High-precision evaluation of contributions to $g-2$ of the electron in sixth order

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Five graphs contributing to $g-2$ of the electron in sixth order are evaluated to high precision. With these results, the dominant error in the theoretical prediction for $g-2$ arises from the eighth-order uncertainty. Updated comparisons of theory and experiment are given. The techniques employed are briefly characterized.

I. INTRODUCTION

Continued improvement in the experimental precision in determining the g factor of the electron requires further refinement of the theoretical predictions. The predictions to order α and α^2 are known analytically.[1] In order α^3, 27(46) of the 40(72) Feynman graphs (the number in parentheses refers to a graph counting scheme in which a graph and its mirror image are distinguished) have been evaluated analytically[2]; 3(5) more have been reduced to one-dimensional integrals[3] and are known to eight decimal places; another 2(6) (the light-by-light) graphs[4] have been evaluated to better than three decimal places. We report here a careful evaluation of 5(10) of the 8(15) remaining graphs.

In the next section, we give the results and the implications for $g-2$. Using Kinoshita's preliminary estimate[5] for the eighth-order result we show that theory and experiment are in reasonably good agreement. The greatest source of theoretical error is now from the eighth order.

In Sec. III, we outline the technique used. It is a hybrid of analytical and numerical methods. By a combination of hyperspherical and dispersive techniques, we reduce the evaluation of each graph to a three-dimensional integral, which is then evaluated numerically. The usual Feynman parameter techniques require the evaluation of a seven-dimensional integral. To our dismay, the round-off problems in our three-dimensional approach were very severe and necessitated the use of multiple-precision floating-point techniques designed by one of us (MJL). These multiple-precision evaluations are quite slow compared to hardware double precision evaluations, and so our running times were quite long.

II. RESULTS

The graphs we evaluated are shown in Fig. 1 and the results are given in Table I, together with a comparison with previous evaluations. We can see that the error has been reduced by a factor of 30 compared to the most precise previous evaluation, and that our result is slightly more than one standard deviation below that estimate. If we write

$$\frac{g-2}{2}=a_2\frac{\alpha}{\pi}+a_4\left[\frac{\alpha}{\pi}\right]^2+a_6\left[\frac{\alpha}{\pi}\right]^3+a_8\left[\frac{\alpha}{\pi}\right]^4+\cdots$$

$$+\text{ weak-interaction effects}$$

$$+\text{ strong-interaction effects}$$

$$+\text{ effects due to }\mu\text{'s, }\tau\text{'s},\ldots,$$

we know that[1]

$$a_2=\tfrac{1}{2}\,,$$

$$a_4=\frac{197}{144}+\frac{\pi^2}{12}-\frac{\pi^2\ln 2}{2}+\frac{3\zeta(3)}{4}$$

$$=-0.328\,479\ldots.$$

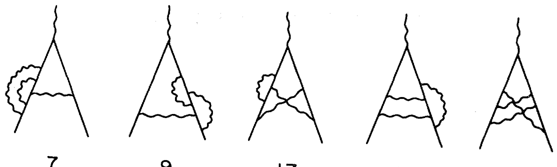

FIG. 1. The Feynman graphs for which new values are reported in this paper. Graph numbers follow Ref. 6.

Combining our results with the previously quoted results, and a numerical evaluation[8] of the 3(5) remaining graphs, we find

$$a_6 = 1.1765 \pm 0.0013 ,$$

where the error is essentially all due to the graphs of Ref. 8. If we combine this result with Kinoshita's preliminary estimate[5]

$$a_8 = -0.8 \pm 2.5$$

and using the value[9]

$$\alpha^{-1} = 137.035\,963 \pm 0.000\,015 ,$$

we find

$$\left[\frac{g-2}{2}\right]_{\text{theory}} = (1\,159\,652\,455 \pm 127 \pm 17 \pm 73) \times 10^{-12} ,$$

where the errors come from the errors in α, a_6, and a_8, respectively. (Weak- and strong-interaction effects, and the contribution of μ's and τ's, amount to less than 5×10^{-12}.)

The best experimental value for $(g-2)/2$ is[10]

$$\left[\frac{g-2}{2}\right]_{\text{exp}} = (1\,159\,652\,200 \pm 40) \times 10^{-12} .$$

Thus theory and experiment are in reasonable agreement. Earlier discrepancies between theory and experiment have been reduced in two ways. First, our new value for the 5(10) graphs is smaller (by slightly more than one standard deviation) than the most accurate previous estimate. Secondly, Kinoshita's estimate for a_8 is negative. Both these effects operate in the same direction and reduce the disagreement between theory and experiment.

Another way of reporting the results is to assume the validity of QED, and to estimate α from the experimental value of $g-2$ and the theoretical calculations. The result is

$$\alpha^{-1} = 137.035\,993 \pm 0.000\,005 \pm 0.000\,002 \pm 0.000\,009 ,$$

where the errors come from experiment, a_6, and a_8, respectively. This estimate for α^{-1} agrees roughly with that used above.

III. TECHNIQUE

To illustrate the technique, we concentrate on one of the graphs in Fig. 1, for example graph 9. The momenta are chosen as in Fig. 2. The particular routing is explained below. One writes the usual Feynman rules for the amplitude and projects out the F_2 form factor.[2] Since the magnetic moment is the coefficient of the linear term in q in the overall vertex function, it suffices to keep only terms linear in q. That is, we expand a term like

$$\frac{1}{(p-m-q/2)^2} \approx \frac{1}{(p-m)^2 - q\cdot(p-m)} \approx \frac{1}{(p-m)^2}\left[1 + \frac{q\cdot(p-m)}{(p-m)^2}\right] . \tag{3.1}$$

TABLE I. Contributions of the graph and its mirror image to a_6, in the Feynman gauge. Infrared divergences proportional to $\log\lambda$ or $\log^2\lambda$ have been omitted.

Graph No.	7	9	17	19	27	Total
Our new values	−2.670 546 ±0.000 030	0.617 727 ±0.000 121	0.607 660 ±0.000 240	−0.334 698 ±0.000 011	1.861 992 ±0.000 240	0.082 065 ±0.000 362
Previous evaluations						
Levine and Wright (Ref. 6)	−2.664 ±0.020	0.625 ±0.006	0.613 ±0.013	−0.330 ±0.013	1.854 ±0.013	0.097 0 ±0.023 0
Cvitanovic and Kinoshita (Ref. 7)	−2.670 7 ±0.001 9	0.618 9 ±0.006 4	0.609 7 ±0.003 4	−0.318 2 ±0.007 2	1.857 2 ±0.008 6	0.089 3 ±0.006 0

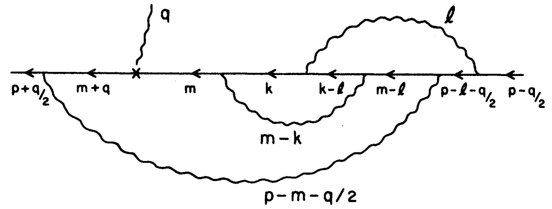

FIG. 2. Momentum routing chosen for graph 9.

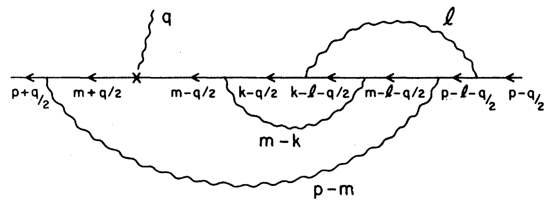

FIG. 3. Bad choice of momentum routing, necessitating evaluation of many more discontinuities.

After projecting out the integrand contributing to $(g-2)$, we isolate the terms dependent on k (the inner loop). For each of these, we write a dispersive integral of the form[11]

$$\int \frac{d^4k f(k^2,k\cdot l,k\cdot m)}{(k^2-1)[(k-l)^2-1](m-k)^2} = \frac{1}{\pi}\int_1^\infty \frac{dA\rho(A,l^2,m^2)}{A-(l-m)^2} \qquad (3.2)$$

where $f(k^2, k\cdot l, k\cdot m)$ is a polynomial in its arguments. f is not more than quadratic in k. (This reflects the choice of momentum routings. If we had chosen the routing of Fig. 3, f could have had up to quartic terms in k, and several of the denominators on the left side of (3.2) would have been squared.) Appropriate ultraviolet counterterms are subtracted, as well as infrared pieces which render the entire integral finite.

Having inserted the dispersion integral into the entire expression, it is straightforward to continue analytically to Euclidean space and to use the hyperspherical techniques[2] to perform the remaining angular integrations over the directions of l and m. What remains is a threefold integral over A, L, and M where L and M are the magnitudes of the (Euclidean) vectors l and m. The integrand involves rational functions, square roots, logarithms, and Spence functions of its arguments.

We could not see how to perform any of the remaining integrations analytically. So we resorted to numerical integration. We wrote the integrand as a sum of terms, where each term was a product of a function of L^2 and M^2, times one of the discontinuities which occurred from the right side of (3.2), times a factor occurring from performing the angular integrations in L and M.

To give an idea of the size of the final integrand, evaluating it at one point required 1/45 sec in double precision on the PDP-10 at the University of Pittsburgh. Thus to do the integration at $40\times40\times40$ points requires less than 25 min, a very manageable time. Unfortunately, we ran into severe round-off problems and had to resort to multiple-precision routines which one of us (MJL) wrote for running on a VAX 11/780. The arithmetic precision is automatically varied across the integration space to guarantee negligible round-off error in the final results.

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[11]See, for example, Appendix 2 of Ref. 3.