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Monte Carlo study of compact U(1) four-dimensional lattice gauge theory

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Expectation values of the specific heat and of Wilson loops are presented for compact U(1) gauge theory on 4^4, 5^4, and 6^4 lattices, calculated by averages over up to 7500 iterations through the lattice generated by Monte Carlo methods. The calculations show clear evidence for a single second-order phase transition.

I. INTRODUCTION

Compact quantum electrodynamics (QED) is known[1] to undergo a transition from a phase at strong coupling, where electrons are confined, to a phase at weak coupling where free electrons and photons are found. In this paper, we present the results of a Monte Carlo investigation of compact U(1) gauge theory on four-dimensional lattices.[2-4] A second-order phase transition is seen in which external sources are unconfined, which is signaled by the Wilson loops changing from an area-law to a perimeter-law behavior.[5] It is hoped that the introduction of fermions into the system does not essentially alter this result.

II. MONTE CARLO ALGORITHM

A Monte Carlo process generates a series of configurations of link variables on the lattice in such a way that the mean value of an observable measured over all configurations should converge to the expectation value. We define our action as a sum over unoriented plaquettes p, such that

$$S[U]=\sum_p (1-\mathrm{Re}U_p),$$

where U_p is the product around a plaquette of link variables $U=\exp(i\theta)$. The partition function is

$$Z=\int [dU]\exp(-\beta S[U]).$$

To simulate this system we use the heat bath,[2] in which link variables are updated by selection from the Boltzmann distribution without direct reference to their current values. Given a link, we generate its angle with probability

$$P(\theta)d\theta=\exp[-\beta\Delta S(\theta)]d\theta,$$

where $\Delta S(\theta)$ is the contribution that would be made to the action by that link. A practical technique for reproducing this distribution is as follows.[6] Making the change of variable

$$x=1-\frac{2\theta}{\pi},$$

$$P(\theta)d\theta=\exp(\alpha\cos\theta)d\theta,\quad 0\leq\theta\leq 2\pi$$
$$=Q(x)d\{Z(x)\},\quad 0\leq Z(x)\leq 1$$

where

$$Z(x)=\frac{e^{\alpha x}-e^{-\alpha}}{e^{\alpha}-e^{-\alpha}},$$

$$Q(x)=\exp\left\{\alpha\left[\cos\frac{\pi}{2}(1-x)-x\right]\right\},$$

and α is a constant arising from the current state of the links that interact with the link to be updated. Two pseudo-random numbers are selected from the uniform distribution on the interval 0 to 1, which we denote by $\hat{Z}$ and $\hat{Z}'$. The variable x is calculated from the inverse function to $Z(x)$,

$$x=-1+\frac{1}{\alpha}\ln[1+(e^{2\alpha}-1)\hat{Z}]$$

then to produce the weighting $Q(x)$ this value of x is kept only if

$$\frac{Q(x)}{Q_{\max}}>\hat{Z}',$$

where $Q_{\max}=\exp(0.210\,513\,7\alpha)$ is the maximum of $Q(x)$. This sequence is repeated until a value of x is accepted. We refer to the application of this updating procedure to every link on the lattice as one iteration. Periodic boundary conditions are used in all our calculations. In our calculations, the first ten iterations through the lattice are ignored.

III. THE RESULTS

In Fig. 1 we show the average action per plaquette $\langle E\rangle$ as a function of the inverse temperature β. For $\beta<0.88$ and $\beta>1.13$ we have used a 6^4 lattice. To achieve convergence in this region, 400 iterations through the lattice were found to be sufficient. However, for $0.88\leq\beta\leq 1.13$ we sam-

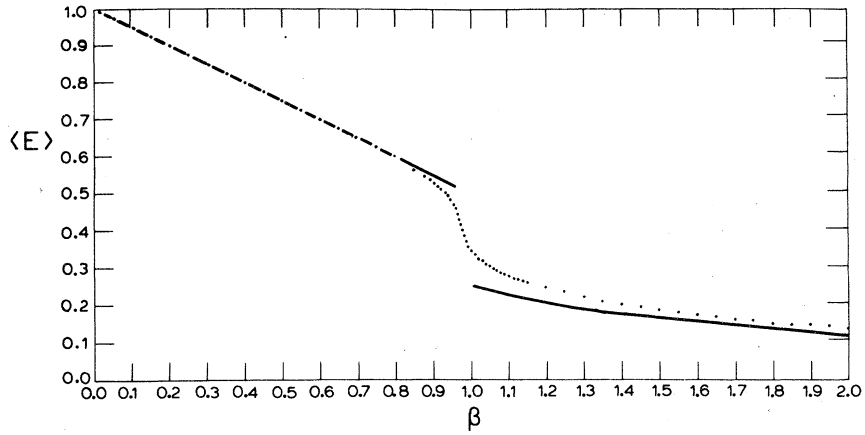


FIG. 1. The U(1) average action per plaquette $\langle E \rangle$ as a function of the inverse temperature β for a lattice of size 6^4. The curves are the leading-order high- and low-temperature expansions of Ref. 2.

pled up to 7 500 iterations through the lattice. Also shown in Fig. 1 are the leading high- and low-temperature expansions[2] given by

$$\langle E \rangle = 1 - \frac{\beta}{2} + O(\beta^2)$$

and

$$\langle E \rangle = \frac{1}{4\beta},$$

respectively.

The results for the average action per plaquette $\langle E \rangle$ for an ordered and a disordered 4^4 starting lattice are shown in Fig. 2(a). The data in this diagram result from averaging over 7 500 iterations through the lattice. We immediately note the fact that there is more dispersion in the disordered starting data than in the ordered starting data. This is somewhat surprising after such long runs. A possible explanation[7] may be that on cooling from disorder some sort of dislocations get frozen into the lattice in a random manner. For example, if one rapidly freezes water a lot of internal energy is likely to be locked up in boundaries between separate crystal grains. However, this cannot be the whole explanation for this effect because the disordered starting data points are almost as often below as above the ordered starting data. This can be clearly seen in Fig. 2(b).

To illustrate how quickly the lattice comes to equilibrium outside the critical region, we show in Fig. 3(a) a plot of the average action per plaquette versus the number of iterations through the lattice for the inverse temperature $\beta = 0.55$. At $\beta = 0.55$, we obtain the same result for the average action per plaquette if we average over the first 50 iterations, the first 500 iterations, or any higher number of iterations through the lattice. We can thus equilibrate the lattice outside the transition region with very few iterations through the lattice.

It was found by Lautrup and Nauenberg[8] that, in the region of the critical point, the approach to equilibrium is not always monotonic but has some discontinuous jumps. We illustrate this in Fig. 3(b) by plotting the first 360 iterations through a 6^4 lattice at an inverse temperature of $\beta = 1.000$. The rapid change near 180 iterations is clearly evident. Because of this it was found necessary to make 7 500 iterations through the lattice in order to obtain smooth results. In order to cut down the computation time for 7 500 iterations through the lattice, we used a 4^4 lattice.

In Fig. 3(c) we show the average action per plaquette against β for the inverse temperature $\beta = 5.50$ which is well beyond the transition region. As was found in Fig. 3(a), the lattice soon comes to statistical equilibrium. Here the average over the first 30 iterations is substantially the same as the average over 80 iterations through the lattice.

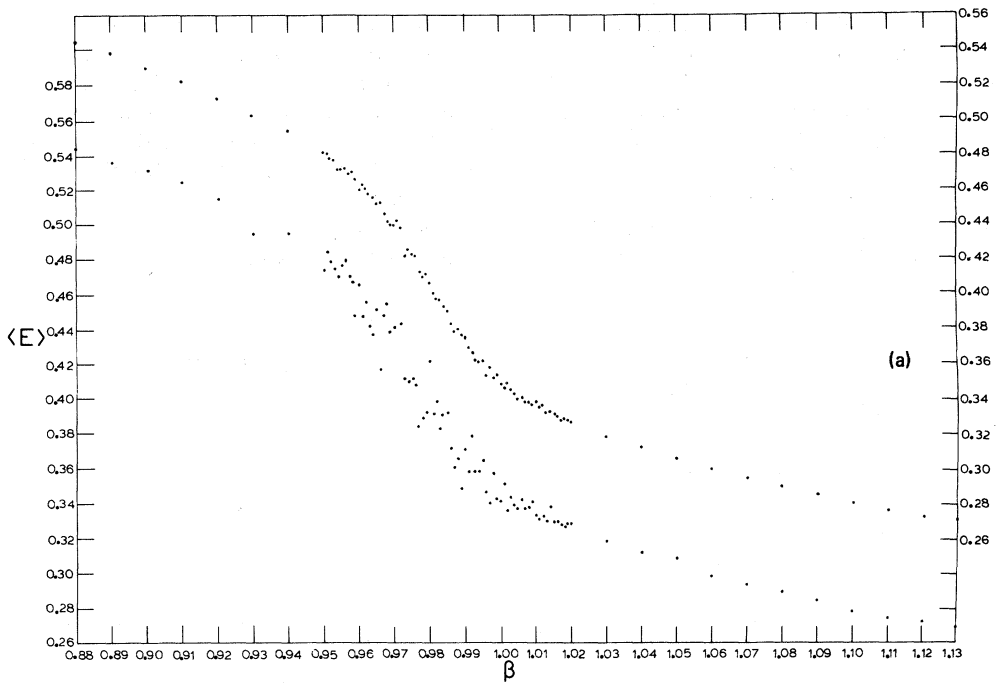
We would next like to investigate the effect on the measurement of the average action per plaquette of the lattice size. Figures 4 and 5 show the results for 5^4 and 6^4 lattices obtained by averaging over 6000 and 3000 iterations, respectively, through the lattice. We only plot the data over the range $0.88 \leq \beta \leq 1.13$.

The Wilson loop $W(S, S)$ is defined by

$$W(S, S) = \langle \mathrm{Tr} U(C) \rangle,$$

where C is a closed path of links and S is the linear dimension of the loop that is being calculated. In general, we know that

$$0 \leq W \leq 1,$$



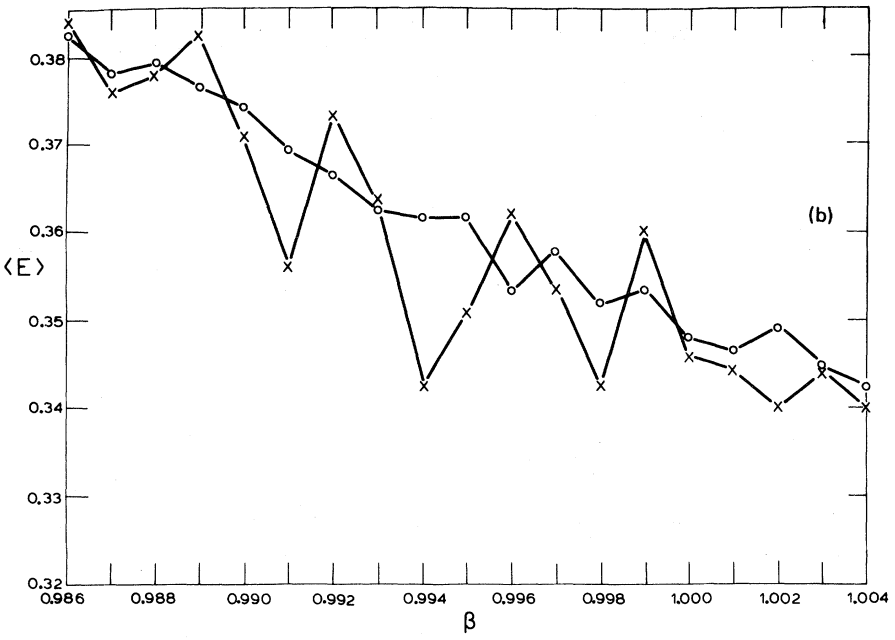


FIG. 2. (a) The U(1) average action per plaquette $\langle E \rangle$ as a function of the inverse temperature β for a lattice of size 4^4. The data points result from averaging over 7 500 iterations through the lattice. (The uppermost data set represents an ordered starting lattice while the lowermost data set represents a disordered starting lattice.) (b) The U(1) average action per plaquette $\langle E \rangle$ for an ordered starting lattice (open circles) and a disordered starting lattice (crosses).

where $\lim_{\beta\to\infty} W \to 1$ and $\lim_{\beta\to 0} W \to 0$ are the limiting behavior of the Wilson loops. As the loop gets bigger, W gets smaller.

We present the square Wilson loops against the inverse temperature β in Fig. 6. All these results have been obtained for a 6^4 lattice by averaging 400 iterations through the lattice.

In Fig. 7 we show the expectation values of square Wilson loops as a function of the lattice size for $\beta = 1.50$ and $\beta = 3.00$, respectively. We see that the loops of up to one link stabilize after a 3^4 lattice, while those of up to two links stabilize after a 5^4 lattice. The loops of up to three links have almost stabilized after a 6^4 lattice, but loops of up to four and five links are still changing at a 6^4 lattice. This gives the rough rule that finite-size effects show up in Wilson loops larger than half the lattice size.

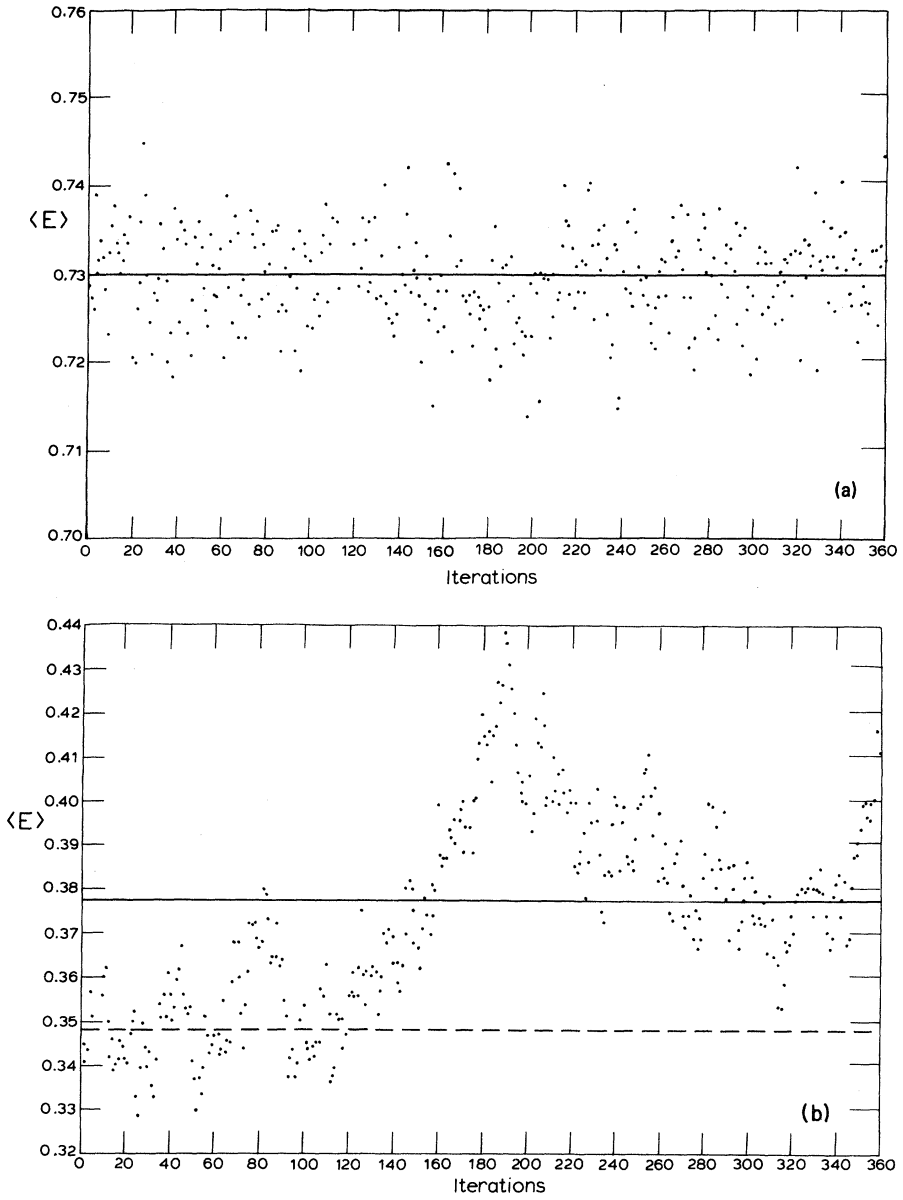


FIG. 3. The U(1) average action per plaquette $\langle E\rangle$ as a function of the number of Monte Carlo iterations through a 4^4 lattice with an ordered start. (a) $\beta = 0.55$ (the solid line results from averaging over 400 iterations through the lattice), (b) $\beta = 1.000$ (the solid and dashed lines result from averaging over 400 and 7 500 iterations through the lattice, respectively), and (c) $\beta = 5.50$ (the solid line results from averaging over 80 iterations through the lattice).

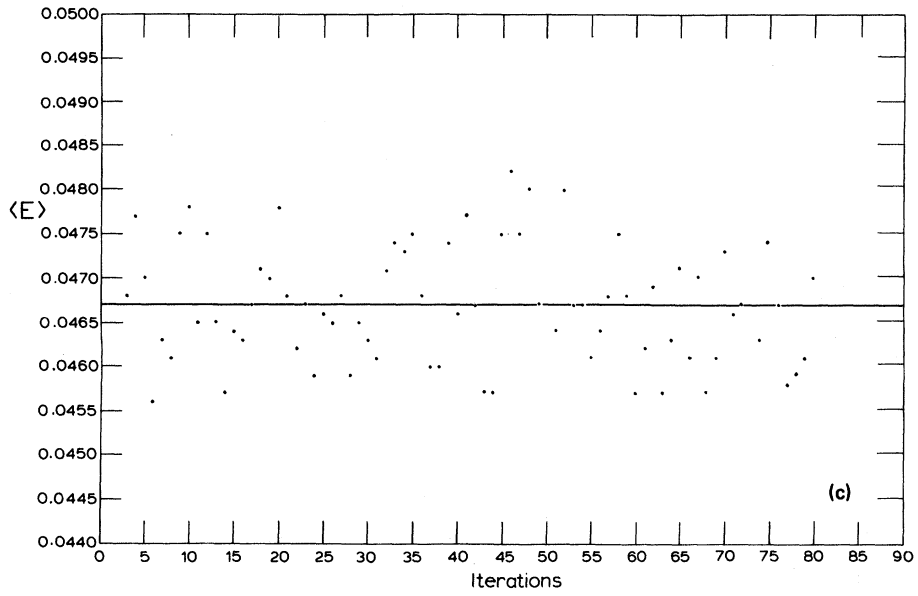


FIG. 3. (*Continued*.)

Figure 8(a) presents the square Wilson loop $W(S,S)$ versus the loop side S for various values of β. We see that this is almost a straight line so that

$$W_\beta(S,S) \simeq \exp[-B(\beta)S],$$

where B is the slope of the curve. Since Fig. 8(a) is almost a straight line this proves that in the low-temperature region the square Wilson loops obey the perimeter law.

In Fig. 8(b) we present a plot of the square Wilson loop against the loop side squared S^2 for the inverse temperature below the critical temperature. Again we observe an approximate straight-line behavior indicating

$$W_\beta(S,S) \simeq \exp[-\chi(\beta)S^2],$$

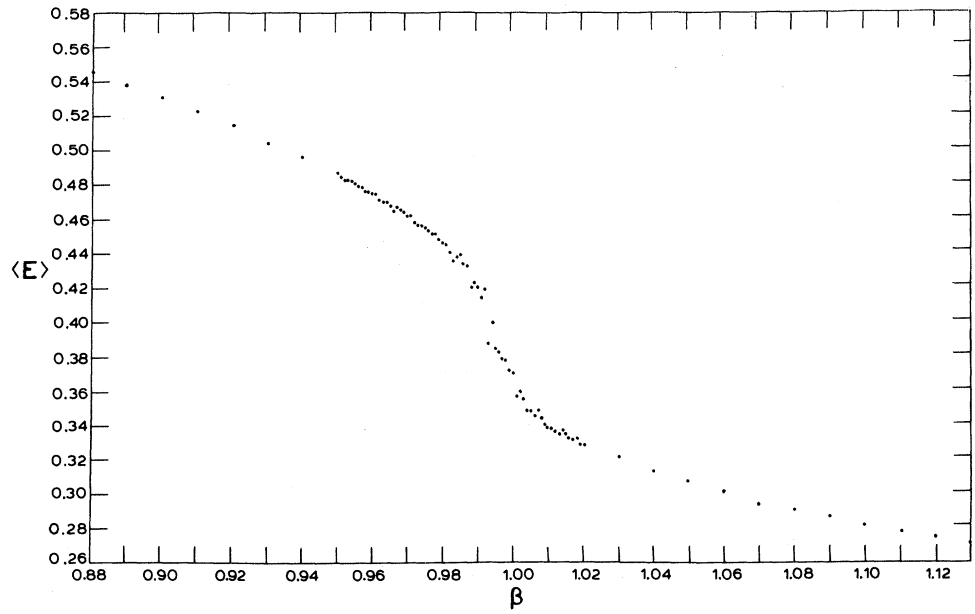


FIG. 4. The U(1) average action per plaquette $\langle E\rangle$ as a function of the inverse temperature β for a 5^4 lattice with an ordered start. The data points result from averaging over 6000 iterations through the lattice.

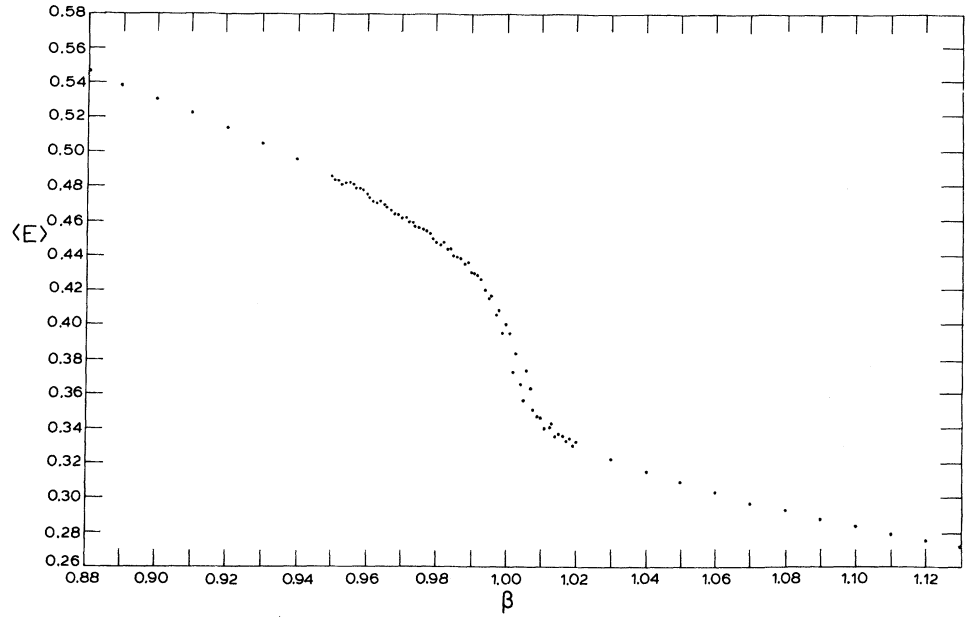


FIG. 5. The U(1) average action per plaquette $\langle E \rangle$ as a function of the inverse temperature β for a 6^4 lattice with an ordered start. The data points result from averaging over 3000 iterations through the lattice.

where χ is the slope of the curve. Figure 8(b) shows that the square Wilson loops obey the area law below the critical region.

In Fig. 9 we show the fitted slope of the square Wilson loop versus the inverse temperature β. The sudden crossover from an area decay to a perimeter decay is evident near $\beta \simeq 1.0$. This change of the loop behavior is a clear indication of a phase transition.[5]

In order to see the phase transition more clearly, we plot for comparison the smoothed average action per plaquette data for 4^4, 5^4, and 6^4 lattices in Fig. 10. These curves were obtained from the data shown in Figs. 2(a), 4, and 5, respectively. These data consisted of 89 data points in the range $0.88 \leq \beta \leq 1.13$ resulting from averaging 3 000, 6 000, and 7 500 iterations through 6^4, 5^4, and 4^4 lattices, respectively. We must keep in

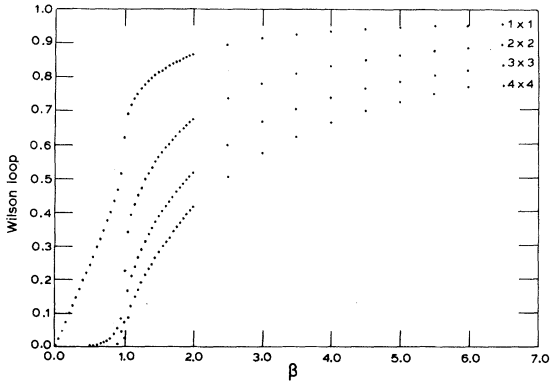


FIG. 6. The square Wilson loops $W(S,S)$ for $S=1$, 2, 3, 4, and 5 as a function of the inverse temperature β for U(1).

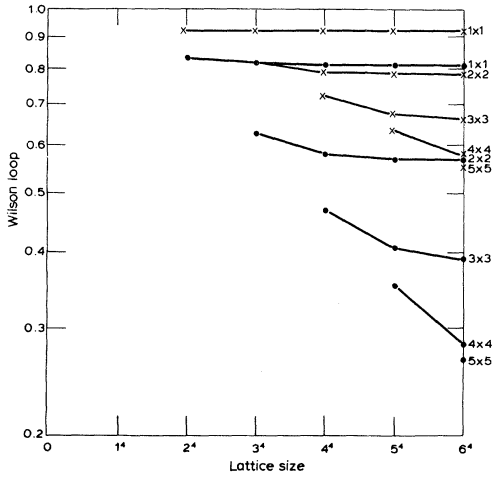


FIG. 7. The square Wilson loops $W(S,S)$ for $S=1$, 2, 3, 4, and 5 as a function of the lattice size for U(1). (The dot represents the data for $\beta=1.50$ while × represents the data for $\beta=3.00$.)

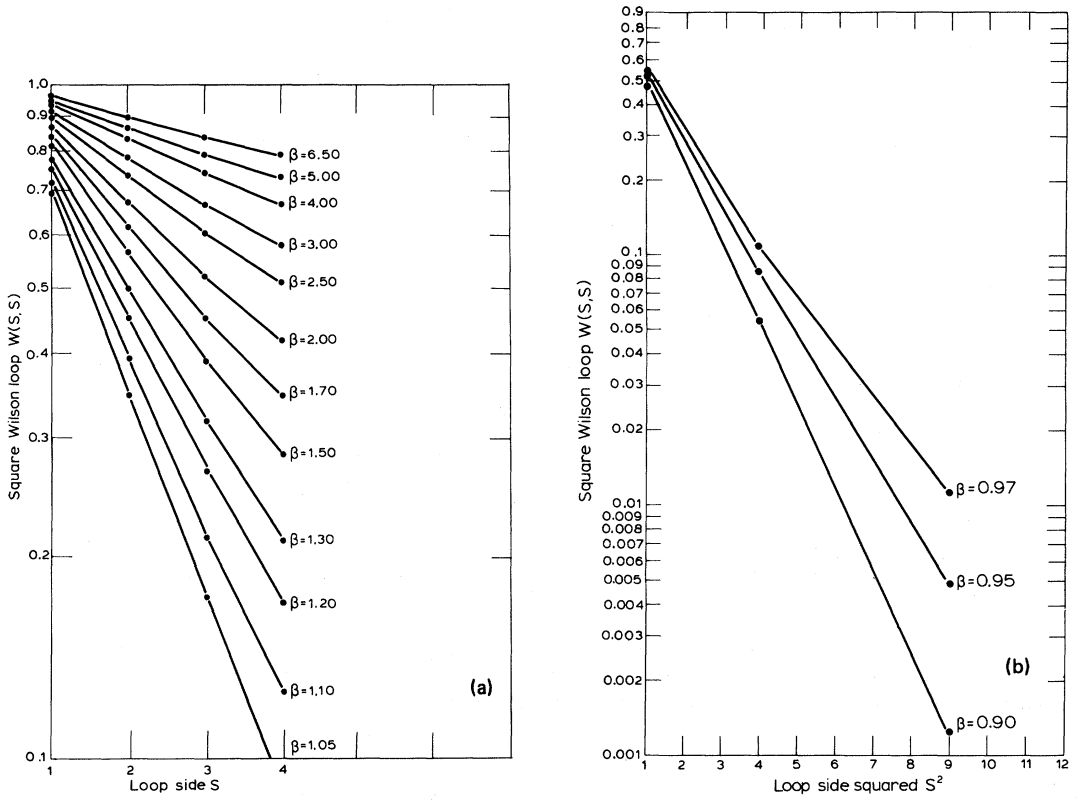


FIG. 8. (a) The square Wilson loops $W(S,S)$, for the inverse temperature β above the critical point, as a function of the lattice size S. (b) The square Wilson loops $W(S,S)$, for the inverse temperature β below the critical point, as a function of the lattice size S.

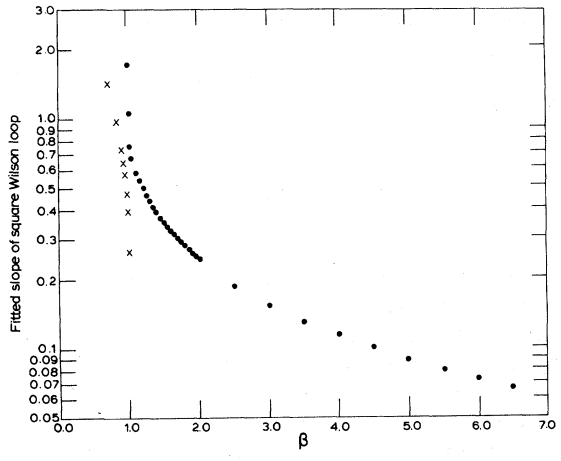


FIG. 9. The coefficients of the area- and perimeter-law behavior of the square Wilson loops $W(S,S)$ as a function of the inverse temperature β. The crosses and solid circles represent the coefficients of the area law $\chi(\beta)$ and the perimeter law $B(\beta)$, respectively.

mind that the scale in Figs. 2(a), 4, and 5 is considerably enlarged over that of Fig. 1. There are some fluctuations in the system in the critical region still remaining after the averaging over 7 500 iterations through the lattice. These are due both to residual thermal fluctuations in the system which cannot be eliminated and to finite-size effects. The critical temperatures are given by the values $\beta_c = 0.978 \pm 0.005$, $\beta_c = 0.993 \pm 0.002$, and $\beta_c = 1.003 \pm 0.001$ for 4^4, 5^4, and 6^4 lattices, respectively.

Lautrup and Nauenberg[8] found similar results from a quantity they call the normalized specific heat. The normalized specific heat is not the same as the specific heat, the quantity we calculate. The specific heat for constant volume C_V is defined by

$$C_V \equiv \frac{\partial \langle E \rangle}{\partial T} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} .$$

In order to evaluate this expression, we first smooth the average action per plaquette data. These smoothed data are then interpolated by

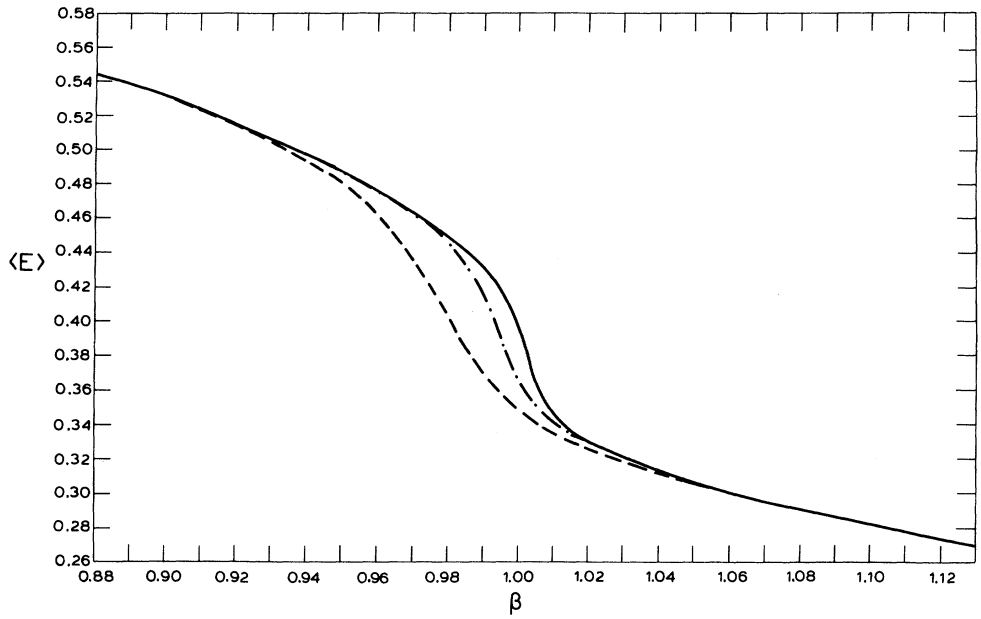


FIG. 10. The smoothed U(1) average action per plaquette $\langle E \rangle$ as a function of the inverse temperature β. (The dashed curve represents the smoothed data for a 4^4 lattice, the dot-dashed curve represents the smoothed data for a 5^4 lattice, while the solid curve represents the smoothed data for a 6^4 lattice.)

means of cubic splines.[9] Once one has obtained the cubic spline interpolation of the smoothed data, the first derivative results immediately. The equation above then yields the specific heat for constant volume. The results are shown in Fig. 11 for 4^4, 5^4, and 6^4 lattices. Using these curves we calculate values for the critical inverse temperature β_c which agree with those quoted previously in relation to our discussion of Fig. 10. Finite-size scaling theory has been applied[10] to our specific heat results and the results given in Ref. 10. Preliminary results on the string tension are reported in Ref. 11.

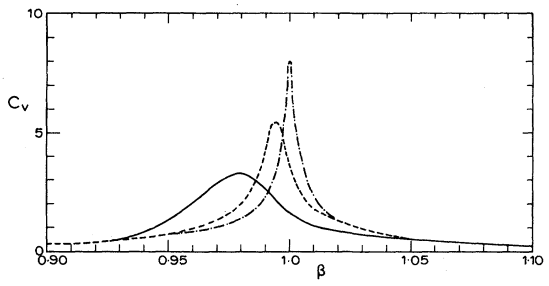


FIG. 11. The specific heat for constant volume C_V as a function of the inverse temperature β. (The solid curve represents the data for a 4^4 lattice, the dashed curve represents the data for a 5^4 lattice, while the dot-dashed curve represents the data for a 6^4 lattice.)

We have studied the average action per plaquette, square Wilson loops, and the specific heat for a lattice gauge theory with the gauge group U(1). We see clear evidence for a second-order phase transition in all three quantities near $\beta_c \simeq 1.0$. Guth[1] has shown, using Griffiths-Kelly-Sherman inequalities, that U(1) lattice gauge theory has at least one phase transition. Since U(1) is the gauge group of quantum electrodynamics, this confirms that photons and electrons can exist as free particles.

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