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Construction of non-Abelian solutions to the classical Yang-Mills equations with sources

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The SU(2) Yang-Mills equations with static external sources are considered. They are rewritten in a suitable form so as to permit easy construction of the magnetic multipole solutions. It is shown that under certain conditions magnetic multipole solutions can lead to non-Abelian Coulomb solutions.

I. INTRODUCTION

Recently much attention has been paid to the study of classical solutions of the Yang-Mills (YM) field equations in the presence of external sources.[1-9] The hope is that analysis of the classical problems may provide insights into the structure of the corresponding quantum theory, particularly with respect to the nonperturbative property.

There are essentially two classes of classical solutions: (A) solutions with sources of arbitrary strength and (B) solutions with sources of critical strength.[5] The first example of class (B) solutions was provided by Sikivie and Weiss[2] and is known as the magnetic dipole solution, since for this solution the non-Abelian magnetic field possesses the long-range dipole behavior. The magnetic dipole solution can in fact be generalized so that one obtains magnetic multipole solutions.[2,7] Recently numerical solutions were presented by Jackiw, Jacobs, and Rebbi[4] for both class (A) and (B) solutions. Their class (A) solution, in a suitable gauge frame, appears similar to the Abelian Coulomb solution and is hence known as the non-Abelian Coulomb solution or the type-I solution. Their class (B) solution possesses two branches when the external source strength exceeds a certain critical value and is hence known as the bifurcating solution or the type-II solution. Generalization to the SU(3) gauge field configurations has also been performed numerically in Ref. 6. In Ref. 8, explicit closed-form expressions for the non-Abelian Coulomb solution and the type-II solution are given. On comparing magnetic multipole solutions in Refs. 2 and 7 and the non-Abelian Coulomb solutions in Ref. 8, one cannot help feeling that there exists a relation between the two kinds of solutions so that given a multipole solution, one can readily write down a corresponding non-Abelian Coulomb solution.

In this paper, we rewrite the equations in Ref. 2 in a suitable form so that explicit magnetic multipole solutions can be more easily constructed. We then establish a relation between magnetic multipole solutions and non-Abelian Coulomb solutions so that by virture of it the latter can be obtained from the former. However, the converse is not true. The relation proves useful because magnetic multipole solutions are easier to obtain than the non-Abelian Coulomb solutions. Finally we illustrate our results by constructing an explicit non-Abelian Coulomb solution and end with some remarks.

II. YANG-MILLS EQUATIONS WITH EXTERNAL SOURCES

The SU(2) YM equations in the presence of an external source are

$$(D_\mu F^{\mu\nu})_a = j_a^\nu = \delta_0^\nu \rho_a\,, \tag{1a}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c\,, \tag{1b}$$

and our metric is $g_{ii} = -g_{00} = 1$. Here ρ_a is the external charge density and g the coupling constant. We shall only consider solutions with finite total energy and finite total external charge.

Substituting the following ansatz[2] into the YM equations,

$$A_0^a = \delta_3^a \phi(\rho, x_3)/g\,, \tag{2a}$$

$$A_i^a = \delta_1^a \epsilon_{i3j} \frac{x_j}{g\rho} A(\rho, x_3)\,, \tag{2b}$$

$$\rho_a = \delta_a^3 q/g\,, \tag{2c}$$

where $\rho^2 = x_1^2 + x_2^2$, one obtains two coupled nonlinear equations for the functions $\phi(\rho, x_3)$ and $A(\rho, x_3)$,

$$-\nabla^2\phi + A^2\phi = q\,, \tag{3a}$$

$$\nabla^2 A - \frac{1}{\rho^2} A + \phi^2 A = 0\,. \tag{3b}$$

An expression for $A(\rho, x_3)$ can be constructed by first linearizing Eq. (3b) to

$$\nabla^2 \mathcal{A} - \frac{1}{\rho^2} \mathcal{A} = 0\,, \tag{4}$$

and assuming that $A(\rho, x_3)$ tends to $\mathcal{A}$ at large distances in such a way so as to ensure finite total energy and finite total charge. One then derives $\phi(\rho, x_3)$ and q, respectively, from Eqs.

(3b) and (3a).

The expression for the function $A(\rho, x_3)$ can be written as[2,10]

$$A(\rho, x_3) = c r_0 \frac{\rho}{r^3} \bar{a}(y), \quad y = r/r_0. \tag{5}$$

Here the constant c is the norm of $A(\rho, x_3)$ and the parameter r_0 indicates the size of the external charge distribution. The function $\bar{a}(y)$ can actually also depend on the spherical coordinate θ but for our purpose here we shall restrict it to be a function of y only. From Eqs. (3) and (5), one gets

$$-(\phi'' + 2\phi'/y) + \left(\frac{c\rho}{r}\right)^2 \phi(\bar{a}/y^2)^2 = q, \tag{6a}$$

$$\bar{a}'' - \frac{2}{y}\bar{a}' + r_0^2\phi^2\bar{a} = 0. \tag{6b}$$

The prime means differentiation with respect to y. In order to ensure the total energy to be finite, $\bar{a}(y)$ must vanish as

$$\bar{a}(y) \approx y^3 - O(y^5) \tag{7a}$$

at small y, and it must tend to one at least as fast as

$$\bar{a}(y) \approx 1 - O\left(\frac{1}{y}\right) \tag{7b}$$

at large y. Explicit expressions for magnetic multipole solutions are then constructed by choosing the appropriate function $a(y)$ which satisfies the boundary conditions (7) and yields a real function $\phi(x)$ from Eq. (6b). Of course $\bar{a}(y)$ must be regular everywhere. In passing we note that boundary conditions of $\phi(x)$ and $q(x)$ are governed by those of $\bar{a}(y)$.

The spherically symmetric non-Abelian Coulomb solution is derived by employing the following ansatz for the gauge field potentials[4]:

$$A_0^a = n^a f(y)/(gr), \quad n^a = x^a/r, \tag{8a}$$

$$A_i^a = \epsilon_{iaj} n^j [a(y) - 1]/(gr). \tag{8b}$$

The external source is specified in the radial frame[4,5]

$$\rho_a = n_a q(y)/(g r_0^3). \tag{8c}$$

This ansatz simplifies Eqs. (1a) and (1b) and the following nonlinear differential equations result:

$$-f'' + 2a^2 f/y^2 = yq, \tag{9a}$$

$$-a'' + (a^2 - 1 - f^2)a/y^2 = 0. \tag{9b}$$

The finite-energy requirement imposes boundary conditions on the functions $a(y)$ and $f(y)$, and for the non-Abelian Coulomb solution, the following asymptotic behavior must be fulfilled[4]: for small y, $a(y)$ must go to one and $f(y)$ must approach zero at least as fast as

$$a(y) \approx 1 + a_0 y^2, \tag{10a}$$

$$f(y) \approx f_0 y^2, \tag{10b}$$

where a_0 and f_0 are constants. For large y, $a(y)$ must tend to one and $f(y)$ must vanish at least as fast as

$$a(y) \approx 1 + a_1 y^{-1}, \tag{11a}$$

$$f(y) \approx f_1 y^{-1}. \tag{11b}$$

Here a_1 and f_1 are constants. To construct the non-Abelian Coulomb solution, the method is again to solve Eq. (9b) first subject to the asymptotic conditions (10) and (11). One then evaluates the source density $q(y)$ from Eq. (9a). In this way the expressions $a(y)$ and $f(y)$ are the solution corresponding to the charge distribution as obtained through Eq. (9a). We note that the asymptotic conditions (10b) and (11b) of the function $f(y)$ can in fact be derived from those of the function $a(y)$ by using Eq. (9b).

III. MAGNETIC MULTIPOLE SOLUTIONS

Equation (6b) can be written as

$$r_0\phi(y) = \left[\left(\frac{2}{y}\bar{a}' - \bar{a}''\right)\Big/\bar{a}\right]^{1/2}. \tag{12}$$

This equation does not readily allow us to write down an explicit expression for $\bar{a}(y)$ in terms of elementary functions such that the boundary conditions (7) are fulfilled and the function $\phi(y)$ is real, although undoubtedly a large class of the required $\bar{a}(y)$ exist. However, by a suitable choice of variable we now show that Eq. (12) can be written in a convenient form which is more easily solvable. Trading the variable u for y, where u is given by

$$u = ey^3 \tag{13}$$

and e is a positive real parameter, we find that Eq. (12) can be simplified as

$$r_0\phi = 3(eu^2)^{1/3}\left(-\frac{1}{\bar{a}}\frac{d^2\bar{a}}{du^2}\right)^{1/2}. \tag{14}$$

This implies that

$$-\frac{1}{\bar{a}}\frac{d^2\bar{a}}{du^2} = K(u), \tag{15}$$

where $K(u)$ is a function which is non-negative everywhere. The asymptotic behavior of $\bar{a}(y)$, as given by Eqs. (7), permits us to make a further simplification by requiring $\bar{a}(u)$ to be positive everywhere. With this constraint for $\bar{a}(u)$, we observe that an explicit magnetic multipole solution emerges whenever $\bar{a}(u)$ possesses the asymptotic

TABLE I. Explicit magnetic multipole solutions and their asymptotic behaviors.

$\bar{a}(u)$	Asymptotic behaviors $u\approx 0$	$u\approx\infty$
(a) $\tanh u$	$u-\frac{1}{3}u^3$	$1-2\exp(-2u)$
(b) $u\tanh(1/u)$	$u-2u\exp(-2/u)$	$1-(3u^2)^{-1}$
(c) $u(b^n+u^{n/3})^{-3/n}$, $b>0,\ n=2,3,4,\ldots$	$\frac{u}{b^3}\left(1-\frac{3}{n}(u^{1/3}/b)^n\right)$	$1-\frac{3}{n}(b/u^{1/3})^n$
(d) $1-(1+bu)^{-1}\exp(-u)$, $b\geqslant 0$	$(b+1)u-(\frac{1}{2}+b+b^2)u^2$	$1-\exp(-u)/(bu)$
(e) $\frac{2}{\pi}\tan^{-1}(u)$	$\frac{2}{\pi}(u-\frac{1}{3}u^3)$	$1-2/(\pi u)$
(f) $u\tan^{-1}(1/u)$	$u\left(\frac{\pi}{2}-u\right)$	$1-(3u^2)^{-1}$

behavior (7) and obeys the following inequality:

$$\frac{d^2\bar{a}}{du^2}<0. \tag{16}$$

Thus the construction of *explicit* magnetic multipole solutions is made relatively easier by requiring the function $\bar{a}$ to fulfill conditions (16) and (7) instead of (12) and (7). In passing we observe that the asymptotic expressions (7) also satisfy the inequality (16) as they should. We emphasize that in the above consideration, it is assumed that $\bar{a}(u)$ is positive everywhere.

In Table I, we list some explicit expressions for $\bar{a}(u)$ whose second derivatives are negative. The list is of course not exhaustive. Solution (a) was first presented by Sikivie and Weiss,[2] and solutions (b) and (c) were given in Ref. 7.

IV. THE RELATION

In Ref. 8, we obtained analytic closed-form expressions for the non-Abelian Coulomb solutions

$$a(y)=1+(b/y)\tanh(u),\quad b>0 \tag{17}$$

and

$$a(y)=1+by^2\tanh(1/u),\quad b>0\,. \tag{18}$$

On comparing with the magnetic multipole solutions (a) and (b) in Table I, it is easy to see that

$$a(y)=1+b\,\bar{a}(u)/y,\quad b>0 \tag{19}$$

where b is a positive parameter. We now wish to show that the relation (19) is valid in general; that is, given a magnetic multipole solution one can immediately write down the non-Abelian Coulomb solution by virtue of Eq. (19). Before commencing the proof, we observe that asymptotic conditions of $\bar{a}$ as given by Eqs. (7) will guarantee the correct asymptotic behavior of $a(y)$. However, the converse is not necessarily true. The asymptotic behavior of $a(y)$ will not in general lead, via Eq. (19), to the correct asymptotic conditions for $\bar{a}$. Hence one cannot expect in general to construct, via relation (19), the magnetic multipole solution from the non-Abelian Coulomb solution.

A non-Abelian Coulomb solution is obtained if we can write down an appropriate expression for $a(y)$ which satisfies the boundary conditions (10a) and (11a) and which gives rise to a real function $f(y)$ from the Eq. (9b). Equation (9b) can be rewritten as

$$af^2=-y^2a''+a^3-a\,. \tag{20}$$

Applying the relation (19), this becomes

$$af^2=9b(eu^5)^{1/3}\left(-\frac{d^2\bar{a}}{du^2}\right)+(b\,\bar{a}/y)^3+3(b\,\bar{a}/y)^2\,. \tag{21}$$

For the magnetic multipole solution, the inequality (16) holds and $\bar{a}$ is positive everywhere. This implies that $a(y)$ is also positive everywhere, and as a consequence a real function $f(y)$ results from Eq. (21). The function $f(y)$ so constructed will have the correct asymptotic behavior as this is determined from the asymptotic conditions of $a(y)$ and the Eq. (20).

Thus starting from a magnetic multipole solution, a non-Abelian Coulomb solution is easily derived by using Eqs. (19) and (20). In other words, one obtains $a(y)$ and $f(y)$ from a valid $\bar{a}(y)$.

Although Eq. (19) establishes a relation between $\bar{a}$ and $a(y)$ it has no bearing on the properties of the magnetic multipole solution and the non-Abelian

Coulomb solution. In fact they are quite distinct as the magnetic multipole solution requires a source of critical strength to sustain itself whereas the non-Abelian Coulomb solution exists for a source of arbitrary strength. However, the relation (19) is useful because in view of the inequality (16), the magnetic multipole solution is easier to construct than the non-Abelian Coulomb solution, and the relation (19) enables us to obtain the latter from the former.

V. NON-ABELIAN COULOMB SOLUTION

As an illustration of the results obtained in Sec. IV, we construct a non-Abelian Coulomb solution from the magnetic multipole solution (d) given in Table I,

$$\bar{a} = 1 - (1+bu)^{-1}\exp(-u), \quad b \geq 0 . \tag{22}$$

From Eqs. (19) and (20), the non-Abelian Coulomb solution is

$$a = 1 + \frac{1}{y}\left(1 - \frac{e^{-u}}{1+bu}\right), \quad b \geq 0 \tag{23a}$$

$$af^2 = 9\frac{u^2}{y}\left[1 + \left(\frac{2b}{1+bu}\right)\left(1 + \frac{b}{1+bu}\right)\right]\frac{e^{-u}}{1+bu} + 3\left(\frac{\bar{a}}{y}\right)^2 + \left(\frac{\bar{a}}{y}\right)^3 , \tag{23b}$$

$$u = ey^3, \quad e > 0 . \tag{23c}$$

The asymptotic behavior can be easily found. Thus at small y, we have

$$a \approx 1 + (b+1)ey^2 - O(y^5), \tag{24a}$$

$$f \approx 3^{1/2}(b+1)ey^2 + O(y^3), \tag{24b}$$

and at large y,

$$a \approx 1 + \frac{1}{y}, \tag{25a}$$

$$f \approx 3^{1/2}/y . \tag{25b}$$

One can also estimate the asymptotic behavior of the source density $q(y)$. At small y, $q(y)$ behaves as

$$q \approx -12(3)^{1/2}\,\frac{(\frac{1}{2}+b+b^2)}{1+b}\,e + O(y), \tag{26a}$$

and at large y,

$$q \approx 4(3^{1/2}+3^{-1/2})/y^5 . \tag{26b}$$

The graphs of $a(y)$, $f(y)$, and $q(y)$ (see Figs. 1–4) are as shown and they appear similar to the numerical solution of Ref. 4. The general shapes of $a(y)$ and $f(y)$ are not very sensitive to the change of the source distribution. We have also plotted graphs for fixed b but different e. There is no new feature.

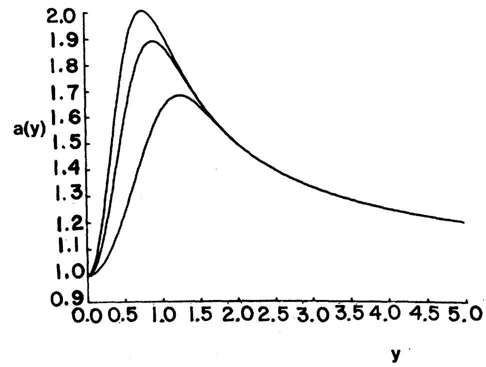

FIG. 1. The function $a(y)$ for the solution (23) with $e=1$. Starting from the lowest curve these correspond to $b=0,2,4$.

VI. REMARKS

We make a few comments:

(a) In Sec. IV, we have shown that construction of magnetic multipole solutions is made easier by using the inequality (16). One might enquire whether the same technique can be used to find non-Abelian Coulomb solution. The answer is no. By writing $a(y)$ as

$$a(y) = 1 + V(y), \tag{27}$$

then from Eq. (9b) we obtain

$$f(y) = \left(V^2 + 2V - \frac{y^2 V''}{1+V}\right)^{1/2} . \tag{28}$$

Assuming $V(y)$ is positive everywhere one might think that the inequality

$$V''(y) < 0 \tag{29}$$

will result in a real function $f(y)$. But this will not occur because the function $V(y)$ must possess the following asymptotic behavior:

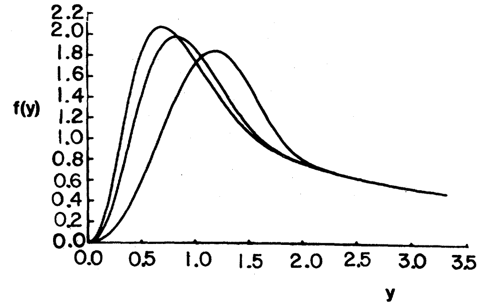

FIG. 2. The function $f(y)$ for the solution (23) with $e=1$. Starting from the lowest curve these correspond to $b=0,2,4$.

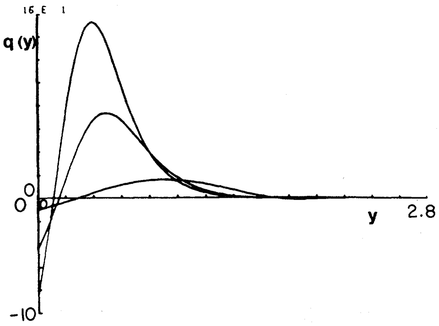

FIG. 3. The source density $q(y)$ for the solution (23) with $e=1$, $b=0,2,4$. The curve with the smallest (largest) amplitude corresponds to $b=0$ (4). Here $E1$ means the q axis is scaled by a factor 10, e.g., $16E1=160$.

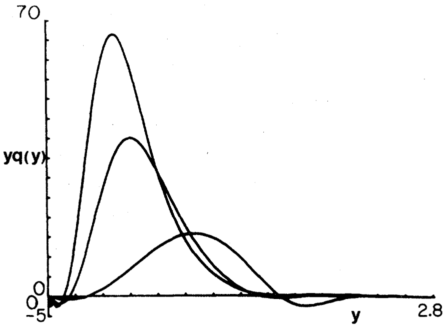

FIG. 4. Profiles of the function $yq(y)$ for the solution (23) with $e=1$, $b=0,2,4$. The curve with the smallest (largest) amplitude corresponds to $b=0$ (4).

$$V(y)\approx a_0 y^2 \tag{30a}$$

at $y\approx 0$, and

$$V(y)\approx a_{-1} y^{-1} \tag{30b}$$

at $y\approx\infty$. The requirement that $V(y)$ be positive everywhere restricts the constants a_0 and a_{-1} to positive values only. Hence neither condition (30a) nor (30b) can ever fulfill the inequality (29). Thus the non-Abelian Coulomb solution cannot be obtained as easily as the magnetic multipole solution and the relation (19) is useful.

(b) We have constructed analytic expressions for the non-Abelian Coulomb solution in the SU(2) case.[8] The method can be extended to the SU(3) case.

(c) In Ref. 9 it is shown that if the source density vanishes exponentially at large distances, then the field strengths, provided they also decrease exponentially, have to vanish at least as fast as the source. It is well known that the source in the SU(2) Yang-Mills equations does not determine the solutions uniquely. Thus for magnetic dipole solutions[2] given as solution (a) in Table I, although the source density q vanishes exponentially at large distances, the magnetic field has the dipole behavior $1/y^3$. For the non-Abelian Coulomb solution (23) we find that at large y the field strengths approach zero as y^{-3}. From Eq. (26b), we see that the source density decreases faster than the field strengths at large distances.

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[10]The function $\bar{a}(y)$ in Eq. (5) is the same as the function $f(y)$ in Ref. 2.