=0} = 25.0$ and 29.6, using Eq. (7) and the approximate result of Eq. (10), respectively. As these values bracket the result of Hennecke *et al.*, the discrepancy between the 2ν and experimental ratios may indicate the presence of 0ν decay.

The reliability of this conclusion, that a lepton-number-violating mechanism is contributing to these decays, depends on the correctness of the Hennecke *et al.* measurement and on the assumption of equal matrix elements for ^{128}Te and ^{130}Te. One expects the geochemical result to be accurate as uncertainties in the ore age and some effects of Xe diffusion will cancel in the ratio. However, one point we find worrisome in the results of Ref. 21: the Hennecke *et al.* half-lives for ^{130}Te, 0.89×10^{21} and 1.05×10^{21} yr, are considerably outside the range $(2.03-3.09)\times10^{21}$ yr of other recent measurements. Absolute half-lives are determined from the ratio of ^{130}Xe to ^{132}Xe, which is not radiogenic, and thus some part of this discrepancy could be attributed to an error in dating the Hennecke *et al.* ore samples, which would then not affect the ^{128}Xe/^{130}Xe ratio. Yet it should be noted that the apparent difference in the ^{130}Xe/^{132}Xe ratio from other recent measurements is similar to the anomaly ^{128}Xe/^{130}Xe on which the argument for lepton-number violation rests.

If we accept the result of Hennecke *et al.*, the conclusion that lepton number is violated follows only after assuming the matrix elements for ^{128}Te and ^{130}Te are roughly equal. The support for this assumption, as cited in the work of Bryman and Picciotto,[8] Doi *et al.*,[9] and Minkowksi,[10] is the nuclear-matrix-element calculation of Vergados,[7] in which

$$\left|\frac{M_{\rm GT}(128)}{M_{\rm GT}(130)}\right|=\frac{0.284}{0.248}=1.15\cong1\ .$$

(A value of 1.78 would yield a ratio of 2ν-$\beta\beta$-decay rates in agreement with experiment.) It has been noted that the Vergados matrix element for ^{130}Te, whose decay proceeds predominantly by the 2ν mechanism for any reasonable $\langle m^{\rm Maj}\rangle_\nu$, yields $\tau_{1/2}^{2\nu,{\rm PR}}=2.73\times10^{21}$ yr, in good agreement with the general geochemical range. This has been construed as circumstantial evidence for the validity of the Vergados nuclear-structure approximations.

However, one must consider carefully the implications of the approximate treatment of phase space in Eq. (4). We illustrate this point in Table I, where $f_{\rm GT}$ and $f_{\rm GT}^{\rm PR}$ are compared for a variety of $\beta\beta$-decay nuclei and where $|M_{\rm GT}|$ is extracted directly from experiment under the assumption of purely GT 2ν $\beta\beta$ decay [$|M|_F \underset{\rm set}{=} 0$ in Eq. (2)]. Noting that $f_{\rm GT}^{\rm PR}$ underestimates the 2ν phase space by a factor of 5 (see Appendix), we find that the upper bound on $|M_{\rm GT}(130)|$ of $0.104-0.129$ can be obtained from the recent geochemical estimates of $\tau_{1/2}^{130}$. Thus the Vergados matrix element underestimates $\tau_{1/2}^{130}$ by a factor of at least $3.7-5.7$. Effectively, if one uses the Vergados calculation to argue in favor of lepton-number violation in the decay of ^{128}Te, one chooses to exacerbate an existing discrepancy in the theoretical rates in order to reproduce a ratio of rates. (That is, the Vergados estimate of $\tau_{1/2}^{128}$, roughly correct in the limit of lepton-number conservation, also becomes too small once the 0ν mechanism is introduced to improve the ratio of rates.) On more general grounds, one also sees that a correct treatment of phase space shows, directly from experiment, that $|M_{\rm GT}(130)|$ could be nearly a factor of 2 smaller than the celebrated suppressed matrix element of ^{48}Ca.[22] Certainly with such suppression, it is not obvious that the matrix element ratio will be unity.

One possible escape from this predicament is to find a consistent theoretical description of both the absolute rates and the ratio of rates. Clearly there are fundamental difficulties with the Vergados shell-model approximation. The neutron holes are restricted to the $h_{11/2}$ subshell in *both* initial and

TABLE I. The approximate and full phase-space factors $f_{\rm GT}^{\rm PR}$ and $f_{\rm GT}$ for 2ν $\beta\beta$ decay are compared. Average excitation energies $\langle E_{\rm GT}^N\rangle$, calculated in a statistical model (Refs. 6 and 12), and our shell-model results for the double Gamow-Teller matrix elements $|M_{\rm GT}|_{\rm th}$ are also shown. In the last column maximum experimental values for $|M_{\rm GT}|$ are derived by dividing total $\beta\beta$-decay rates by $f_{\rm GT}$.

	$f_{\rm GT}^{\rm PR}$ (sec^{-1})	$f_{\rm GT}$ (sec^{-1})	$\langle E_{\rm GT}^N\rangle$ (MeV)	$\|M_{\rm GT}\|_{\rm th}$	$\|M_{\rm GT}\|_{\rm exp}$
^{130}Te	1.31×10^{-28}	6.48×10^{-28}	11.50	1.483	0.104−0.129 (Refs. 18, 29, 20)
^{128}Te	2.34×10^{-32}	1.29×10^{-31}	11.59	1.474	0.185−0.230 (Ref. 21)
^{82}Se	5.82×10^{-28}	1.08×10^{-27}	8.07	0.938	1.430 (Ref. 27)
					0.272 (Ref. 28)
^{76}Ge	2.23×10^{-29}	3.67×10^{-29}	7.88	1.278	
^{48}Ca	1.45×10^{-26}	1.65×10^{-26}	5.07	0.222	<0.193 (Ref. 22)

final states, so that the *only two-body density matrix elements* which contribute in his first-order calculation are of the form $(h_{11/2})^{-2}J;n \rightarrow (h_{11/2})^2 J;p$. Yet the $(h_{11/2})^2$ configuration is expected to be a minor proton configuration, the $g_{7/2}$ and $d_{5/2}$ subshells being primarily occupied. *A priori* one might expect contributions of the form $(d_{3/2})^{-2}J;n \rightarrow (d_{5/2})^2 J;p$ connecting a minor neutron configuration to a major proton component to be of similar importance. More generally, to properly estimate $M_{\rm GT}$, one should insist that all Pauli-allowed configurations which can be reached by operating on the initial state with the double Gamow-Teller operator be included in the final wave function.

We have performed such a calculation, allowing both proton and neutron holes to range throughout the model space $(2d_{5/2}, 1g_{7/2}, 3s_{1/2}, 2d_{3/2}, 1h_{11/2})$. (We considered extending the model space beyond the magic numbers 50 and 82 in order that remaining spin partners $1h_{9/2}$ and $1g_{9/2}$ could be treated. Inclusion of these subshells, however, introduces wave-function spuriosity which could have serious effects as our effective interaction is not translationally invariant.) Details of these calculations, in which we use a potential derived by Baldridge and Vary from the Kuo bare G matrix,[23] are given in Ref. 12. Although we employ weak coupling, we do not take the extreme approach of Vergados, where the ground state is the simple product of the lowest neutron and proton state. Instead we take the lowest 50 proton and 50 neutron states from the full, identical-particle shell model calculations, form a weak-coupling basis of definite spin and parity by combining these states in all allowable combinations, and diagonalize the proton-neutron interaction in this basis. The two-body density matrix is then calculated and the various $\beta\beta$-decay matrix elements evaluated.

The results exacerbate the discrepancy between theory and experiment in the absolute rates. The new matrix elements $|M_{\rm GT}|$ (see Table I) are $5-6$ times *larger* than those calculated by Vergados due to a coherent addition of many components of the density matrix. The ratio $|M_{\rm GT}(128)/M_{\rm GT}(130)|$ is unity to better than 1%. Again, then, this provides theoretical support for matrix element equality only if one is willing to ignore an alarming problem with the theoretical rates. (We believe at least two points should be studied before this improved theoretical treatment is accepted at face value. We are concerned, in view of recent experimental evidence for considerable collectivity in Gamow-Teller strengths,[24] that the approximate treatment of the pn interaction implicit in the weak-coupling approximation may be inadequate. We are also concerned about many-body modifications of the β-decay vertices. The quenching of F_A via Δ-hole excitations has been discussed frequently in the context of β-decay systematics of heavy nuclei.[25])

If both the result of Hennecke *et al.* and the assumption of equal matrix elements are accepted, the Te ratio indicates lepton-number violation. If the γ_5 invariance of the weak current is broken by a Majorana mass, rather than by a right-handed coupling of the Majorana neutrino, then a calculation of the 0ν mechanism of Fig. 1(b) will place an upper limit on $\langle m^{\rm Maj}\rangle_\nu$. (In the limit where only a single Majorana mass eigenstate contributes to ψ_{ν_e}, $\langle m^{\rm Maj}\rangle_\nu = C^2 m^{\rm Maj}$, with C the mixing amplitude and $m^{\rm Maj}$ the mass eigenvalue. See the note in Ref. 14 for the constraints $\beta\beta$ decay can place on more general mass matrices.) If, in addition, we assume that no other mechanism for lepton violation contributes (no superweak decay, no 0ν $\beta\beta$ decay with light scalar particles, etc.), that upper bound is the neutrino mass. We now consider the calculation of that mass.

In the conventional approach, we find from Eqs. (4) and (10),

$$\frac{\tau_{1/2}^{128}}{\tau_{1/2}^{130}} = \frac{\omega_{2\nu}^{\rm PR}(130) + \omega_{0\nu}^{\rm approx}(130)}{\omega_{2\nu}^{\rm PR}(128) + \omega_{0\nu}^{\rm approx}(128)} = \frac{f_{\rm GT}^{\rm PR}(130) + f_{xx}^{\rm GW}(130) F_A{}^4 / R_0{}^2(130) (\langle m^{\rm Maj}\rangle_\nu / m_e)^2}{f_{\rm GT}^{\rm PR}(128) + f_{xx}^{\rm GW}(128) F_A{}^4 / R_0{}^2(128) (\langle m^{\rm Maj}\rangle_\nu / m_e)^2} \,, \tag{11}$$

where we have assumed $|M_{\rm GT}(128)| = |M_{\rm GT}(130)|$. If we take $F_A = -1.25$, $R_0 = 1.2 A^{1/3} f$, and $\tau_{1/2}^{128}/\tau_{1/2}^{130} = (1.59 \pm 0.05) \times 10^3$, we find

$$\langle m^{\rm Maj}\rangle_\nu^{\rm approx} = (21.6 \pm 0.4)\ {\rm eV} \tag{12}$$

with the error reflecting only the uncertainty in the geochemical ratio. Note that in the work of Doi *et al.*, the GT matrix element ratio and $\langle E_N \rangle$ are taken from Vergados and $g(r_{ij})/r_{ij} \cong 0.6/R_0$, leading to $\langle m^{\rm Maj} \rangle_\nu^{\rm Approx} = 30$ eV, a value with which we agree given these approximations.

Now we consider the effect of using the more correct expressions of Eqs. (2) and (7). We find for equal GT matrix elements,

$$\frac{\tau_{1/2}^{128}}{t_{1/2}^{130}} = \frac{f_{\rm GT}(130) + f_{xx}(130) F_A{}^4/\tilde{R}_0{}^2(130)[1-\kappa(130)]^2(\langle m^{\rm Maj}\rangle_\nu/m_e)^2}{f_{\rm GT}(128) + f_{xx}(128) F_A{}^4/\tilde{R}_0{}^2(128)[1-\kappa(128)]^2(\langle m^{\rm Maj}\rangle_\nu/m_e)^2}$$

with

$$\tilde{R}_0 = |M_{\rm GT}| / |M_2''| \; , \qquad \kappa = F_1{}^2 M_1'' / F_A{}^2 M_2'' \; . \tag{13}$$

This differs from Eq. (11) to the extent that κ and $\tilde{R}_0$ differ from 0 and R_0, respectively, and due to the more careful treatment of phase space, though the substantial phase-space effects discussed earlier largely cancel in taking the ratio. In Table II we show the numerical results for κ and $\tilde{R}_0$ for the Te isotopes and for several $\beta\beta$-decay nuclei we have discussed elsewhere.[6,12] As κ is not negligible, we see that the factorization of the radial dependence from M_1'', yielding $\langle 1/r \rangle$ times the isospin-forbidden matrix element M_F, is unjustified.

However, the difference in various earlier estimates of $\tilde{R}_0$ account primarily for the broad range of $\langle m^{\rm Maj} \rangle_\nu$ values extracted from the Te ratio. Doi *et al.*[9] adopted the Primakoff-Rosen approximation that $\langle 1/r_{ij} \rangle \approx 1/R_0$ and then estimated that proper account of $g(r_{ij})$ at such large distance weakened this matrix element by a factor of 0.6, yielding $\tilde{R}_0^{\rm Doi} = 1.67\ R_0$. By explicit calculation we find that, for independent-particle wave functions and with $g(r_{ij}) = 1$, $\tilde{R}_0 = 0.37\ R_0$, which is not surprising as $1/r_{ij}$ preferentially weights smaller separations. If we then include the effects of $g(r_{ij})$ on the matrix element, $\tilde{R}_0$ becomes 0.47 R_0. [Of course, this does not imply that contributions at larger separations are enhanced. On the contrary, $g(r_{ij})$ reduces those contributions, but we absorb some average reduction factor for the numerator $g(r_{ij})$ into an effective $\tilde{R}_0$.] The effect of $g(r_{ij})$ is smaller than that estimated by Doi *et al.* because $g(r_{ij})$ is closer to unity for $r_{ij} \sim \tilde{R}_0 < R_0$. Finally, we correct for the absence of short-ranged correlations in our shell-model wave functions by modifying the two-body densities

$$\psi_{1,2}(\vec{r}_i, \vec{r}_j) \rightarrow \left[1 - \beta(r_{ij})\right] \psi_{1,2}(\vec{r}_i, \vec{r}_j) \; , \qquad \beta(r) = e^{-ar^2}(1 - br^2) \; , \tag{14}$$

with $a = 1.1\ {\rm fm}^{-2}$ and $b = 0.68\ {\rm fm}^{-2}$.[26] This yields $\tilde{R}_0 = 0.59\ R_0$, which then is used in Eq. (13). We note that Minkowski[10] has recently advocated using a much smaller $\tilde{R}_0$, reducing the Doi *et al.* mass estimate by a factor of 13 to 20. Minkowski maintains that $\tilde{R}_0$ must be so drastically scaled to account for the $1/r_{ij}$ weighting of small separations and the loss of pair-spin correlations for $r_{ij} \geq 1$ fm. The explicit calculations we have performed do not support such drastic renormalizations of the Primakoff-Rosen estimate. Clearly long-range components of the nuclear force also in-

TABLE II. The matrix-element ratios $\kappa = F_1{}^2 M_1''/F_A{}^2 M_2''$ and $\tilde{R}_0 = |M_{\rm GT}/M_2''|$ resulting from shell-model calculations. We take $F_1 = 1$ and $F_A = -1.25$, and we give $\tilde{R}_0$ in units of $R_0 = 1.2A^{1/3}$. The third and fourth columns give values for $\tilde{R}_0$ with $g = 1$ and with the full g of Eq. (6) employed in M_2''. The fifth column shows the effects of modifying the shell-model two-particle densities by the correlation function of Eq. (14). It is this value that is employed in the present calculations.

	κ	$\tilde{R}_0 \left[\left\langle \frac{1}{r} \right\rangle\right]$	$\tilde{R}_0 \left[\left\langle \frac{g}{r} \right\rangle\right]$	$\tilde{R}_0 \left[\left\langle \frac{g}{r} \right\rangle_{\rm corr}\right]$
^{130}Te	−0.23	0.37	0.47	0.59
^{128}Te	−0.23	0.38	0.47	0.59
^{82}Se	−0.18	0.36	0.43	0.54
^{76}Ge	−0.20	0.41	0.49	0.61
^{48}Ca	−0.15	0.26	0.29	0.44

duce strong spin-singlet correlations in $T=1$ pairs.

Our numerical values for κ and $\tilde{R}_0$ then yield

$$\langle m^{\rm Maj} \rangle_\nu = 10.1 \pm 0.2 \text{ eV} \quad (15)$$

with the error again reflecting only the uncertainty in the geochemical ratio. This can be compared with the mass limit obtained from laboratory bounds on the neutrinoless decay of ^{76}Ge and ^{82}Se, $\langle m^{\rm Maj} \rangle_\nu \lesssim 15$ eV.[6] [If the prescription outlined here is followed precisely in calculating the 0ν decay rates of these nuclei, the values of Ref. 6 are changed slightly to $\langle m^{\rm Maj} \rangle_\nu \lesssim 17$ eV and $\lesssim 13$ eV, respectively, for ^{76}Ge and ^{82}Se.] Previously, noting that the Doi *et al.*[9] estimate of $\langle m^{\rm Maj} \rangle_\nu$ was in apparent conflict with these limits, Georgi, Glashow, and Nussinov[5] had pointed out that masses derived from the Te ratio cannot be compared directly to the results of laboratory searches for the 0ν mechanism of Fig. 1(b). In particular, in the context of a model suggested by Gelmini and Roncadelli,[5] in which neutrino masses are generated by virtue of the spontaneous breakdown of global $B-L$ symmetry with the appearance of a massless Goldstone particle, they discuss a second 0ν-$\beta\beta$-decay mode in which the emitted electrons are accompanied by a very light boson. This process would contribute to the inclusive test of lepton-number violation provided by the Te ratio, but would not be seen in laboratory searches for the two-electron mode. Thus, if the Doi result and the two-electron mass limits are accepted at face value, this might suggest the existence of such additional 0ν modes. Our result of $\langle m^{\rm Maj} \rangle_\nu \simeq 10$ eV no longer requires any such adjustment to accommodate the two-electron mode mass limit. We also recall that the derivation of the 15 eV mass limit, which requires a calculation of the matrix element X, was made plausible by the agreement of the calculated 2ν rate with the laboratory measurement of Moe and Lowenthal.[27] However, that laboratory rate disagrees with the geochemical rate.[28] As the relative constancy of κ and $\tilde{R}_0$ in Table II indicates a rough scaling of 2ν and 0ν matrix elements, a more conservative mass limit could be obtained by renormalizing all matrix elements so that the geochemical result is reproduced. This would yield $\langle m^{\rm Maj} \rangle_\nu \lesssim 52$ eV. We believe a second, more definitive laboratory measurement of the ^{82}Se 2ν decay rate is extremely important in clarifying whether systematic difficulties exist in theoretical half-life estimates.

In summary, we have argued that the evidence for lepton-number violation in the Te $\beta\beta$-decay ratio may not be as strong as commonly believed. As a proper treatment of phase space demonstrates that the 2ν matrix elements for ^{128}Te and ^{130}Te must be suppressed to account for the absolute geochemical rates, the argument that the matrix element ratio is near unity is not obvious *a priori.* While the present careful theoretical calculations of these matrix elements do support this argument, an alarming discrepancy in predicting the absolute geochemical rates remains troublesome. Stipulating that the matrix element ratio is unity, we have faithfully treated the radial dependence of the 0ν operators in order to determine the scaling of 0ν and 2ν matrix elements. This yields $\langle m^{\rm Maj} \rangle_\nu \cong 10$ eV if we assume no explicit right-handed coupling of Majorana neutrinos to electrons; this mass is not inconsistent with the limits derived from laboratory searches for the two-electron mode. We emphasize again that the reliability of this result is difficult to assess until the origin of the discrepancy between theoretical and geochemical estimates of absolute rates is uncovered.

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APPENDIX

We briefly compare the relativistic and nonrelativistic treatments of the Coulomb functions $\mathscr{F}(Z,\epsilon)$ which correct the electron plane waves for distortion due to the nuclear charge.

The nonrelativistic point Coulomb correction is derived by taking the square of the ratio of the Schrödinger scattering solution for a point charge Z to a plane wave, evaluated at the origin. For an electron,

$$\mathscr{F}^{\rm NR}(Z,\epsilon) = \frac{2\pi\eta}{1-e^{-2\pi\eta}} = e^{\pi\eta} \, | \Gamma(1+i\eta) |^2 , \quad (A1)$$

with $\eta = \alpha Z/\beta$ and $\beta = k/\epsilon$, $k = |\vec{k}|$ the electron momentum. In much of the $\beta\beta$-decay literature this formula has been further simplified by setting $\beta=1$ in the exponent, yielding Eq. (3), as this permits analytic integration over the $\beta\beta$-decay phase space.

In the present work we take $\mathscr{F}(Z,\epsilon)$ from Ref. 13, where the Coulomb correction is derived by numerical solution of the Dirac equation for an extended nuclear charge. We can write this correction factor as

$$\mathscr{F}(Z,\epsilon)=\mathscr{F}^R(Z,\epsilon)L_0 \tag{A2}$$

with $\mathscr{F}^R(Z,\epsilon)$ the square of the ratio of the Dirac scattering solution for a point charge Z to a plane wave, evaluated at the nuclear surface (the relativistic density for a point nucleus is infinite at the origin). The numerical factor L_0, generally small, takes into account the finite charge distribution and screening corrections, while

$$\mathscr{F}^R(Z,\epsilon)=2(1+\gamma)(2kR_0)^{2(\gamma-1)} \times e^{\pi\eta}\left|\frac{\Gamma(\gamma+i\eta)}{\Gamma(2\gamma+1)}\right|^2 , \tag{A3}$$

with $\gamma=[1-(\alpha Z)^2]^{1/2}$ and R_0 the nuclear radius. Thus

$$\frac{\mathscr{F}^{NR}(Z,\epsilon)}{\mathscr{F}(Z,\epsilon)} \cong \frac{\mathscr{F}^{NR}(Z,\epsilon)}{\mathscr{F}^R(Z,\epsilon)} = \frac{\Gamma(3+2\xi)^2}{2(2+\xi)}\frac{1}{(2kR_0)^{2\xi}} \times\left|\frac{\Gamma(1+i\eta)}{\Gamma(1+\xi+i\eta)}\right|^2 , \tag{A4}$$

with each of the three terms in this expression approaching unity in the limit of small Z, $\xi=\gamma-1\rightarrow 0$.

To illustrate the behavior of this ratio we expand the first and third terms in Eq. (A4) to $O(\xi)$ [note $\xi(^{130}\mathrm{Xe})=-0.081$]. We find

$$\frac{\Gamma(3+2\xi)^2}{2(2+\xi)}=1+3.191\xi+O(\xi^2) ,$$

$$\frac{\Gamma(1+i\eta)}{\Gamma(1+i\eta+\xi)}=1-\xi\psi^0(1+i\eta)+O(\xi^2) .$$

The digamma function ψ^0 is tabulated in Ref. 29. Using $R_0=1.2A^{1/3}f$, we obtain for ^{130}Xe with $k/m_e=0.2$

$$\frac{\mathscr{F}^{NR}(Z,\epsilon)}{\mathscr{F}^R(Z,\epsilon)} \cong (0.742)(0.440)(1.131) =0.369$$

and with $k/m_e=3.0$

$$\frac{\mathscr{F}^{NR}(Z,\epsilon)}{\mathscr{F}^R(Z,\epsilon)} \cong (0.742)(0.682)(0.939) =0.475 .$$

Such factors generate the factor-of-5 enhancement in the ^{130}Te $\beta\beta$-decay phase space which we discussed in the main text.

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