Relativistic duality, and relativistic and radiative corrections for heavy-quark systems

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We give a JWKB proof of a relativistic duality relation which relates an appropriate energy average of the physical cross section for $e^+e^- \to q\bar{q}$ bound states $\to$ hadrons to the same energy average of the perturbative cross section for $e^+e^- \to q\bar{q}$. We show that the duality relation can be used effectively to estimate relativistic and radiative corrections for bound-quark systems to order $\alpha_s{}^2$. We also present a formula which relates the square of the "large" ${}^3S_1$ Salpeter-Bethe-Schwinger wave function for zero space-time separation of the quarks to the square of the nonrelativistic Schrödinger wave function at the origin for an effective potential which reproduces the relativistic spectrum. This formula allows one to use the *nonrelativistic* wave functions obtained in potential models fitted to the ψ and Υ spectra to calculate *relativistic* leptonic widths for $q\bar{q}$ states via a relativistic version of the Van Royen—Weisskopf formula.

I. INTRODUCTION

Duality as used in the study of e^+e^- annihilation equates an appropriate energy average of the physical cross section for $e^+e^- \to q\bar{q}$ bound states$\to$hadrons to the same energy average of the perturbative cross section for $e^+e^- \to$ free $q\bar{q}$ systems calculated in QCD,[1–3]

$$\langle \sigma_{\text{bound}} \rangle = \langle \sigma_{\text{free}} \rangle + \text{small corrections} . \qquad (1)$$

This relation has been used extensively in the analysis of data on heavy-quark systems, e.g., to extract quark masses from the data, to predict leptonic widths, and to test QCD.[2–9]

The nonrelativistic version of duality has been shown to hold in the JWKB approximation for single-channel potential models of the $q\bar{q}$ system,[10–14] and has been checked by numerical calculations.[15] We recently extended the proof of nonrelativistic duality to multichannel processes, and showed using methods that go beyond the JWKB approximation that the correction terms in Eq. (1) are calculable.[16]

In this paper, we give a JWKB proof of the *relativistic* duality relation. As part of our proof, we investigate the connection of the Salpeter-Bethe-Schwinger (SBS) wave function for the relativistic bound $q\bar{q}$ system with the Schrödinger wave function used in the conventional nonrelativistic description of the ψ and Υ systems.[17] We show, in particular, that for SBS and Schrödinger descriptions of the $q\bar{q}$ system which have the *same* spectra,

$$|\psi_{nS}(\vec{0},0)|^2 = \frac{M_n'^2}{4m_q^2}\frac{v'}{v'_{\text{nonrel}}}|\psi_{nS}^{\text{nonrel}}(\vec{0})|^2 , \qquad (2)$$

where v' and v'_{nonrel} are the relativistic and nonrelativistic velocities of a free quark in a pair with total energy $M_n' = 2m_q + E_n - V(0)$. Here $\psi_{nS}(\vec{0},0)$ is the "large" ${}^3S_1$ component of the SBS wave function for zero space-time separation of the quarks, and $\psi_{nS}(0)$ is the Schrödinger wave function of the equivalent nonrelativistic system evaluated for zero spatial separation of the quarks. This remarkable formula allows one to use *nonrelativistic* wave functions obtained in potential models fitted to the ψ and Υ spectra to calculate *relativistic* leptonic widths for $q\bar{q}$ states via a relativistic version of the usual Van Royen—Weisskopf formula.[18]

The closely related proofs of the relativistic duality relation, and of Eq. (2), are given in Sec. II of the paper. We discuss the background of the problem in Sec. II A. In Sec. II B, we analyze the structure of the complete SBS wave function for a $J^{PC}=1^{--}$ $q\bar{q}$ bound state, and derive a relativistic expression for the leptonic decay width in terms of $|\psi_{nS}(0)|^2$. Some of the results on the SBS wave function are apparently new. In Secs. II C and II D, we consider the reduction of the SBS equation to Schrödinger form, and derive a JWKB expression for $|\psi_{nS}(0)|^2$ in terms of the density of states. In Sec. II E, we summarize our results and discuss the connection of the nonrelativistic and relativistic wave functions.

We conclude the paper by showing in Sec. III that the duality relation in Eq. (1) can be used with the radiatively corrected free-quark cross section[19] for $e^+e^- \to q\bar{q}$ to estimate relativistic and gluonic radiative corrections to the bound $q\bar{q}$ system. In particular, we recalculate approximately the known order-α_s gluonic correction to the leptonic widths of the $^3S_1$ $q\bar{q}$ bound states,[20,21] and estimate the corrections of order $\alpha_s{}^2$. The relativistic and radiative corrections are important for the $c\bar{c}$ and $b\bar{b}$ systems ($\alpha_s \sim 0.2$, $\langle v^2/c^2 \rangle \gtrsim 0.23$ for the $c\bar{c}$ system, and $\gtrsim 0.08$ for the $b\bar{b}$ system[22]).

II. PROOF OF RELATIVISTIC DUALITY

A. Schrödinger versus relativistic duality

The nonrelativistic free cross section for $e^+e^- \to q\bar{q}$ is given by

$$W^2\sigma_{\rm free}^{\rm nonrel} = 6\pi\alpha^2 e_q{}^2 v^{\rm nonrel} \,|\, \psi_E^{\rm nonrel}(\vec{0}) \,|^2 \,, \tag{3}$$

where $\psi_E^{\rm nonrel}(0)$ is the free $q\bar{q}$ wave function for energy $E = W - 2m_q$ and $v^{\rm nonrel} = (E/m_q)^{1/2}$. With conventional plane-wave normalization, $|\,\psi_E^{\rm nonrel}(\vec{0})\,|^2 = 1$. The nonrelativistic cross section for producing $q\bar{q}$ bound states in a confining potential is given by

$$W^2\sigma_{\rm bound}^{\rm nonrel} = \sum_n 6\pi^2\Gamma_n^{\rm nonrel}(e^+e^-)\delta(E - E_n) \,, \tag{4}$$

where $\Gamma_n^{\rm nonrel}$ and E_n are the leptonic width and energy of the nth $q\bar{q}$ bound state. $\Gamma_n^{\rm nonrel}$ is related to the bound-state wave function at the origin $\psi_{nS}^{\rm nonrel}(\vec{0})$, by the nonrelativistic Van Royen—Weisskopf formula[18,23]

$$\Gamma_n^{\rm nonrel}(e^+e^-) = (4\pi\alpha^2 e_q{}^2/m_q{}^2) \,|\, \psi_{nS}^{\rm nonrel}(\vec{0}) \,|^2 \,. \tag{5}$$

It has been shown by JWKB methods[10–14] that $|\,\psi_{nS}^{\rm nonrel}(\vec{0})\,|^2$ is related for nonsingular potentials to the density of states dn/dE_n by

$$|\,\psi_{nS}^{\rm nonrel}(\vec{0})\,|^2 = \frac{m_q{}^{3/2}}{4\pi^2}[E_n - V(0)]^{1/2}\frac{dE_n}{dn} = \frac{m_q{}^2}{4\pi^2} v_n^{\rm nonrel}\frac{dE_n}{dn} \,. \tag{6}$$

We note that $v_n^{\rm nonrel} = [E_n - V(0)]^{1/2}/m_q{}^{1/2}$ is the velocity of a *free* quark with energy $\frac{1}{2}[E_n - V(0)]$.

The nonrelativistic duality relation, Eq. (1), is obtained by combining the above results, averaging the cross sections over some energy range, and approximating the sum in Eq. (4) by an integral. Singular potentials can be handled by modifying $\sigma_{\rm free}$ to include effects of the short-range (singular) part of the interaction. As we remarked in the Introduction, the duality relation can be proven beyond the JWKB approximation, and is quite accurate when the calculable corrections are included.[16]

The relativistic duality relation has heretofore not been proven. The relativistic free cross section (without radiative corrections) is given by

$$W^2\sigma_{\rm free} = 6\pi\alpha^2 e_q{}^2 v(1 - \tfrac{1}{3}v^2) \,. \tag{7}$$

The relativistic bound-state cross section is given by

$$W^2\sigma_{\rm bound} = \sum_n 6\pi^2\Gamma_n(e^+e^-)\delta(W - M_n) \,, \tag{8}$$

where $M_n = 2m_q + E_n$ is the mass of the nth $q\bar{q}$ bound state, and $\Gamma_n(e^+e^-)$ is given by a relativistic version of the Van Royen—Weisskopf formula which we will derive in the next section,

$$\Gamma_n(e^+e^-) = (16\pi\alpha^2 e_q{}^2/M_n{}^2) \,|\, \psi_{nS}(0) \,|^2 (1 - \Delta_n) \,. \tag{9}$$

Here $\psi_{nS}(0)$ is the "large component" of the S-state $q\bar{q}$ Salpeter-Bethe-Schwinger (SBS) wave function evaluated for zero space-time separation of the quarks. Δ_n incorporates in D-state effects and terms which arise from spinor factors and "small components" in the full relativistic amplitude, as discussed by Bergström *et al.*,[24] and approaches $(1 - \frac{1}{3}v_n{}^2)$ for nearly free quarks.

It is customary in most work on $q\bar{q}$ systems to assume that the SBS wave function in Eq. (9) can be replaced by a Schrödinger wave function. [Furthermore, the factor $(1 - \Delta_n)$ is usually omitted.] With this replacement, $|\,\psi_{nS}(0)\,|^2$ is related to dE_n/dn by Eq. (6), and the relativistic free and bound-state cross sections of Eqs. (7) and (8) are *not* dual.[10,23]

We will show that with a proper treatment of the relativistic wave function, Eq. (6) is replaced for relativistic systems by

$$|\psi_{nS}(0)|^2 \simeq \frac{M_n'^2 v_n'}{16\pi^2}\frac{dM_n}{dn}(1-\Delta_n') , \tag{10}$$

where v_n' is the relativisitic velocity of a free quark with energy $\frac{1}{2}M_n' = \frac{1}{2}[M_n - V(0)]$, and Δ_n' is a correction for retardation and radiative gluonic effects. The relativistic duality relation, Eq. (1), is then obtained by substituting Eqs. (9) and (10) in Eq. (8), averaging the result over an appropriate energy range, and replacing the sum on n by an integral. We will discuss this relation and give an application in Sec. III. [The expression for $|\psi_{nS}(0)|^2$ in Eq. (10) with M_n' replaced by M_n and Δ_n' omitted was recently proposed *ad hoc* by Tainov[25] as necessary to restore duality for relativistic systems.]

B. $\Gamma_n(e^+e^-)$ and the Salpeter-Bethe-Schwinger wave function

The leptonic width $\Gamma_n(l^+l^-)$ for the decay of a $J^{PC}=1^{--}$ $q\bar{q}$ system (vector meson) of mass M_n into a lepton pair through a single intermediate photon is given by

$$\Gamma_n(l^+l^-) = (16\pi\alpha^2 e_q{}^2/M_n{}^2) v_l [1+2(m_l/M_n)^2] \frac{1}{3}\sum_\lambda \vec{\mathrm{j}}_\lambda \cdot \vec{\mathrm{j}}_\lambda^* , \tag{11}$$

where $\vec{\mathrm{j}}_\lambda$ is essentially the matrix element of the quark current between the vacuum and the meson rest state with spin projection λ,[26]

$$\vec{\mathrm{j}}_\lambda = \left\langle 0 \left| \frac{1}{\sqrt{2}}(\bar{\psi}_q \vec{\gamma}\psi_q)(0) \right| n,\lambda \right\rangle . \tag{12}$$

(The factor $1/\sqrt{2}$ is introduced for later convenience.) This matrix element can be expressed in terms of the momentum-space SBS wave function $\tilde{\psi}_\lambda(M_n,p) = \psi_{\alpha,\beta;\lambda}(M_n,p)$, $\alpha,\beta = 1,\ldots,4$, by[27]

$$\vec{\mathrm{j}}_\lambda = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[-\frac{1}{\sqrt{2}} C^{-1}\vec{\gamma}\tilde{\psi}_\lambda(M_n,p) \right] , \tag{13}$$

where $C = i\gamma_2\gamma_0$ is the usual charge-conjugation matrix, and the trace is over the Dirac indices.

The 16-component wave function $\tilde{\psi}_\lambda(M_n,p)$ is a solution of the (relativistic) SBS equation given below as Eq. (28). We would like to relate this function to the wave functions which appear in nonrelativistic Schrödinger models for the $q\bar{q}$ system. We will first study the kinematic structure of $\tilde{\psi}_\lambda$, and will show in the remainder of this section that $\tilde{\psi}_\lambda$ can be described in terms of eight independent wave functions for interacting quarks and antiquarks: two ${}^3S_1$, ${}^3D_1$ combinations for the "large-large" and "small-small" components of $\tilde{\psi}_\lambda$, and two ${}^2P_{1/2}$, ${}^4P_{1/2}$ combinations which describe quark—vector-meson and antiquark—vector-meson interactions in the "large-small" components. Only the ${}^3S_1$, ${}^3D_1$ combinations enter $\vec{\mathrm{j}}_\lambda$ and Γ_n. We will consider the reduction of the SBS equation to a Schrödinger-type equation for these wave functions in Sec. II C, will complete the proof of Eq. (10) in Sec. II D, and will discuss the connection of our results with the usual Schrödinger description of the $q\bar{q}$ system in Sec. II E.

The general structure of $\tilde{\psi}_\lambda$ is easily determined. We first separate $\tilde{\psi}_\lambda$ into "positive-energy" and "negative-energy," or "large" and "small," parts with respect to the first and second Dirac indices using the usual projection operators $\Lambda_\pm(\vec{\mathrm{p}})$ for energy $E(\vec{\mathrm{p}}) = (\vec{\mathrm{p}}^2 + m_q{}^2)^{1/2}$ and momentum $\vec{\mathrm{p}}(-\vec{\mathrm{p}})$,

$$\tilde{\psi}_\lambda = \tilde{\psi}_\lambda^{++} + \tilde{\psi}_\lambda^{+-} + \tilde{\psi}_\lambda^{-+} + \tilde{\psi}_\lambda^{--} . \tag{14}$$

Here

$$\tilde{\psi}_\lambda^{\pm\pm}(W_n,\vec{\mathrm{p}},p_0) = \Lambda_\pm^{(1)}(\vec{\mathrm{p}})\Lambda_\pm^{(2)}(-\vec{\mathrm{p}}) \times \tilde{\psi}_\lambda(W_n,\vec{\mathrm{p}},p_0) \tag{15}$$

with

$$\Lambda_+(\vec{\mathrm{p}}) = [E(\vec{\mathrm{p}}) + \vec{\alpha}\cdot\vec{\mathrm{p}} + \beta m_q]/2E(\vec{\mathrm{p}}) = \sum_s (m_q/E) u(\vec{\mathrm{p}},s) u^\dagger(\vec{\mathrm{p}},s) , \tag{16a}$$

$$\Lambda_-(\vec{\mathrm{p}}) = [E(\vec{\mathrm{p}}) - \vec{\alpha}\cdot\vec{\mathrm{p}} - \beta m_q]/2E(\vec{\mathrm{p}}) = \sum_s (m_q/E) v(-\vec{\mathrm{p}},s) v^\dagger(-\vec{\mathrm{p}},s) . \tag{16b}$$

The Dirac spinors u and v are given in terms of two-component Pauli spinors χ_s by

$$u(\vec{\mathrm{p}},s) = \left[\frac{E+m_q}{2m_q}\right]^{1/2} \begin{bmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{\mathrm{p}}}{E+m_q} \end{bmatrix} \chi_s , \tag{17a}$$

$$v(-\vec{p},s)=\left[\frac{E+m_q}{2m_q}\right]^{1/2}\begin{bmatrix}-\dfrac{\vec{\sigma}\cdot\vec{p}}{E+m_q}\\ 1\end{bmatrix}\times(-i\sigma_2)\chi_s\ . \tag{17b}$$

We next extract the spinor structure of $\tilde{\psi}_\lambda$ implied by Eqs. (14)–(17), and write $\psi_\lambda^{\pm\pm}$ in terms of a set of functions $\chi_{ss'\lambda}^{\pm\pm}$ which depend on the spin projections s, s' of particles 1, 2. Thus,

$$\tilde{\psi}_{\alpha\beta;\lambda}^{++}(W_n,\vec{p},p_0)=\sum_{ss'}(m_q/E)u_\alpha^{(1)}(\vec{p},s)u_\beta^{(2)}(-\vec{p},s')\chi_{ss'\lambda}^{++}(W_n,\vec{p},p_0)$$
$$=\sum_{ss'}(m_q/E)u_\alpha^{(1)}(\vec{p},s)[-\bar{v}^{(2)}(-\vec{p},s')C]_\beta\chi_{ss'\lambda}^{++} \tag{18a}$$

with

$$\chi_{ss'\lambda}^{++}=\sum_{\alpha\beta}u_\alpha^{(1)\dagger}(\vec{p},s)u_\beta^{(2)\dagger}(-\vec{p},s')\tilde{\psi}_{\alpha\beta;\lambda} \tag{18b}$$
$$=u^{(1)\dagger}(\vec{p},s)\tilde{\psi}_\lambda C\gamma_0 v^{(2)}(-\vec{p},s')\ , \tag{18c}$$

$$\tilde{\psi}_{\alpha\beta;\lambda}^{+-}(W_n,\vec{p},p_0)=\sum_{ss'}(m_q/E)u_\alpha^{(1)}(\vec{p},s)v_\beta^{(2)}(\vec{p},s')\chi_{ss'\lambda}^{+-}(W_n,\vec{p},p_0)$$
$$=\sum_{ss'}(m_q/E)u_\alpha^{(1)}(\vec{p},s)[-\bar{u}^{(2)}(\vec{p},s')C]_\beta\chi_{ss'\lambda}^{+-} \tag{19a}$$

with

$$\chi_{ss'\lambda}^{+-}=\sum_{\alpha\beta}u_\alpha^{(1)\dagger}(\vec{p},s)v_\beta^{(2)\dagger}(-\vec{p},s')\tilde{\psi}_{\alpha\beta;\lambda} \tag{19b}$$
$$=u^{(1)\dagger}(\vec{p},s)\tilde{\psi}_\lambda C\gamma_0 u^{(2)}(-\vec{p},s')\ , \tag{19c}$$

etc., where we have used the relation $u=C\bar{v}^T$, and have treated $\tilde{\psi}_{\alpha\beta;\lambda}$ as a 4×4 matrix in Eqs. (18c) and (19c). We will also regard the functions $\chi_{ss'\lambda}^{\pm\pm}$ as elements of 2×2 matrices in the two-particle spin space.

With this notation, the matrix element $\vec{j}_\lambda$, Eq. (13), is easily reduced to a trace of products of Pauli spin matrices with the χ's,

$$\vec{j}_\lambda=\frac{1}{\sqrt{2}}\int\frac{d^4p}{(2\pi)^4}\frac{m_q}{E(\vec{p})}\sum_{ss'}[\bar{v}^{(2)}(-\vec{p},s')\vec{\gamma}u^{(1)}(\vec{p},s)\chi_{ss'\lambda}^{++}$$
$$+\bar{u}^{(2)}(\vec{p},s')\vec{\gamma}u^{(1)}(\vec{p},s)\chi_{ss'\lambda}^{+-}+\bar{v}^{(2)}(-\vec{p},s')\vec{\gamma}v^{(1)}(-\vec{p},s)\chi_{ss'\lambda}^{-+}$$
$$+\bar{u}^{(2)}(\vec{p},s')\vec{\gamma}v^{(1)}(-\vec{p},s)\chi_{ss'\lambda}^{--}] \tag{20a}$$

$$=\frac{1}{\sqrt{2}}\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[i\sigma_2\left[\vec{\sigma}-\frac{E-m_q}{E}\hat{p}\vec{\sigma}\cdot\hat{p}\right]\chi_\lambda^{++}+\left[\vec{\sigma}-\frac{E-m_q}{E}\hat{p}\vec{\sigma}\cdot\hat{p}\right](-i\sigma_2)\chi_\lambda^{--}\right.$$
$$\left.+\frac{\vec{p}}{m_q}(\chi_\lambda^{+-}-\chi_\lambda^{-+})\right]\ . \tag{20b}$$

The physical interpretation of the various terms in Eq. (20) is shown graphically in Figs. 1(a)–1(d). The χ^{++} term describes dissociation of the vector meson into a quark and antiquark [Fig. 1(a)]. The χ^{+-} term describes the annihilation of a quark and the vector meson into a quark [Fig. 1(b)], while χ^{-+} [Fig. 1(c)] describes the corresponding process $\bar{q}V\to\bar{q}$. Finally, the χ^{--} term describes

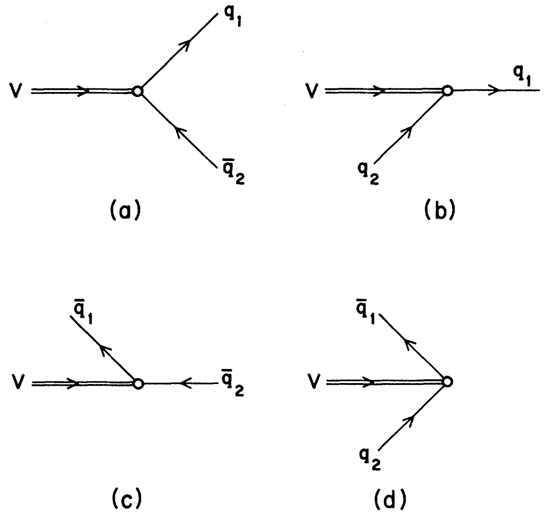

FIG. 1. Schematic representation of the χ functions in Eq. (20). (a) The dissociation of a vector meson V into a ${}^3S_1$-${}^3D_1$ $q\bar{q}$ pair described by χ^{++}. (b), (c) The annihilation of q ($\bar{q}$) with V in a ${}^2P_{1/2}$-${}^4P_{1/2}$ configuration to give a final q ($\bar{q}$) described by χ^{+-} (χ^{-+}). (d) The annihilation of a ${}^3S_1$-${}^3D_1$ $q\bar{q}$ pair with V described by χ^{--}.

the annihilation to the vacuum of the vector meson with the quark-antiquark pair [Fig. 1(d)]. Given the above interpretation, we expect the χ^{++} and χ^{--} wave functions to correspond to ${}^3S_1$-${}^3D_1$ $q\bar{q}$ states, and the χ^{+-} and χ^{-+} wave functions to correspond to ${}^2P_{1/2}$-${}^4P_{1/2}$ qV and $\bar{q}V$ states, respectively. The structure of the matrices $\chi^{\pm\pm}$ is determined by conservation of angular momentum, parity, and charge conjugation. The wave function $\tilde{\psi}_\lambda$ for a $J^{PC}=1^{--}$ state must be proportional to the polarization vector ϵ_λ^μ of the vector meson. In the meson rest frame, $\epsilon_\lambda^\mu=(0,\vec{\epsilon}_\lambda)$, so $\tilde{\psi}_\lambda$ must be of the form $\vec{\epsilon}_\lambda\cdot\vec{\psi}$, with $\vec{\psi}$ a polar vector with $C=-1$. Also, each antiparticle spinor in Eqs. (18)–(20) introduces a factor $-i\sigma_2$ when reduced to two components [see Eq. (17b)]. The χ's are consequently of the form

$$\begin{aligned}\chi_\lambda^{++}&=\vec{\epsilon}_\lambda\cdot\vec{A}^{++}(-i\sigma_2),\quad \chi_\lambda^{--}=i\sigma_2\vec{\epsilon}_\lambda\cdot\vec{A}^{--}\,,\\ \chi_\lambda^{+-}&=\vec{\epsilon}_\lambda\cdot\vec{A}^{+-},\quad \chi_\lambda^{-+}=i\sigma_2\vec{\epsilon}_\lambda\cdot\vec{A}^{-+}(-i\sigma_2)\,,\end{aligned}\tag{21}$$

where the $\vec{A}$'s are 2×2 matrix functions of p_0 and $\vec{p}$. Because of the odd relative parity of the q and $\bar{q}$ in χ^{++} and χ^{--}, $\vec{A}^{++}$ and $\vec{A}^{--}$ must be axial vectors, and are therefore expressible as linear combinations of the two independent axial vectors $\vec{\sigma}$ and $\hat{p}\vec{\sigma}\cdot\hat{p}$. Similarly, $\vec{A}^{+-}$ and $\vec{A}^{-+}$ must be polar vectors, and therefore expressible as linear combinations of $\hat{p}\underline{1}$ and $\vec{\sigma}\times\hat{p}$.

Under charge conjugation, the quarks and antiquarks in Eq. (20) and Fig. 1 are interchanged. As a result, $\vec{p}\to-\vec{p}$, $p_0\to-p_0$, the initial and final spin indices on the $\vec{A}$'s are interchanged, and (for $C=-1$) we must have

$$\mathscr{C}^{-1}\vec{A}^{++}(\vec{p},p_0)\mathscr{C}=-\vec{A}^{++T}(-\vec{p},-p_0)\,,\tag{22a}$$

$$\mathscr{C}^{-1}\vec{A}^{--}(\vec{p},p_0)\mathscr{C}=-\vec{A}^{--T}(-\vec{p},-p_0)\,,\tag{22b}$$

$$\mathscr{C}^{-1}\vec{A}^{+-}(\vec{p},p_0)\mathscr{C}=-\vec{A}^{-+T}(-\vec{p},-p_0)\,,\tag{22c}$$

where $\vec{A}^T$ is the transposed matrix, and $\mathscr{C}=-i\sigma_2$ is the 2×2 charge-conjugation matrix.

The most general form of the χ matrices consistent with these constraints is[28]

$$\chi_\lambda^{++}=\Big[\frac{1}{\sqrt{2}}\vec{\epsilon}_\lambda\cdot\vec{\sigma}\tilde{\psi}_S(|\vec{p}|,|p_0|)+\tfrac{1}{2}(3\vec{\epsilon}_\lambda\cdot\hat{p}\vec{\sigma}\cdot\hat{p}-\vec{\epsilon}_\lambda\cdot\vec{\sigma})\tilde{\psi}_D(|\vec{p}|,|p_0|)\Big](-i\sigma_2)\,,\tag{23a}$$

$$\chi_\lambda^{--}=i\sigma_2\Big[\frac{1}{\sqrt{2}}\vec{\epsilon}_\lambda\cdot\vec{\sigma}\tilde{\psi}_S'(|\vec{p}|,|p_0|)+\tfrac{1}{2}(3\vec{\epsilon}_\lambda\cdot\hat{p}\vec{\sigma}\cdot\hat{p}-\vec{\epsilon}_\lambda\cdot\vec{\sigma})\tilde{\psi}_D'(|\vec{p}|,|p_0|)\Big]\,,\tag{23b}$$

$$\begin{aligned}\chi_\lambda^{+-}=&\,(\vec{\epsilon}_\lambda\cdot\hat{p}\underline{1}+i\vec{\epsilon}_\lambda\cdot\vec{\sigma}\times\hat{p})[\tilde{\psi}_{2P}(|\vec{p}|,|p_0|)+p_0\tilde{\psi}_{2P}'(|\vec{p}|,|p_0|)]\\&+(2\vec{\epsilon}_\lambda\cdot\hat{p}\underline{1}-i\vec{\epsilon}_\lambda\cdot\vec{\sigma}\times\hat{p})[\tilde{\psi}_{4P}(|\vec{p}|,|p_0|)+p_0\tilde{\psi}_{4P}'(|\vec{p}|,|p_0|)]\,,\end{aligned}\tag{23c}$$

$$\begin{aligned}\chi^{-+}=&\,i\sigma_2(\vec{\epsilon}_\lambda\cdot\hat{p}\underline{1}-i\vec{\epsilon}_\lambda\cdot\vec{\sigma}\times\hat{p})(-i\sigma_2)[\tilde{\psi}_{2P}(|\vec{p}|,|p_0|)-p_0\tilde{\psi}_{2P}'(|\vec{p}|,|p_0|)]\\&+i\sigma_2(2\vec{\epsilon}_\lambda\cdot\hat{p}\underline{1}+i\vec{\epsilon}_\lambda\cdot\vec{\sigma}\times\hat{p})(-i\sigma_2)[\tilde{\psi}_{4P}(|\vec{p}|,|p_0|)-p_0\tilde{\psi}_{4P}'(|\vec{p}|,|p_0|)]\,.\end{aligned}\tag{23d}$$

χ^{++} and χ^{--} are expressed, as expected, as combinations of ${}^3S_1$ and ${}^3D_1$ $q\bar{q}$ wave functions (the spin-angle functions are normalized). The spin structure of χ^{+-} (χ^{-+}) similarly involves a combination of ${}^2P_{1/2}$ and ${}^4P_{1/2}$ states for the vector meson and the incoming q ($\bar{q}$), expressed in each case with particle 1 outgoing. However, there are *two* "radial" wave functions $\tilde{\psi}_P$ and $\tilde{\psi}_P'$ for each of these states, rather than the single functions which would be expected nonrelativistically. The extra functions are multiplied by p_0, and therefore lead, upon Fourier transformation, to space-time wave functions which are odd under the interchange of t_1 and t_2 and vanish for $t_1=t_2$. While the complete equal-time wave function therefore involves only the six independent "radial" functions expected naively, the extra functions will contribute in processes in which the time ordering is important.

The decomposition of the full SBS wave function given by Eqs. (14), (18), (19), and (23), and our interpretation of the terms, are apparently new.

We now return to the current $\vec{j}_\lambda$ in Eq. (20b). It is easily seen from Eqs. (23) that the χ^{+-} and χ^{-+} terms do not contribute to $\vec{j}_\lambda$ since the terms proportional to $\hat{p}$ cancel or integrate to zero, and those containing $\vec{\sigma}$ have a vanishing trace. This is illustrated in Fig. 2. The photon current acts at a single space-time point, to which the two quarks must connect, and there are consequently no graphs which incorporate Figs. 1(b) and 1(c). After calculation of the remaining traces, $\vec{j}_\lambda$ reduces to

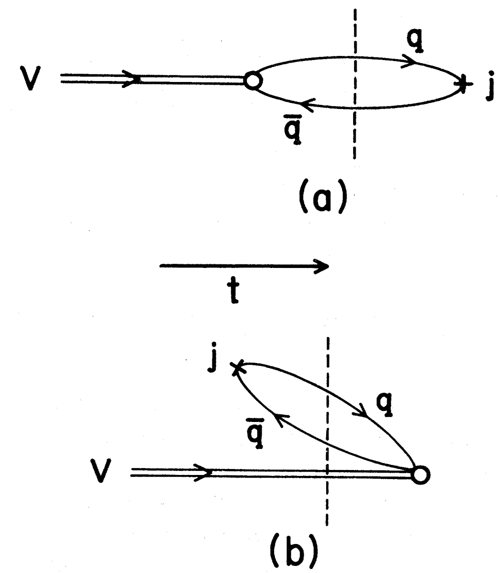

FIG. 2. (a) and (b) give schematic representations of the contributions of χ^{++} and χ^{--} to $\vec{j}_\lambda$, Eqs. (20b) and (24). Because the current acts at a point, the q and $\bar{q}$ in (a), (b) must emerge from (or annihilate with) V at equal times. The χ^{+-} and χ^{-+} configurations in Figs. 1(b) and 1(c) cannot appear in equal-time diagrams, and therefore do not contribute to $\vec{j}_\lambda$.

$$\begin{aligned}\vec{j}_\lambda=\vec{\epsilon}_\lambda \int \frac{d^4p}{(2\pi)^4}\Bigg[& [\tilde{\psi}_S(|\vec{p}|,|p_0|)+\tilde{\psi}_S'(|\vec{p}|,|p_0|)]\left[1-\frac{1}{3}\frac{E-m_q}{E}\right] \\ & -\frac{\sqrt{2}}{3}\frac{E-m_q}{E}[\tilde{\psi}_D(|\vec{p}|,|p_0|)+\tilde{\psi}_D'(|\vec{p}|,|p_0|)]\Bigg] \\ \equiv{}& \vec{\epsilon}_\lambda \mathcal{J} ,\end{aligned} \tag{24}$$

where $\mathcal{J}$ involves only equal-time SBS wave functions.

Finally, the leptonic width $\Gamma_n(l^+l^-)$ from Eq. (11) is

$$\Gamma_n(l^+l^-)=\frac{16\pi\alpha^2 e_q^2 v_l}{M_n^2}\left[1+2\left[\frac{m_l}{M_n}\right]^2\right]|\mathcal{J}_n|^2 . \tag{25}$$

It will be convenient for S states to extract the dominant term $\psi_{nS}(0)$ from $\mathcal{J}$, and write $\mathcal{J}$ as

$$\mathcal{J}_n=\psi_{nS}(0)(1-\tfrac{1}{2}\Delta_n) , \tag{26}$$

where the correction term Δ_n is of order v^2/c^2. To this order, $\Gamma_n(e^+e^-)$ is given by the modified Van Royen—Weisskopf formula of Eq. (9),

$$\Gamma_n(e^+e^-)\approx\frac{16\pi\alpha^2 e_q^2}{M_n^2}|\psi_{nS}(0)|^2(1-\Delta_n) . \tag{27}$$

We will show in the next sections that $|\psi_{nS}(0)|^2$ is related to the density of $^3S_1$ states by Eq. (10). The relativistic duality relation then follows. Our method involves the reduction of the SBS equation for the exact relativistic wave functions to an approximate Schrödinger equation for ψ_{nS}. This equation involves an energy-dependent, slightly nonlocal potential. We then derive the desired expression for $|\psi_{nS}(0)|^2$ by a modification of the JWKB argument given by Quigg and Rosner.[11]

We consider first the reduction of the SBS equation to Schrödinger form.

C. Reduction of the Salpeter-Bethe-Schwinger equation to Schrödinger form

The SBS equation[29,30] is the wave equation for the bound state ψ of two Dirac particles, in our case the q and $\bar{q}$,

$$[S^{(1)}(\tfrac{1}{2}P+p)S^{(2)}(\tfrac{1}{2}P-p)]^{-1}\tilde{\psi}_\lambda(M_n,p) = i\int \frac{d^4k}{(2\pi)^4}K(M_n,p,k)\tilde{\psi}_\lambda(M_n,k)\ . \quad (28)$$

$S^{(i)}$ is the free one-particle propagator, $P=(M_n,\vec{0})$, M_n is the total energy of the nth bound state, and p is the relative four-momentum of the two particles. The kernel K includes the self-mass corrections to the quark propagator (all one-particle-irreducible graphs) and all two-particle-irreducible interactions. We assume that this equation still holds in the presence of a confining interaction, as it does in the corresponding nonrelativistic case.[31] The amplitude $\tilde{\psi}_\lambda$ is normalized according to the condition

$$\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\bar{\tilde{\psi}}_\lambda(M_n,p)\frac{\partial}{\partial M_n}\{\,[iS^{(1)}(\tfrac{1}{2}P+p)S^{(2)}(\tfrac{1}{2}P-p)]^{-1}(2\pi)^4\delta^4(p-k)-K(M_n,p,k)\,\}\tilde{\psi}_{\lambda'}(M_n,k)=\delta_{\lambda\lambda'}\ . \quad (29)$$

Equation (28) is a set of coupled integral equations for the eight wave functions ψ_S, ψ_D, . . . , discussed in the preceding section. However, to calculate Γ_n, we actually need only the equal-time (or "instantaneous") projections of the S- and D-state wave functions. We can obtain an equation for these quantities if, following Salpeter,[32] we omit the self-mass corrections and ignore the retardation effects in the kernel in Eq. (28), that is, if we treat K as independent of p_0 and k_0, $K=K(M,\vec{p},\vec{k})$. The corrections associated with these approximations can be calculated in perturbation theory.[20,22,24,27]

We again introduce the positive- and negative-energy projections $\tilde{\psi}_\lambda^{\pm\pm}(M_n,\vec{p},p_0)$ of $\tilde{\psi}$, Eq. (15), and define the corresponding equal-time amplitudes $\phi_n^{\pm\pm}(\vec{p})$ (suppressing the subscript λ) by

$$\phi_n^{\pm\pm}(\vec{p})=\int\frac{dp_0}{2\pi}\tilde{\psi}^{\pm\pm}(M_n,\vec{p},p_0)\ . \quad (30)$$

Then, following Salpeter,[32] we shift the inverse propagators to the right-hand side of Eq. (28) and integrate on p_0, k_0 to obtain the Salpeter equations

$$[M_n-2E(\vec{p})]\phi_n^{++}(\vec{p})=\Lambda_+^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p})\int\frac{d^3k}{(2\pi)^3}\gamma_0^{(1)}\gamma_0^{(2)}K(M_n,\vec{p},\vec{k})\phi_n(\vec{k})\ , \quad (31a)$$

$$[M_n+2E(\vec{p})]\phi_n^{--}(\vec{p})=-\Lambda_-^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p})\int\frac{d^3k}{(2\pi)^3}\gamma_0^{(1)}\gamma_0^{(2)}K(M_n,\vec{p},\vec{k})\phi_n(\vec{k})\ , \quad (31b)$$

$$\phi_n^{+-}=\phi_n^{-+}=0\ . \quad (31c)$$

It is convenient to extract the spinor dependence of ϕ_n^{++} and ϕ_n^{--} as in Eq. (18) and to introduce wave functions $\phi_{n,ss'}^{\pm}(\vec{p})$ defined as

$$\phi_n^{++}(\vec{p})=\sum_{s,s'}\frac{m_q}{E(\vec{p})}u_s^{(1)}(\vec{p})u_{s'}^{(2)}(-\vec{p})\phi_{n,ss'}^{+}(\vec{p})\ , \quad (32a)$$

$$\phi_n^{--}(\vec{p})=\sum_{s,s'}\frac{m_q}{E(\vec{p})}v_s^{(1)}(-\vec{p})v_{s'}^{(2)}(\vec{p})\phi_{n,ss'}^{-}(\vec{p})\ , \quad (32b)$$

where

$$\Phi_{n,ss'}^{+}=\left\{\left[\frac{1}{\sqrt{2}}\vec{\epsilon}_\lambda\cdot\vec{\sigma}\tilde{\psi}_{n,S}(|\vec{p}|)+\tfrac{1}{2}(3\vec{\epsilon}_\lambda\cdot\hat{p}\vec{\sigma}\cdot\hat{p}-\vec{\epsilon}_\lambda\cdot\vec{\sigma})\tilde{\psi}_{n,D}(|\vec{p}|)\right](-i\sigma_2)\right\}_{ss'}\ , \quad (32c)$$

$$\Phi_{n,ss'}^{-}=\left\{i\sigma_2\left[\frac{1}{\sqrt{2}}\vec{\epsilon}_\lambda\cdot\vec{\sigma}\tilde{\psi}'_{n,S}(|\vec{p}|)+\tfrac{1}{2}(3\vec{\epsilon}_\lambda\cdot\hat{p}\vec{\sigma}\cdot\hat{p}-\vec{\epsilon}_\lambda\cdot\vec{\sigma})\tilde{\psi}'_{n,D}(|\vec{p}|)\right]\right\}_{ss'}\ , \quad (32d)$$

with

$$\tilde{\psi}_n(|\vec{p}|)=\int\frac{dp_0}{2\pi}\tilde{\psi}_n(|\vec{p}|,|p_0|)\ . \quad (32e)$$

We can rewrite Eqs. (31) as integral equations for the wave functions $\Phi^{\pm}(\vec{p})$ or $\tilde{\psi}_{n,S},\tilde{\psi}_{n,D}$,

$$[M_n-2E(\vec{p})]\Phi^+_{n,ss'}(\vec{p})=(\mathcal{K}^{++}\Phi_n^+)_{ss'}(\vec{p})+(\mathcal{K}^{+-}\Phi_n^-)_{ss'}(\vec{p})\ , \tag{33a}$$

$$[M_n-2E(\vec{p})]\Phi^-_{n,ss'}(\vec{p})=(\mathcal{K}^{-+}\Phi_n^-)_{ss'}(\vec{p})+(\mathcal{K}^{--}\Phi_n^-)_{ss'}(\vec{p})\ , \tag{33b}$$

where the integral operators $\mathcal{K}$ are defined by

$$(\mathcal{K}^{++}\Phi_n^+)_{ss'}(\vec{p})=\sum_{rr'}\int\frac{d^3k}{(2\pi)^3}\frac{m_q}{E(\vec{p})}\bar{u}_s^{(1)}(\vec{p})\bar{u}_{s'}^{(2)}(-\vec{p})K(M_n,\vec{p},\vec{k})\frac{m_q}{E(\vec{k})}u_r^{(1)}(\vec{k})u_{r'}^{(2)}(-\vec{k})\Phi^+_{n,rr'}(\vec{k})\ , \tag{34a}$$

$$(\mathcal{K}^{+-}\Phi_n^-)_{ss'}(\vec{p})=\sum_{rr'}\int\frac{d^3k}{(2\pi)^3}\frac{m_q}{E(\vec{p})}\bar{u}_s^{(1)}(\vec{p})\bar{u}_{s'}^{(2)}(-\vec{p})K(M_n,\vec{p},\vec{k})\frac{m_q}{E(\vec{k})}v_r^{(1)}(-\vec{k})v_{r'}^{(2)}(\vec{k})\Phi^-_{n,rr'}(\vec{k})\ , \tag{34b}$$

with similar expressions for $\mathcal{K}^{-+}$ and $\mathcal{K}^{--}$. In matrix notation

$$\begin{bmatrix}(M_n-2E)\delta-\mathcal{K}^{++} & -\mathcal{K}^{+-}\\ -\mathcal{K}^{-+} & -(M_n+2E)\delta-\mathcal{K}^{--}\end{bmatrix}\begin{bmatrix}\Phi_n^+\\ \Phi_n^-\end{bmatrix}(\vec{p})\equiv(\mathcal{M}\Phi)(\vec{p})=0\ , \tag{35}$$

where

$$\delta=\int\frac{d^3k}{(2\pi)^3}(2\pi)^3\delta(\vec{p}-\vec{k})\delta_{rs}\delta_{r's'}\ . \tag{36}$$

The normalization of the momentum-space wave functions is given in terms of the matrix $\mathcal{M}$ by

$$\int\frac{d^3p}{(2\pi)^3}\Phi_n^\dagger(\vec{p})\left[\frac{\partial\mathcal{M}}{\partial M_n}\Phi_n\right](\vec{p})=1\ . \tag{37}$$

Equation (33b) determines Φ^- in terms of Φ^+,

$$\Phi_n^-=-[(M_n+2E)\delta+\mathcal{K}^{--}]^{-1}\mathcal{K}^{-+}\Phi^+\ . \tag{38}$$

For the usual models for $q\bar{q}$ systems with only scalar- and vector-exchange interactions, $\mathcal{K}^{+-}$ and $\mathcal{K}^{-+}$ are of order v^2/c^2 relative to $\mathcal{K}^{++}$ in the nonrelativistic region. Φ^- is consequently of order v^4/c^4 relative to Φ^+ and can be neglected. [Even for $\mathcal{K}^{+-},\mathcal{K}^{-+}\sim\mathcal{K}^{++}$, $\Phi^-\sim O(v^2/c^2)\Phi^+$.] Φ^+ then satisfies the equation

$$[2E(\vec{p})-M_n]\Phi_n^+(\vec{p})+(\mathcal{K}^{++}\Phi_n^+)(\vec{p})=0 \tag{39}$$

to order v^2/c^2, and is normalized so that

$$\int\frac{d^3p}{(2\pi)^3}(\Phi^+)^\dagger\left[\delta-\frac{\partial\mathcal{K}^{++}}{\partial M_n}\right]\Phi^+=1\ . \tag{40}$$

We will work in the rest of the paper with Eq. (39), although the results could be generalized.

Our final step in the reduction of the SBS equation of Eq. (28) is to expand $E(\vec{p})=(\vec{p}^{\,2}+m_q^{\,2})^{1/2}$ in Eq. (39) to obtain a Schrödinger equation with relativistic corrections,

$$\left[\frac{p^2}{m_q}-\frac{p^4}{4m_q^{\,3}}+\cdots+\mathcal{K}^{++}\right]\Phi_n^+=E_n\Phi_n^+\ , \tag{41}$$

where $E_n=M_n-2m_q$. This approximation is correct to $O(v^2/c^2)$. Although Eq. (41) is in Schrödinger form, it is not an "ordinary" Schrödinger equation because the "potential" $\mathcal{K}^{++}$ is in general nonlocal and energy-dependent. In addition, the p^4 term should properly be treated as a perturbation.

D. Evaluation of $|\psi_{nS}(0)|^2$

In Eqs. (25)–(27), we related the leptonic width $\Gamma_n(e^+e^-)$ for the decay of a bound $J^P=1^-$ $q\bar{q}$ system to $|\psi_{nS}(0)|^2$, the square of the relativistic large component S-state SBS wave function at the origin (zero space-time separation of the q and $\bar{q}$). Krammer and Leal Ferriera[10] and Quigg and Rosner[11] used JWKB arguments to show for the ordinary Schrödinger equation that $|\psi_{nS}(\vec{0})|^2$ is related to the density of states dn/dE_n as shown in Eq. (6). (A higher-order JWKB discussion was given recently by Pasupathy and Singh.[14] The corresponding relation for arbitrary orbital angular momentum was derived by Bell and Pasupathy.[12]) We now extend this result to the relativistic case.

The coordinate-space analog of Eq. (41) for the Salpeter wave function $\Psi_n(\vec{r})\equiv\Psi_n(\vec{r},0)$ is

$$\left[-\frac{\nabla^2}{m_q}-\frac{\nabla^4}{4m_q^{\,3}}+\cdots+V\right]\Psi_n=E_n\Psi_n\ , \tag{42}$$

where Ψ_n and V are the Fourier transforms of Φ^+ and $\mathscr{K}^{++}$. V is a nonlocal, energy-dependent integral operator,

$$(V\Psi_n)(\vec{r}) = \int d^3r' V(E_n,\vec{r},\vec{r}\,')\Psi_n(\vec{r}\,') \,. \tag{43}$$

Ψ_n is to be normalized so that

$$\int d^3r\, \Psi_n^\dagger \left[1 - \frac{\partial V}{\partial E_n}\right]\Psi_n = 1 \,. \tag{44}$$

The real form of the SBS kernel $K(M_n,\vec{p},\vec{k})$, hence of $V(E_n,\vec{r},\vec{r}\,')$, is of course not known. However, models which treat the $q\bar{q}$ interaction as local are strikingly successful in explaining the properties of the $c\bar{c}$ and $b\bar{b}$ systems.[17] This suggests that any intrinsic nonlocality in K is small, or is well approximated in the empirical potential. The additional nonlocality associated with the spinor factors connecting K to $\mathscr{K}^{++}$ and V pertains only over a range on the order of the Compton wavelength of a quark. This will also have very little effect in the region of the observed $q\bar{q}$ resonances, where the empirical potential varies slowly on the scale of a Compton wavelength. We therefore conclude that it is reasonable to treat V as a local, but possibly energy-dependent, potential in the following arguments.

Equation (42) is still a matrix equation in spin space, and couples S- and D-state wave functions. Since the S-D coupling and D-state effects are empirically quite small in the $c\bar{c}$ and $b\bar{b}$ systems, we will assume for simplicity that V is spin independent, and will consider only the $^3S_1$ states.

It is convenient to iterate Eq. (42) once to eliminate ∇^4 in terms of $E_n - V$. The modified S-state equation for the radial wave functions $u_n(r) = \sqrt{4\pi}\Psi_{nS}(r)$ is then given to order v^2/c^2 by

$$\frac{d^2u_n}{dr^2} = -m_q(E_n - V)u_n + \frac{1}{4m_q}\frac{d^2}{dr^2}[(E_n - V)u_n] + \cdots \tag{45a}$$

$$= -\frac{1}{2m_q}\frac{dV}{dr}\frac{du_n}{dr} - m_q\left[E_n - V + \frac{1}{4m_q}(E_n - V)^2 + \frac{1}{4m_q{}^2}\frac{d^2V}{dr^2}\right]u_n + \cdots \,. \tag{45b}$$

We can eliminate the term in du_n/dr in Eq. (45b) by the substitution $u_n = w_n e^{-V/4m_q}$, and find that to order v^2/c^2

$$\frac{d^2w_n}{dr^2} + m_q(E_n - V_{\text{eff}})w_n = 0 \tag{46}$$

with

$$V_{\text{eff}} = V - \frac{1}{4m_q}(E_n - V)^2 \,. \tag{47}$$

The $(E_n - V)^2$ perturbation term in V_{eff} is significant only where it is small, $(E_n - V)/4m_q << 1$. This condition is satisfied for the confining potentials used to describe heavy-quark systems except near the color-Coulomb singularity at the origin, and for $r\to\infty$, and is always satisfied on the average. As a practical matter, we note that the Schrödinger wave function for a confining potential V decays to zero before the perturbation becomes large. We can therefore cut off the perturbation at large r without changing our results to order v^2/c^2, and with this proviso can treat V_{eff} as a new confining potential "close to" V. The color-Coulomb singularity is naturally smeared out over a distance $m_q{}^{-1}$ by the nonlocalities discussed above, but can also be treated separately.[15,16]

The Quigg-Rosner[11] derivation of Eq. (6) for local potentials proceeds in two steps. The first step relates $|\Psi_{nS}(0)|^2$ to the expectation value of $\partial V/\partial r$, and is essentially unchanged for V nonlocal. The second step uses a JWKB argument to evaluate $\langle \partial V/\partial r\rangle$ and requires some modification for an energy-dependent effective potential. We will sketch both steps.

We first write $|\Psi_{nS}(0)|^2$ in terms of the radial wave function by using the identity

$$|\Psi_{nS}(0)|^2 = -\frac{1}{4\pi}\int_0^\infty \frac{d}{dr}\left[\frac{du_n}{dr}\right]^2 dr \,. \tag{48}$$

We next use Eq. (45a) to eliminate the second derivative of $u_{n'}$ and find after some partial integrations that

$$|\Psi_{nS}(0)|^2 \simeq \left[1 + \frac{E_n - V(0)}{2m_q}\right]\frac{m_q}{4\pi}\int_0^\infty |u_n(r)|^2\frac{\partial V}{\partial r}dr = \frac{M_n'}{8\pi}\left\langle\frac{\partial V}{\partial r}\right\rangle_n \,, \tag{49}$$

$M_n' = M_n - V(0)$, where we assume $V(0)$ is finite. [If this is not the case, one must treat the perturbation terms in Eq. (45) more carefully.] This expression is identical to the nonrelativistic formula for $|\psi_{nS}^{\text{nonrel}}(0)|^2$ except for an overall factor $M_n'/2m_q$.

The normalization of the radial wave functions is determined by Eq. (44). It will be convenient to introduce the normalization integral explicitly in Eq. (48), and write $|\Psi_{nS}(0)|^2$ as

$$|\Psi_{nS}(0)|^2 = \frac{M_n'}{8\pi} \frac{\int_0^\infty dr\, |u_n(r)|^2 \frac{\partial V}{\partial r}}{\int_0^\infty dr\, |u_n(r)|^2 \left[1 - \frac{\partial V}{\partial E_n}\right]} \tag{50a}$$

$$\simeq \frac{M_n'}{8\pi} \frac{\int_0^\infty dr\, |w_n(r)|^2 \left[1 + \frac{E_n - V}{2m_q}\right] \frac{\partial V}{\partial r}}{\int_0^\infty dr\, |w_n(r)|^2 \left[1 + \frac{E_n - V}{2m_q}\right] \left[1 - \frac{\partial V}{\partial E_n}\right]}, \tag{50b}$$

where we have expanded the exponential in the relation $u_n = w_n e^{-V/4m_q}$ and have introduced factors of $M_n/2m_q$ in both the numerator and denominator.

To evaluate the ratio of integrals, we follow the procedure of Ref. 11, approximate the w_n's in Eq. (50b) by the JWKB solutions to Eq. (46),

$$w_n \simeq (E_n - V_{\text{eff}})^{-1/4} \cos\phi\,, \tag{51}$$

integrate only to the classical turning point r_0, and replace $\cos^2\phi$ by its "average" value $\frac{1}{2}$. We again assume for simplicity that V is nonsingular at $r = 0$. Singular potentials require a more elaborate treatment.[13] We can then evaluate the integral in the numerator explicitly,

$$\int_0^\infty dr\, |u_n|^2 \frac{\partial V}{\partial r} \simeq \left[E_n' \left[1 + \frac{E_n'}{4m_q}\right]\right]^{1/2}, \tag{52}$$

where $E_n' = E_n - V(0)$. The integral in the denominator can be related to the JWKB expression for the energy eigenvalue E_n,

$$\pi(n - \tfrac{1}{4}) \simeq m_q^{1/2} \int_0^{r_0} dr (E_n - V_{\text{eff}})^{1/2}. \tag{53}$$

Differentiating with respect to E_n, we find that

$$\frac{2\pi}{m_q^{1/2}} \frac{dn}{dE_n} \simeq \int_0^{r_0} dr \left[E_n - V_{\text{eff}}\right]^{-1/2} \left[1 - \frac{\partial V_{\text{eff}}}{\partial E_n}\right] = \int_0^{r_0} dr \left[E_n - V_{\text{eff}}\right]^{-1/2} \left[1 + \frac{E_n - V}{2m_q}\right] \left[1 - \frac{\partial V}{\partial E_n}\right]. \tag{54}$$

Then, combining terms, we obtain our final result:

$$|\Psi_{nS}(0)|^2 \simeq M_n' \frac{m_q^{1/2}}{8\pi^2} \left[E_n' \left[1 + \frac{E_n'}{4m_q}\right]\right]^{1/2} \frac{dE_n}{dn} \tag{55a}$$

$$= \frac{M_n'^2 v_n'}{16\pi^2} \frac{dM_n}{dn}, \tag{55b}$$

where $M_n' = 2m_q + E_n - V(0)$ is the total energy of the pair at the origin and v_n' is the velocity of a free quark with energy $\frac{1}{2}M_n'$,

$$v_n' = \left[1 - \frac{4m_q^2}{M_n'^2}\right]^{1/2}. \tag{56}$$

Although the proof of Eqs. (55) would seem, as given, to involve rather drastic approximations in the JWKB evaluation of $\langle \partial V/\partial r \rangle$ for low-lying states, and in the (unnecessary[13]) restriction to nonsingular potentials, the corresponding nonrelativistic result in Eq. (6) has been checked numerically and is remarkably accurate even for n small.[15] We believe that Eqs. (55) are equally reliable. Higher-order JWKB results are discussed in Ref. 14.

E. Summary of results and relativistic duality

It will be useful to summarize our results and comment on the connection between the nonrelativistic and relativistic descriptions of the bound $q\bar{q}$ system before deriving the relativistic duality relation.

The relativistic leptonic width for the decay of a $q\bar{q}$ bound state is given in Eqs. (25)–(27). For the e^+e^- decay

$$\Gamma_n(e^+e^-)=\frac{16\pi\alpha^2 e_q{}^2}{M_n{}^2}\,|\,\psi_{nS}(0)\,|^2(1-\Delta_n)\,, \tag{57}$$

where $\psi_{nS}(0)$ is the large S-state component of the exact $q\bar{q}$ Salpeter-Bethe-Schwinger wave function at the origin, and Δ_n includes the relativistic and D-state corrections to the current matrix element given in Eq. (24). Alternatively,

$$\Gamma_n(e^+e^-)=\frac{16\pi\alpha^2 e_q{}^2}{M_n{}^2}\,|\,\Psi_{nS}(0)\,|^2 \times(1-\Delta_n)(1-\Delta_n')\,, \tag{58}$$

where $\Psi_{nS}(0)$ is the Salpeter wave function for an *instantaneous* $q\bar{q}$ interaction, and Δ_n' incorporates retardation effects and gluonic radiative corrections. These corrections are discussed in detail in Ref. 24.

The value of $|\,\Psi_{nS}(0)\,|^2$ is related to the density of states dn/dM_n (or dn/dE_n) in Eqs. (55). The corresponding result for a nonrelativistic Schrödinger system is given in Eq. (6). If we compare Eqs. (6) and (55a), we see that the wave functions for relativistic and Schrödinger systems with (necessarily different) spin-independent effective potentials fitted to the *same spectrum* are related by

$$|\,\Psi_{nS}(\vec{0},0)\,|^2=\frac{M_n'^{\,2}}{4m_q{}^2}\frac{v'}{v'_{\text{nonrel}}}\,|\,\psi_{nS}^{\text{nonrel}}(\vec{0})\,|^2\,, \tag{59}$$

where $v'_{\text{nonrel}}=(E_n'/m_q)^{1/2}$. This formula allows us to convert results obtained in nonrelativistic potential theory to "equivalent" results for a relativistic theory, and should be a useful tool in heavy-quark phenomenology.

The correction factor Δ_n in Eqs. (57) and (58) may be obtained from Eq. (24). To order v^2/c^2,

$$\begin{aligned}\Delta_n&=\tfrac{2}{3}[\Psi_{nS}(0)]^{-1}\int\frac{d^4p}{(2\pi)^4}\frac{E-m_q}{E}[\tilde{\Psi}_{nS}(\,|\,\vec{p}\,|\,,p_0)+\sqrt{2}\tilde{\Psi}_{nD}(\,|\,\vec{p}\,|\,,p_0)]\\&=\frac{4m_q}{3\pi}[\Psi_{nS}(0)]^{-1}\int_0^\infty dr\,K_0(m_qr)\left[\Psi_{nS}(0,0)-\frac{d}{dr}[r\Psi_{nS}(r,0)]+\sqrt{2}\Psi_{nD}(0,0)-\sqrt{2}\frac{d}{dr}[r\Psi_{nD}(r,0)]\right],\end{aligned} \tag{60}$$

where $K_0(m_qr)$ is the exponentially decreasing hyperbolic Bessel function.[33] For nonsingular potentials, we can expand the wave functions in Eq. (60) in Taylor series, and find that

$$\begin{aligned}\Delta_n&=\frac{1}{3m_q}[E_n-V(0)]+\frac{\sqrt{2}}{m_q{}^2}\frac{\Psi_{nD}''(0)}{\Psi_{nS}(0)}+O\left[\frac{1}{m_q{}^3}\right]\\&=\frac{1}{3}\frac{p_n{}^2(0)}{m_q{}^2}+\frac{\sqrt{2}}{m_q{}^2}\frac{\Psi_{nD}''(0)}{\Psi_{nS}(0)}+O\left[\frac{1}{m_q{}^3}\right],\end{aligned} \tag{61}$$

where $p_n(0)$ is the momentum of the quark at the origin. This is just the result which would be obtained by approximating $(E-m)/E$ by $p^2/2m_q$ in Eq. (60) and identifying the resulting integrals with $(\nabla^2\Psi)(0)$. the exact result involves an averaging over a region of radial extent $m_q{}^{-1}$, and can be used for singular potentials.

Relativistic effects on $q\bar{q}$ spectra and leptonic widths have been discussed by many authors,[34] but are generally ignored in potential-model fits to heavy-quark data except in connection with the hyperfine splittings of the states.[35] The usual procedure is to determine the nonrelativistic potential V and wave function $\psi_{n,S}^{\text{nonrel}}(0)$ by fitting the observed spectrum. The leptonic widths are then calculated from the approximate Van Royen–Weisskopf formula

$$\Gamma_n(e^+e^-)\sim\frac{16\pi\alpha^2 e_q{}^2}{M_n{}^2}\,|\,\psi_{nS}^{\text{nonrel}}(0)\,|^2\,(1-\Delta_n')\,. \tag{62}$$

This differs from Eq. (58) by the replacement of the relativistic wave function by the Schrödinger wave function, and by the omission of the correction factor $(1-\Delta_n)$. Nevertheless, as noted by many authors,[35] this (incorrect) procedure gives good results for the ratios of leptonic widths, and absolute widths which are acceptable given the uncertainty in the radiative correction Δ'_n. We obtain a relativistically correct expression for Γ_n in terms of $|\psi_{nS}^{\text{nonrel}}(0)|^2$ by using Eqs. (59) and (61) in Eq. (58). The corrections increase the predicted widths for the first three $^3S_1$ states in the ψ system [assuming $m_c=1.45$ GeV and $V(0)=0$] by factors of 1.037, 1.111, and 1.125, and increase the ratios $\Gamma(2S)/\Gamma(1S)$ and $\Gamma(3S)/\Gamma(1S)$ by factors of 1.071 and 1.085. We conclude that relativistic effects should *not* be ignored in calculations of leptonic widths, and note that they may affect attempts to reconstruct the effective $q\bar{q}$ potential from the data.[36] *We emphasize, however, that Eqs. (58) and (59) provide theoretical justification for the use of a property-corrected nonrelativistic Schrödinger description of $q\bar{q}$ systems.*

We now have all the ingredients for a derivation of the relativistic duality relation for nonsingular potentials. By duality we mean the approximate equality of an appropriate energy average of the relativistic cross section for $e^+e^-\to q\bar{q}$ bound states with the same average of the free (noninteracting) cross section for $e^+e^-\to q\bar{q}$,

$$\langle\sigma_{\text{bound}}\rangle=\langle\sigma_{\text{free}}\rangle+\text{corrections}\ . \tag{63}$$

The free cross section is given in Eq. (7),

$$W^2\sigma_{\text{free}}=6\pi\alpha^2 e_q{}^2 v(1-\tfrac{1}{3}v^2)\ . \tag{64}$$

The bound cross section in the absence of (singular) short-range gluonic interactions is given by Eqs. (8), (58), and (55) with $\Delta'_n=0$ and $\Delta_n=p_n{}^2(0)/3m_q{}^2=\frac{1}{3}v_n{}^2$, Eq. (61),

$$W^2\sigma_{\text{bound}}=\sum_n 6\pi^2\Gamma_n(e^+e^-)\delta(W-M_n)=6\pi\alpha^2 e_q{}^2 v_n\left[1-\tfrac{1}{3}v_n{}^2\right]\frac{dM_n}{dn}\delta(W-M_n)\ . \tag{65}$$

[We assume for simplicity that $V(0)=0$.[37]] We obtain the duality relation correct to order v^2/c^2 by convoluting $W^2\sigma_{\text{bound}}$ with a smooth function of energy and converting the sum on n into an integral,

$$\begin{aligned}\langle W^2\sigma_{\text{bound}}\rangle &= \int f(W-W')W'^2\sigma_{\text{bound}}(W')dW'\\ &=6\pi\alpha^2 e_q{}^2\sum_n f(W-M_n)v_n\left[1-\tfrac{1}{3}v_n{}^2\right]\frac{dM_n}{dn}\\ &=6\pi\alpha^2 e_q{}^2\int f(W-W')v'\left[1-\tfrac{1}{3}v'^2\right]dW'+\text{corrections}\\ &=\int f(W-W')W'^2\sigma_{\text{free}}(W')dW'+\text{corrections}\\ &=\langle W^2\sigma_{\text{free}}\rangle+\text{corrections},\end{aligned} \tag{66}$$

where the corrections may be estimated from the experimental data by using the Euler-MacLaurin summation formula.[16]

While we have derived the relativistic duality relation only for nonsingular potentials, it has been shown elsewhere that nonrelativistic duality holds for singular potentials provided one incorporates the short-range effects of the potential in σ_{free}.[13,16] The present results can be extended to singular interactions in a similar way.

III. APPLICATION: RADIATIVE CORRECTIONS TO LEPTONIC WIDTHS

To illustrate the use of relativistic duality, we estimate the radiative corrections to the leptonic widths $\Gamma_n(e^+e^-)$ for bound $q\bar{q}$ systems by using known results for free $q\bar{q}$ systems. We find that our radiative correction agrees to order α_s with that given by Barbieri *et al.* and Celmaster.[20] We are also able to estimate higher-order corrections.

To estimate the radiative corrections to Γ_n, we use the *local* version of duality, in which the energy average in Eq. (66) is over small intervals ΔW which straddle individual bound states.[16] Using Eq. (65) and assuming that the confining potential vanishes at the origin, we find that

$$\Gamma_n(e^+e^-)\simeq\frac{1}{6\pi^2}\int_{\Delta W}W^2\sigma_{\text{free}}(W)dW\ , \tag{67}$$

where, as noted in Sec. II E, we must use a free cross section in Eq. (67) which includes the effects of the singular short-range gluon exchanges between quarks.[13,16] (In the duality averaging process, only the effects of the long-range confining interaction disappear.) This "free" cross section is given in terms of the Salpeter wave function for a color-Coulomb interaction by

$$W^2\sigma_{\text{free}}=6\pi\alpha^2 e_q{}^2 v\,|\Psi_W(0)|^2(1-\Delta)(1-\Delta')\ , \tag{68}$$

where Δ and Δ' are free analogs of the bound-state relativistic and radiative corrections Δ_n and Δ'_n.

The local duality relation in Eq. (67) has been used in the past without the correction factors to predict leptonic widths for $q\bar{q}$ states.[1,3,4] We are interested in the *corrections* to Γ_n, or equivalently, $|\Psi_{nS}(0)|^2$, so will rewrite Γ_n, using Eq. (58), as

$$\Gamma_n(e^+e^-)=\Gamma_n^{(0)}(e^+e^-)(1-\Delta_n)(1-\Delta'_n)\ , \tag{69a}$$

where $\Gamma_n^{(0)}$ is the width calculated from the uncorrected Van Royen—Weisskopf formula,

$$\Gamma_n^{(0)}(e^+e^-)=\frac{16\pi\alpha^2 e_q{}^2}{M_n{}^2}|\Psi_{nS}(0)|^2\ . \tag{69b}$$

The Salpeter wave functions $\Psi_W(0)$ and $\Psi_{nS}(0)$ in Eqs. (68) and (69) are calculated for the same instantaneous short-range interaction [plus, in the case of $\Psi_{nS}(0)$, the long-range confining interaction], and are dual by the results of Sec. II and Refs. 13 and 16,

$$\Gamma_n^{(0)}(e^+e^-)\simeq\frac{\alpha^2 e_q{}^2}{\pi}\int_{\Delta W} v\,|\Psi_W(0)|^2 dW\ . \tag{70}$$

Finally, then, the corrections to the bound-state leptonic widths can be identified through duality with the corrections in Eq. (68),

$$\Gamma_n^{(0)}(e^+e^-)\Delta_n\simeq\frac{\alpha^2 e_q{}^2}{\pi}\int_{\Delta W} v\,|\Psi_W(0)|^2\Delta\, dW\ , \tag{71a}$$

$$\Gamma_n^{(0)}(e^+e^-)\Delta'_n\simeq\frac{\alpha^2 e_q{}^2}{\pi}\int_{\Delta W} v\,|\Psi_W(0)|^2\Delta' dW\ . \tag{71b}$$

We next identify Δ and Δ'.

In perturbative QCD, the free cross section for $e^+e^-\to q\bar{q}$ is calculated to be[19]

$$W^2\sigma_{\rm free}=6\pi\alpha^2 e_q{}^2 v(1-\tfrac{1}{3}v^2)\times[1+\tfrac{4}{3}\alpha_s f(v)+\frac{C_2}{\pi^2}\alpha_s{}^2+\cdots]\ . \tag{72}$$

The function $f(v)$ is given to good approximation by[38]

$$f(v)=\frac{\pi}{2v}-\frac{4}{\pi}g(v) \tag{73a}$$

with

$$g(v)\simeq 1+0.046v-v(1-v)^2\ . \tag{73b}$$

The velocity dependence of the coefficient C_2 of the $\alpha_s{}^2$ term is not known, as this correction has only been calculated for massless quarks. The numerical value of C_2 also depends on the renormalization scheme used in the calculation. We will use the value obtained by the modified minimal-subtraction scheme,[19] $C_2=1.98-0.115N_f$, where N_f is the number of quark flavors.

The short-range part of the instantaneous interaction between quarks is just the color-Coulomb interaction ion. Thus the potential is

$$V(r)\approx-\frac{4}{3}\frac{\alpha_s}{r}+V'(r)\ , \tag{74}$$

with α_s the value of the strong-coupling constant at the mass scale of the $q\bar{q}$ system, and $V'(r)$ a smooth confining interaction. $\Psi_W(0)$ is the Salpeter wave function calculated using only the short-range part of the interaction, and could be determined to order $\alpha_s{}^2$ by solving the Salpeter equation [Eq. (39)] for a color-Coulomb interaction.[39] In the absence of such a solution, we will approximate $|\Psi_W(0)|^2$ by the standard Coulomb factor

$$|\psi_{\rm Coulomb}(0)|^2=\frac{4\pi\alpha_s}{3v}\,1-e^{-4\pi\alpha_s/3v})^{-1}=1+\frac{2\pi\alpha_s}{3v}+\frac{1}{3}\left[\frac{2\pi\alpha_s}{3v}\right]^2+\cdots\ . \tag{75}$$

This expression sums the leading terms in powers of α_s/v in the complete QCD result.[2] This choice for $|\Psi_W(0)|^2$ *defines* the correction Δ', which must incorporate such distinctive QCD contributions as the three-gluon term of order $\alpha_s{}^2/v$. We note that $4\pi\alpha_s/3v$ is *not* a small parameter for the ψ and Υ systems, so it is desirable to use the exact form of Eq. (75) rather than the expansion in calculations.

We now examine Eq. (72) term by term to bring it into the form of Eq. (68). The factor $(1-\frac{1}{3}v^2)$ is identified with $(1-\Delta)$ by Eq. (61). The factor $[1+(\pi/2v)\frac{4}{3}\alpha_s]$, from the first term in $f(v)$, is clearly to be identified with the expansion of $|\Psi_W(0)|^2$ in Eq. (75) to first order. The contribution from the rest of $f(v)$ must therefore give the first-order radiative correction Δ'

$$\Delta'\simeq\frac{16\alpha_s}{3\pi}g(v)\ . \tag{76}$$

As may be seen from Fig. 3, $g(v)$ does not vary much for $0\le v\le 1$, ranging in value from $g(0)=1$

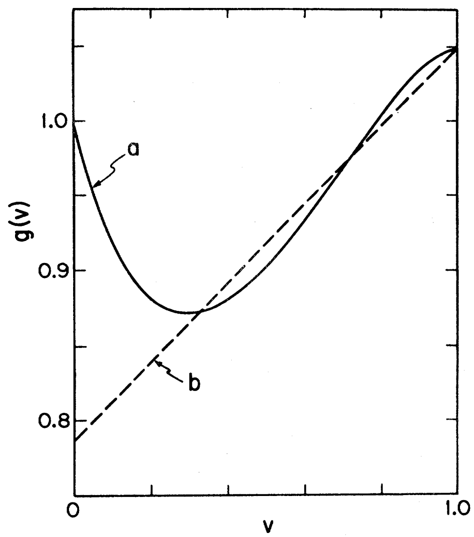

FIG. 3. (a) Plot of the radiative-correction function $g(v)$, Eq. (9) and Ref. 38. Note the suppressed zero on the ordinate. The approximation in Eq. (73b) is barely distinguishable from the exact $g(v)$ on this scale. (b) Schwinger's approximation to $g(v)$, Ref. 38, Eq. (5-4.203).

to a minimum $g(0.3)=0.87$ to a maximum $g(1)=1.05$. As a result, we may approximate the integral in Eq. (71b) by evaluating $g(v)$ at the mass of the resonance in question, and find from Eq. (70) that

$$\Delta_n' \simeq \frac{16\alpha_s}{3\pi} g(v_n) \ . \tag{77}$$

This result is insensitive to the choice of ΔW and the possible corrections to duality, and agrees with the radiative correction obtained in Ref. 20 using the static approximation $v=0$ in bound-state calculations, $\Delta_n'(0)=16\alpha_s/3\pi$. Our result is slightly smaller for the ψ system. Bergström *et al.*[24] obtain a much smaller value of Δ_n' from the authors in Ref. 20. They make a nonrelativistic approximation in part of their calculation but use that result for the entire range of v. The resulting $g(v)$ is negative for $v \gtrsim 0.7$ and gives a much smaller Δ_n' for any $v>0$. They also omit the contributions of $q\bar{q}g$ final states. We do not believe their approximations are valid for heavy-quark systems with large v.[40]

We next estimate the order-α_s^2 contribution to Δ_n'. This has not been calculated previously. Our method is to rewrite the last factor in Eq. (72) as $|\Psi_W(0)|^2$ multiplied by a residual correction factor. Unfortunately, the velocity dependence of C_2 is not known, so we use the $v=1$ limit of Eq. (75) for $|\Psi_W(0)|^2$ to extract the correction. Our result is

$$|\Psi_W(0)|^2(1-\Delta)(1-\Delta') \simeq \left[1+\frac{2\pi\alpha_s}{3v}+\frac{1}{3}\left[\frac{2\pi\alpha_s}{3v}\right]^2\right](1-\tfrac{1}{3}v^2)\left[1-\frac{16\alpha_s}{3\pi}g(v)+\frac{C_2'}{\pi^2}\alpha_s^2\right] , \tag{78a}$$

where

$$C_2'\,|_{v=1}=C_2+22.28 \ . \tag{78b}$$

We then conclude by an argument similar to that above that to order α_s^2

$$\Delta_n' \simeq \frac{16\alpha_s}{3\pi} g(v_n) - (C_2+22.28)\frac{\alpha_s^2}{\pi^2} \tag{79}$$

with $C_2=1.98-0.115\,N_f$.[19] This is a significant correction: for $N_f=5$ and $\alpha_s=0.2$,

$$\Delta_n' \sim 0.340-0.096=0.244 \ . \tag{80}$$

It is interesting to note finally that the radiative corrections do *not* depend to this order on the details of the confining potential $V'(r)$ in Eq. (74), the effects of which appear only in $\Gamma_n^{(0)}$.

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approximation

$$g(v) \simeq \frac{3+v}{4}\left[\frac{\pi}{2}-\frac{3}{4\pi}\right] \simeq \frac{3+v}{4}\frac{\pi}{3}$$

used in many discussions of radiative corrections is good to $\sim 2\%$ for $v > 0.25$, but fails by 20% for $v = 0$.

[39]It is better to use a cutoff Coulomb potential, e.g., the Hulthén potential $V_H = -(4\pi\lambda\alpha_s/3)(e^{\lambda r}-1)$, in this calculation to avoid problems with the infinite number of bound states in the Coulomb potential. The results are changed to order α_s^2 only by a shift in the energy scale (Ref. 37). See Sec. IV C of the first paper of Ref. 16 for a detailed discussion. Any bound states in the cutoff potential must be incorporated individually in the duality relation.

[40]We would like to thank Dr. L. Bergström for comments on these and other points, some of which we will discuss elsewhere.