Relativistic confinement in a Bethe-Salpeter model

S. N. Biswas, K. Datta, and Ashok Goyal
Department of Physics and Astrophysics, University of Delhi, Delhi-110007, India
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Starting with a confinement kernel originally due to Henley, we use the instantaneous approximation to reduce the Bethe-Salpeter equation for bosons to an ordinary second-order differential equation. In the nonrelativistic limit the equation reduces to that for a spherically symmetric simple harmonic oscillator. For the relativistic equation, we obtain an asymptotic series for the energy levels of the bound states valid in the strong-coupling limit. Our results are in qualitative agreement with the numerical results for a similar system studied by Henley.

I. INTRODUCTION

Quantum chromodynamics as a theory of strong interactions has had some significant successes and has been the starting point of numerous investigations on the structure and interactions of hadrons. It is now widely believed that hadrons are "constituted" of more fundamental structures, the quarks, which are confined in the simplest free hadronic systems by strong confining forces described by potentials which increase with the spatial separation of the quarks. While the origin of these confining forces must lie in the nature of the nonlinear quark-gluon and gluon-gluon couplings within the framework of quantum chromodynamics, this picture of hadrons has also motivated quasiphenomenological investigations on the energy levels, level patterns, wave functions, etc., of quark-quark or quark-antiquark systems bound by suitably parametrized phenomenological two-body confining potentials.[1] Simultaneously, there are attempts to relate the parameters characterizing these phenomenological potentials to the fundamental masses and couplings of the theory.[2]

Most calculations of bound-state characteristics of confined two-body systems use the nonrelativistic Schrödinger equation.[3-5] This is not only on account of the simplicity of the formalism: there may be reasons for believing that even though the hadronic processes of interest take place at relativistic energies, the description of the constituent quarks as being in nonrelativistic motion in the confining potential is not devoid of meaning.[6] Nevertheless, quantum chromodynamics being a relativistic field theory of quarks and gluons, the study of confined quark-quark systems in the relativistic domain is of considerable fundamental interest.[7-9] In this work, we present a study of some features of the confined two-body system in the framework of the relativistic two-body bound-state equation. While the Bethe-Salpeter (BS) equation, being an invariant two-body equation may seem an appropriate formalism for the study of the relativistic quark-quark confined system, its use in such a context has long been known to present several difficulties. For a given Lagrangian, such as the one for quantum chromodynamics, the appropriate kernel to use in the integrodifferential BS equation is far from transparent. Even if this question is settled by choosing a kernel on quasiphenomenological grounds, one cannot, for kernels which are otherwise appropriate, reduce the equation to a one-dimensional one. This difficulty is, in turn, connected with that of assigning a meaning to the two-time BS amplitude depending on a relative time variable with no analog in the time-dependent Schrödinger amplitude.

One approach to resolve this difficulty was suggested in the pioneering work of Goldstein and others.[10,11] They showed that this difficulty with the BS equation in the ladder approximation with a kernel corresponding to a massive scalar exchange arises from the high degree of singularity on this relative light cone, which results in a continuum spectrum for the bound-state problem. On the other hand, a point spectrum results if this equation is solved for an interaction which is instantaneous in the c.m. frame.[12] This approximation of the kernel on the relative light cone decreases the degree of singularity. However, even in this approximation the equation can be solved exactly only when the binding energy equals the rest mass of the interacting particles.

In this work, we use the quasiphenomenological approach to choose the kernel for the quark-quark BS equation. Starting from the kernel corresponding to a massive scalar exchange, Henley[13] has recently obtained a sequence of kernels which, in the static limit, have the structure of derivatives of δ functions and which may be regarded as relativistic generalizations of the harmonic oscillator. We use the simplest such kernel proposed by Henley in a scalar-scalar BS equation and adopting the procedure followed by Goldstein study this equation in the instantaneous approximation. The resulting equation is no longer relativistically covariant; relativistic effects are embodied only in the kinematic sense and arise from the fact that the equation used is the Bethe-Salpeter rather than the Schrödinger equation. In the nonrelativistic limit the resulting differential equation reduces to that for a harmonic oscillator; a similar result was also obtained by Henley. In the relativistic case, the differential equation is not exactly soluble: however, using asymptotic formulas due to Titchmarsh,[14] we can study analytically the behavior of the bound-state spectrum in the strong-coupling limit. In Sec. II we set up the BS equation in the instantaneous approximation and in Sec. III reduce this integrodifferential equation to a differential equation by a suitable choice of the kernel. In Sec. IV we study the equation in the nonrelativistic limit whereas in Sec. V we study the equation in the relativistic case and obtain an asymptotic series for the energy eigenvalues valid in the strong-coupling limit. Section VI is devoted to the discussion of our results.

II. THE BOUND-STATE BETHE-SALPETER EQUATION IN THE INSTANTANEOUS APPROXIMATION

We start with the BS equation for the bound state of two scalar particles each of mass m written in momentum space in the form

$$[(p+\tfrac{1}{2}P)^2+m^2][(p-\tfrac{1}{2}P)^2+m^2]\psi(p) = -\frac{ig^2}{\pi^2}\int d^4p' V(p-p')\psi(p') . \quad (1)$$

p is the relative four-momentum of the particles which form the bound state, P is the sum of these momenta, and we use the metric $(-1,1,1,1)$. In writing the equation in this form we assume that the kernel is a function only of the differences of relative momenta: such as, for instance, the situation which obtains when the equation is studied in the ladder approximation for an interaction generated by the exchange of a single scalar particle. The instantaneous approximation consists of replacing the Lorentz-invariant (and hence retarded) Yukawa interaction by one which is instantaneous in the center-of-mass frame. In this frame $P=(E,0)$, E being the total energy of the bound state. In momentum space, this is equivalent to dropping the timelike component of the four-vector $(p-p')$ and replacing $V(p-p')$ by $(2\pi)^{-1}V(\vec{p}-\vec{p}\,')$. In this approximation, (1) reduces to

$$[(\vec{p}^{\,2}+m^2)-(p_0+\tfrac{1}{2}E)^2][(\vec{p}^{\,2}+m^2)-(p_0-\tfrac{1}{2}E)^2]\psi(p) = -\frac{ig^2}{2\pi^3}\int d^4p' V(\vec{p}-\vec{p}\,')\psi(p') . \quad (2)$$

Since the right-hand side of (2) is a function of $\vec{p}$ alone and not of p_0, we define

$$S(\vec{p}) \equiv [(\vec{p}^{\,2}+m^2)-(p_0+\tfrac{1}{2}E)^2] \times[(\vec{p}^{\,2}+m^2)-(p_0-\tfrac{1}{2}E)^2]\psi(p) , \quad (3)$$

which satisfies the homogeneous integral equation

$$S(\vec{p}) = -\frac{ig^2}{2\pi^3}\int d^4p' \frac{S(\vec{p}\,')V(\vec{p}-\vec{p}\,')}{D(p')} , \quad (4)$$

where

$$D(p) \equiv [(\vec{p}^{\,2}+m^2)-(p_0+\tfrac{1}{2}E)^2] \times[(\vec{p}^{\,2}+m^2)-(p_0-\tfrac{1}{2}E)^2] , \quad (5)$$

and it is understood that the squared masses carry negative imaginary parts. The integration over the relative energy variable in (4) is easily performed and we finally obtain the integral equation for the bound state in the instantaneous approximation

$$S(\vec{p}) = \frac{g^2}{(2\pi)^2}\int d\vec{p}\,' \frac{V(\vec{p}-\vec{p}\,')S(\vec{p}\,')}{(\vec{p}\,'^2+m^2)^{1/2}(\vec{p}\,'^2+m^2-\tfrac{1}{4}E^2)} . \quad (6)$$

The equation is no longer a relativistically invariant one owing to the use of the instantaneous approximation: however, it embodies the effects of relativistic kinematics in a fashion which will be manifest once a choice is made for the kernel $V(\vec{p}-\vec{p}\,')$ and a differential equation corresponding to (6) is obtained.

III. CHOICE OF THE KERNEL AND THE DIFFERENTIAL EQUATION

In choosing our kernel function $V(\vec{p}-\vec{p}\,')$ we are guided by the following considerations. A confining potential, rising asymptotically in configuration space, does not result from the static limit of the kernel for a massive scalar exchange: one only obtains the screened Coulomb or Yukawa interaction of the form $e^{-\mu r}/r$, μ being the exchanged mass. However, higher derivatives of this potential with respect to the exchanged mass soften this singularity at the origin and produce interactions rising asymptotically and hence are confining in character. For instance, the third derivative in the zero-exchanged-mass limit will have an oscillatorlike confining character. We are thus led, after Henley,[13] to consider for our kernel the instantaneous limit of

$$V(p-p') = \lim_{\mu\to 0}\frac{\partial^3}{\partial\mu^3}\left[\frac{1}{(p-p')^2+\mu^2-i\epsilon}\right], \quad (7)$$

i.e.,

$$V(\vec{p}-\vec{p}\,') = \lim_{\mu\to 0}\frac{24\mu[(\vec{p}-\vec{p}\,')^2-\mu^2]}{[(\vec{p}-\vec{p}\,')^2+\mu^2]^4} . \quad (8)$$

If we further note the identity

$$\lim_{\mu\to 0}\frac{12\mu}{\pi^2}\frac{[(\vec{p}-\vec{p}\,')^2-\mu^2]}{[(\vec{p}-\vec{p}\,')^2+\mu^2]^4} = \Delta_p\delta^{(3)}(\vec{p}-\vec{p}\,') , \quad (9)$$

the integral equation (6) can easily be converted into a differential equation. Defining

$$\chi(\vec{p}) \equiv \frac{S(\vec{p})}{(\vec{p}^2+m^2)^{1/2}(\vec{p}^2+m^2-\frac{1}{4}E^2)} \tag{10}$$

and using (8) and (9), we obtain for $\chi(\vec{p})$ the differential equation

$$\Delta_p\chi(\vec{p}) = \frac{2}{g^2}(\vec{p}^2+m^2)^{1/2}(\vec{p}^2+m^2-\tfrac{1}{4}E^2)\chi(\vec{p}) \,. \tag{11}$$

IV. NONRELATIVISTIC LIMIT: OSCILLATOR LEVELS

Separating (11) into radial and angular parts, we have

$$\chi(\vec{p}) = \frac{\chi_l(p)}{p} Y_{lm}(\Omega_{\vec{p}}) \,, \quad p = |\vec{p}|$$

where $\chi_l(p)$ satisfies the differential equation

$$\frac{d^2\chi_l}{dp^2} - \frac{l(l+1)}{p^2}\chi_l = \frac{2}{g^2}(p^2+m^2)^{1/2}(p^2+m^2-\tfrac{1}{4}E^2)\chi_l \,. \tag{12}$$

Introducing the dimensionless coupling constant

$$\lambda = (g^2/m^5)^{1/2} \,, \tag{13}$$

we have

$$\frac{d^2\chi_l}{dk^2} - \frac{l(l+1)}{k^2}\chi_l = \frac{2}{\lambda^2}(1+k^2)^{1/2}(k^2-\gamma^2)\chi_l \,, \tag{14}$$

where

$$k = p/m \,, \quad \gamma^2 = -1 + \frac{E^2}{4m^2} \,. \tag{15}$$

To examine (14) in the nonrelativistic limit, we expand the coefficients on the right-hand side of (14) in a power series in k^2. In this limit (14) reduces to

$$\frac{d^2\chi_l}{dk^2} - (ak^2+bk^4+ck^6+\cdots)\chi_l - \frac{l(l+1)}{k^2}\chi_l + \frac{2}{\lambda^2}\gamma^2\chi_l = 0 \tag{16}$$

with

$$a = \frac{2}{\lambda^2}\left(1-\frac{\gamma^2}{2}\right),$$
$$b = \frac{2}{\lambda^2}\left(\frac{1}{2}+\frac{\gamma^2}{8}\right), \tag{17}$$
$$c = -\frac{2}{\lambda^2}\left(\frac{1}{8}+\frac{\gamma^2}{16}\right).$$

The purely nonrelativistic limit of (16) is obtained by neglecting b and c. Since we have a confining potential which rises asymptotically, we expect a positive-energy confined state, i.e., $\gamma^2>0$. The eigenvalues of the equation

$$\frac{d^2\chi_l}{dk^2} + \left(\frac{2\gamma^2}{\lambda^2} - ak^2 - \frac{l(l+1)}{k^2}\right)\chi_l = 0 \tag{18}$$

are given by

$$\frac{\gamma^2}{\lambda^2 a^{1/2}} = 2n_r + l + \tfrac{3}{2} = n + \tfrac{3}{2} \,. \tag{19}$$

If B is the binding energy of the system (>0), i.e., $E \simeq 2m+B$, one obtains $\gamma^2 \simeq B/M$. For the binding energy B, one has the energy spectrum

$$B \simeq \sqrt{2}\,\lambda(n+\tfrac{3}{2}) \,,$$

i.e., (20)

$$E \simeq 2m + \sqrt{2}\,\lambda(n+\tfrac{3}{2}) \,.$$

V. RELATIVISTIC LIMIT

A. Asymptotic momentum dependence of the relativistic wave function

In the extreme relativistic limit ($k \gg 1$) Eq. (14) may be solved exactly to yield the asymptotic momentum dependence of the wave functions for the relativistic system.

From (14) we have, for $k \gg 1$,

$$\frac{d^2\chi_l}{dk^2} \simeq \frac{2}{\lambda^2}k^3\chi_l \,. \tag{21}$$

The solutions to (21) are well known to be modified Bessel functions. Indeed, if $\alpha \equiv 2\sqrt{2}/5\lambda$, we may write

$$\chi_l \underset{k\gg1}{\simeq} A_1\sqrt{k}\,I_{1/5}(\alpha k^{5/2}) + B_1\sqrt{k}\,I_{-1/5}(\alpha k^{5/2}) \,. \tag{22}$$

For a suitable choice of the constants A_1 and B_1, χ_l is asymptotically damped in momentum space: $(I_{1/5}-I_{-1/5}) \sim K_{1/5}(\alpha k^{5/2})$ the modified Bessel function of the second kind has the asymptotic dependence $e^{-\alpha k^{5/2}}$. The relativistic system has strongly damped wave functions which, in momentum space, decay faster than the wave functions for the nonrelativistic oscillator.

B. Asymptotic series for the energy levels in the strong-coupling limit

While we cannot solve (14) exactly except in the nonrelativistic and extreme relativistic limits, we obtain an asymptotic series for the energy levels of this system,[15] valid in the large-coupling limit. Titchmarsh has shown that for the eigenvalue problem characterized by the equation

$$\frac{d^2\chi}{dr^2} + \left(\mu - \frac{(l^2-\frac{1}{4})}{r^2} - q(r)\right)\chi = 0 \,, \quad 0 \le r < \infty \tag{23}$$

with $q(r) \to \infty$ as $r \to \infty$, the eigenvalues $\mu_{n,l}$ are given by the approximate WKB-type formulas:

$$\frac{1}{\pi}\int_0^{R_n}[\mu_{n,l} - q(r)]^{1/2}dr = n + \tfrac{1}{2}l + \tfrac{1}{2} + O\left(\frac{1}{n}\right), \tag{24}$$

where R_n is a positive root of $\mu_{n,l}=q(r)$. To estimate the leading asymptotic dependence of γ^2 on λ in the limit of large λ, we notice that the equation corresponding to (14) with $m=0$, viz.,

$$-\frac{d^2\chi_l}{dk^2}+\left(\frac{2}{\lambda^2}k(k^2-\gamma^2)+\frac{l(l+1)}{k^2}\right)\chi_l=0\,,$$

may, with the scaling transformation

$$k=\lambda^{2/5}\xi\,,\quad \gamma=\lambda^{2/5}\mu\,, \tag{25}$$

be transformed into

$$-\frac{d^2\chi_l}{d\xi^2}+\left(2\xi(\xi^2-\mu^2)+\frac{l(l+1)}{\xi^2}\right)\chi_l=0\,. \tag{26}$$

The leading asymptotic behavior of γ^2 in the limit of large λ is thus $O(\lambda^{4/5})$. Extracting this leading asymptotic behavior by scaling k and γ as in (25), we obtain from (14) the equation

$$-\frac{d^2\chi_l}{d\xi^2}+\left(2(\xi^2-\mu^2)(\xi^2+\lambda^{-4/5})^{1/2}+\frac{l(l+1)}{\xi^2}\right)\chi_l=0\,. \tag{27}$$

Comparing with (23) and (24), we may therefore relate the energy levels $\mu_{n,l}$ of this system by the approximate formula

$$\frac{\sqrt{2}\,\mu_{n,l}{}^{5/2}}{\pi}\int_0^1\left[(1-\eta^2)\left(\eta^2+\frac{1}{\mu_{n,l}{}^2\lambda^{4/5}}\right)^{1/2}\right]^{1/2}d\eta =n+\tfrac{1}{2}l+\tfrac{1}{2}\,, \tag{28}$$

where

$$\eta=\xi/\mu_{n,l}\,.$$

In the lowest approximation, then,

$$\frac{\sqrt{2}\mu_{n,l}{}^{5/2}}{\pi}\int_0^1\eta^{1/2}(1-\eta^2)^{1/2}d\eta\simeq n+\tfrac{1}{2}l+\tfrac{1}{2}\,,$$

i.e.,

$$\mu_{n,l}{}^2=\left[\frac{\pi(2n+l+1)}{\sqrt{2}\,\beta(\frac{3}{4},\frac{3}{2})}\right]^{4/5}\equiv\bar{n}^{4/5}\,. \tag{29}$$

Therefore,

$$\frac{E_{n,l}{}^2}{4m^2}-1=\lambda^{4/5}\left[\frac{\pi(2n+l+1)}{\sqrt{2}\,\beta(\frac{3}{4},\frac{3}{2})}\right]^{4/5}=(\bar{n}\lambda)^{4/5}\,. \tag{29$'$}$$

From the structure of Eq. (28) we may, however, construct an asymptotic series for $\mu_{n,l}$ in the large-λ limit. Consider the integral

$$J(\alpha)\equiv\int_0^1 d\eta[(1-\eta^2)(\eta^2+\alpha^2)^{1/2}]^{1/2}\,. \tag{30}$$

We wish to obtain an expression for $J(\alpha)$ which is asymptotic to $J(\alpha)$ in the limit $\alpha\to 0$. We now use the following result on the asymptotic expansion of integrals. If

$$I(x)=\int_a^b f(t,x)dt$$

and if $f(t,x)$ possesses the asymptotic expansion

$$f(t,x)\sim\sum_{m=0}^{\infty}f_m(t)(x-x_0)^{\delta m}\quad(x\to x_\nu)$$

for some $\delta>0$, uniformly for $a\le t\le b$, then

$$I(x)\sim\sum_{m=0}^{\infty}(x-x_\nu)^{\delta m}\int_a^b f_m(t)dt\,,$$

provided that all the terms on the right-hand side are finite. Consider now $J(\alpha)$ as defined in (30). We write

$$J(\alpha)=\int_\alpha^1 d\eta[(1-\eta^2)(\eta^2+\alpha^2)^{1/2}]^{1/2} +\int_0^\alpha d\eta[(1-\eta^2)(\eta^2+\alpha^2)^{1/2}]^{1/2}\,.$$

In the first term, for the range $0\le\alpha<\eta\le 1$, we have for $(\eta^2+\alpha^2)^{1/4}$ a convergent expansion in powers of α^2/η^2. This expansion, being convergent, is necessarily asymptotic to $(\eta^2+\alpha^2)^{1/4}$. Hence we can write

$$J(\alpha)=\sum_{m=0}^{\infty}\int_\alpha^1 d\eta\,\eta^{1/2}(1-\eta^2)^{1/2}A_m\frac{\alpha^{2m}}{\eta^{2m}} +\int_0^\alpha d\eta[(1-\eta^2)(\eta^2+\alpha^2)^{1/2}]^{1/2}\,,$$

the coefficients A_m being the usual Taylor-Maclaurin expansion coefficients of $(1+\alpha^2/\eta^2)^{1/4}$ at $\alpha=0$. Consider next the limit $\alpha\to 0^+$. Since in this limit the integral which constitutes the second term goes uniformly to zero, and since the integrals $\int_0^1 d\eta\,\eta^{1/2}(1-\eta^2)^{1/2}\eta^{-2m}$ are all finite, we obtain for $J(\alpha)$ the asymptotic expansion

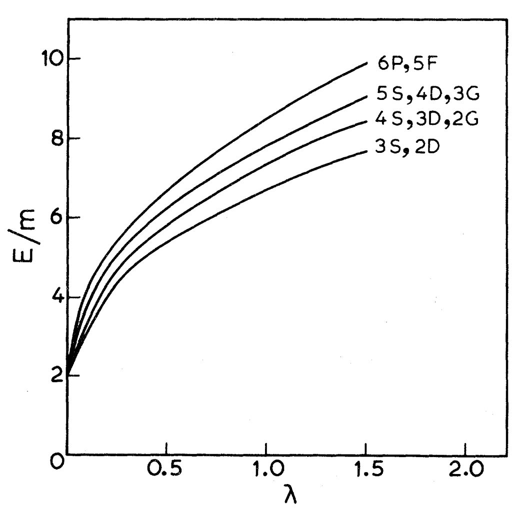

FIG. 1. Energy levels as a function of the coupling constant in the relativistic limit.

$$J(\alpha) \underset{\alpha\to 0^+}{\sim} \sum_{m=0}^{\infty} A_m \alpha^{2m} \int_0^1 \eta^{-2m+1/2}(1-\eta^2)^{1/2} d\eta\,. \quad (31)$$

Using (31) in (28) we obtain for $\mu_{n,l}$ the asymptotic series valid in the large-λ limit:

$$\mu_{n,l}{}^{5/2}\left[\beta(\tfrac{3}{4},\tfrac{3}{2}) + \frac{\beta(-\frac{1}{4},\frac{3}{2})}{4\mu_{n,l}{}^2\lambda^{4/5}} - \frac{3\beta(-\frac{5}{4},\frac{3}{2})}{32\mu_{n,l}{}^4\lambda^{8/5}} + \cdots\right] = (2n+l+1)\frac{\pi}{\sqrt{2}}\,, \quad (32)$$

where the β functions with negative arguments are the usual analytic continuations of the β functions. We solve equation (32) iteratively for $\mu_{n,l}$. Thus if we keep terms of $O(\lambda^{-4/5})$ on the left-hand side, we substitute for $\mu_{n,l}$ in this term the solution obtained for $\mu_{n,l}$ by retaining the lower-order terms in $\lambda^{-4/5}$, i.e., (29). In this way, we obtain for $\mu_{n,l}$ an asymptotic expansion for large λ in powers of $\lambda^{-4/5}$; an iteration which retains terms $O(\lambda^{-4k/5})$ on the left-hand side in (32) does not alter the expansion for $\mu_{n,l}$ up to terms $O(\lambda^{-4(k-1)/5})$. We obtain

$$\mu_{n,l}{}^2 \simeq (\bar{n}\lambda)^{4/5}\left\{1 - \frac{\beta(-\frac{1}{4},\frac{3}{2})}{5\beta(\frac{3}{4},\frac{3}{2})}(\bar{n}\lambda)^{-4/5} + (\bar{n}\lambda)^{-8/5}\left[\frac{3}{40}\frac{\beta(-\frac{5}{4},\frac{3}{2})}{\beta(\frac{3}{4},\frac{3}{2})} + \frac{1}{200}\left(\frac{\beta(-\frac{1}{4},\frac{3}{2})}{\beta(\frac{3}{4},\frac{3}{2})}\right)^2\right] + \cdots\right\}, \quad (33)$$

which gives

$$\frac{E}{M} \simeq 2.799[(2n+l+1)\lambda]^{2/5}\{1 + 0.5105[(2n+l+1)\lambda]^{-4/5} - 0.1042[(2n+l+1)\lambda]^{-8/5} + \cdots\}. \quad (34)$$

In Fig. 1 we plot E/M as a function of λ for several energy levels.

IV. DISCUSSION

We have seen that the Bethe-Salpeter equation for two bosons, when studied in the instantaneous approximation, reduces to an ordinary second-order differential equation for a suitably chosen kernel. The kernel chosen has, in momentum space and in the static limit, the structure of the second derivative of a δ function. It can be regarded as the relativistic generalization of a harmonic-oscillator potential and hence may be expected to be strongly confining in character. Indeed, the resulting differential equation reduces in the nonrelativistic limit to the equation for a spherically symmetric simple harmonic oscillator. The relativistic equation is not exactly soluble but we obtain analytically an asymptotic series for the energy levels of the bound states, valid in the strong-coupling limit.

Comparing Eqs. (20) and (34), we find that the energy levels of the relativistic system are lower than the corresponding values obtained in the nonrelativistic approximation. This is also reflected in the wave functions of the relativistic system which, in momentum space, are more confined than the wave functions of the nonrelativistic system. The relativistic energy levels are no longer equally spaced; however, the oscillatorlike degeneracy in $(2n+l)$ seems to persist. We note that this degeneracy seems to persist because of the use of approximate WKB-type formulas; inclusion of $O(1/n)$ terms in the Titchmarsh formulas would lift this degeneracy. We have not included these terms since we have no analytic estimate of their magnitude. However, our results are in qualitative agreement with those obtained by Henley by numerical integration of a similar differential equation for a relativistic two-boson system.

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