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Consistent quark model with hyperfine interactions for the ground and low-lying excited baryon states

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Isgur and Karl have recently analyzed the ground and low-lying excited baryon states; the negative-parity and positive-parity baryons are treated separately using different parameter sets. In this paper we show how the Hamiltonian introduced by Isgur and Karl to examine the positive-parity baryons can be modified to apply to the negative-parity baryons. In particular, the mass difference between $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ is restored to a value consistent with experiment.

I. INTRODUCTION

Isgur and Karl have used a quark-model framework inspired by quantum chromodynamics (QCD) to describe the spectrum and mixing angles of the ground-state,[1] negative-parity,[2] and positive-parity[3] baryons: these papers will be referred to below as I, II, and III, respectively. Thus far each set of levels of alternating parity has been analyzed as a separate problem with a different parameter set. In this paper we show that it is possible to use a single set of parameters to examine all the low-lying baryon states.

In II, a simple fit to all the baryon levels is made using a harmonic-oscillator potential plus hyperfine splittings. A major triumph[4] is the correct prediction in sign and magnitude of the observed inversion of $\Lambda\frac{5}{2}^-(1830)$ and $\Sigma\frac{5}{2}^-(1765)$ relative to the ground state. In III the harmonic-oscillator pair potential is modified by a term $U(r_{ij})$ which incorporates the effects of a Coulomb-type piece derived from QCD at short range and deviations from the harmonic-oscillator form at large distances. Interband mixing between the ground-state and positive-parity baryons is introduced in I. One of the effects of the modification of the harmonic-oscillator potential in I and III is to reduce the splitting of $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ from 50 to 15 MeV so that it is apparently no longer consistent with experiment. The key to the resolution of this difficulty is the method of calculation of the nonharmonic part of the potential. In III in the $S=-1$ sector, this is calculated in the SU(3) limit ($m_s=m_u$). By using a technique introduced by Kalman, Hall, and Misra,[5] this term can be calculated in a manner which properly takes into account the difference in mass between the strange and nonstrange quarks. Using this we show that the splitting of $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ can be restored to a value in line with experiment.

II. NEGATIVE-PARITY BARYONS

In II Isgur and Karl use a Hamiltonian of the form

$$H=\sum_{i=1}^{3}m_i+H_0+H_{\rm hyp}\ , \tag{2.1}$$

where m_i are the quark masses. In the approximation $m_u\simeq m_d$ at least two of the quarks have equal mass. Thus we set $m_1=m_2=m$:

$$H_0=\sum_i P_i^{\,2}/2m_i+\sum_{i<j}V^{ij}_{\rm conf}-\left[\sum_i P_i\right]^2\Big/\left[2\sum_i m_i\right]+E_0\ , \tag{2.2a}$$

where

$$V^{ij}_{\mathrm{conf}}=1/2K\,|\vec{r}_{ij}|^2\ , \tag{2.2b}$$

$$H_{\mathrm{hyp}}=\sum_{i<j}^{3}\frac{2\alpha_s}{3m_im_j}\left\{\frac{8\pi}{3}\delta^3(\vec{r}_{ij})(\vec{s}_i\cdot\vec{s}_j)+\frac{1}{r_{ij}^{3}}\left[\frac{3(\vec{s}_i\cdot\vec{r}_{ij})(\vec{s}_j\cdot\vec{r}_{ij})}{r_{ij}^{2}}-(\vec{s}_i\cdot\vec{s}_j)\right]\right\}. \tag{2.3}$$

$\vec{r}_{ij}$ is the separation between a pair of quarks. In terms of the Jacobi relative coordinates[2] ρ and λ, Eq. (2) separates[6] into two independent harmonic oscillators with the same spring constant K.

Since in the $S=-1$ sector $m_3 \neq m$, the frequency of the ρ oscillator is higher than that of the λ oscillator by

$$\omega_\rho-\omega_\lambda=\omega\left[1-\left[\frac{2x+1}{3}\right]^{1/2}\right],\quad \omega_\rho=\omega\ , \tag{2.4}$$

where $x=m_d/m_s$ and $\hbar\omega_\rho$ is the harmonic-oscillator spacing in the $S=0$ sector. Isgur and Karl state that "the mass difference between the $J=\frac{5}{2}^-$ Σ and Λ is primarily a spin-independent effect due to the lack of degeneracy between the ρ and λ oscillator .... In the harmonic-oscillator model this mass difference [see Eq. (2.4)] is about 75 MeV $\equiv\hbar(\omega-\omega_\lambda)$ and hyperfine interactions give a relatively small correction ($\sim$20 MeV) to this effect."

III. POSITIVE-PARITY BARYONS

In the $S=0$ sector of a strict harmonic-oscillator model, all the positive baryons have the same mass. A small amount of breaking occurs from the hyperfine interactions, but to reproduce the spectrum in detail an additional hypothesis is necessary. Thus in III, Isgur and Karl use a confining potential of the form

$$V^{ij}_{\mathrm{conf}}=\tfrac{1}{2}kr_{ij}^{2}+U(r_{ij})\ , \tag{3.1}$$

where $U(r_{ij})$ is some unknown potential which incorporates an attractive potential at short range (a Coulomb-type piece derived from QCD) and deviations from the harmonic-oscillator form at large distances. Treating U by first-order perturbation theory using the harmonic-oscillator wave functions, it follows that

$$E(S_s)=E(56,0^+)=E_0\ , \tag{3.2a}$$

$$E(P_m)=E(70,1^-)=E_0+\Omega\ , \tag{3.2b}$$

$$E(S_s')=E(56',0^+)=E_0-\Delta+2\Omega\ , \tag{3.2c}$$

$$E(S_m)=E(70,0^+)=E_0-\tfrac{1}{2}\Delta+2\Omega\ , \tag{3.2d}$$

$$E(D_s)=E(56,2^+)=E_0-\tfrac{2}{5}\Delta+2\Omega\ , \tag{3.2e}$$

$$E(D_m)=E(70,2^+)=E_0-\tfrac{1}{5}\Delta+2\Omega\ , \tag{3.2f}$$

$$E(P_A)=E(20,1^+)=E_0+2\Omega\ . \tag{3.2g}$$

In I Isgur and Karl include the mixing between the ground-state and the positive-parity baryons using the masses and compositions of the excited states obtained previously.[3] The major effect is the decrease in the value of

$$\delta=\frac{4\alpha_s(m\omega)^{3/2}}{3(\pi)^{1/2}m^2 2^{1/2}} \tag{3.3}$$

from 300 to 260 MeV.

The effective oscillator spacing Ω works out to 440 MeV, not far from the value of the oscillator frequency obtained for the p-wave baryons (520 MeV). The value of the true oscillator spacing (250 MeV), however, reduces the splitting between the ρ and λ oscillators in the $S=-1$ sector [Eq. (2.4)] from 75 to about 35 MeV. Since the hyperfine interaction remains the same at about 20 MeV, the mass difference between $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ is reduced from 50 to 15 MeV and is apparently no longer consistent with experiment.

IV. THE NONHARMONIC PART OF THE POTENTIAL

In III, Isgur and Karl calculate the effect of the nonharmonic part of the potential [Eq. (3.1)] in the SU(3) limit. That is, they take no account of the difference between

$$\alpha_\rho=(3km)^{1/4}=\alpha \tag{4.1}$$

and

$$\alpha_\lambda=(3km_\lambda)^{1/4}\ , \tag{4.2}$$

where

$$m_\lambda=3mm_3/(2m+m_3)\ .$$

By taking this difference into account, the mass difference between $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ can be restored to a value consistent with experimental values.

Calculations of the contributions from the nonharmonic part of the potential for different values of α have been discussed by Kalman, Hall, and Misra.[5] Based on this work, we set

$$a(t)=(3\alpha^3 t^{3/2}/\pi^{3/2}) \times \int d^3\rho\, U(\sqrt{2}\rho)\exp(-t\alpha^2\rho^2)\ , \tag{4.3a}$$

$$b(t)=(3\alpha^5 t^{5/2}/\pi^{3/2}) \times \int d^3\rho\, U(\sqrt{2}\rho)\rho^2\exp(-t\alpha^2\rho^2)\ , \tag{4.3b}$$

and

$$c(t)=(3\alpha^7 t^{7/2}/\pi^{3/2}) \times \int d^3\rho\, U(\sqrt{2}\rho)\rho^4\exp(-t\alpha^2\rho^2)\ . \tag{4.3c}$$

Setting $a=a(1)$, $b=b(1)$, and $c=c(1)$, it follows that (see Appendix A)

$$E(S)=2m_u+m_s+\tfrac{3}{2}\omega\{1+[(2x+1)/3]^{1/2}\}+\langle U\rangle_s\ , \tag{4.4}$$

$$E(\rho)=2m_u+m_s+\tfrac{5}{2}\omega+\tfrac{3}{2}\omega[(2x+1)/3]^{1/2}+\langle U\rangle_\rho\ , \tag{4.5}$$

$$E(\lambda)=2m_u+m_s+\tfrac{3}{2}\omega+\tfrac{5}{2}\omega[(2x+1)/3]^{1/2}+\langle U\rangle_\lambda\ , \tag{4.6}$$

where

$$\langle U\rangle_s=\tfrac{1}{3}a+2\langle U(r_{13})\rangle_s =\tfrac{1}{3}a+2a(t)/3\ , \tag{4.7a}$$

$$\langle U\rangle_\rho=\tfrac{2}{9}b+\langle 2U(r_{13})\rangle_\rho =\tfrac{2}{9}b+\alpha^2 ta(t)/2\alpha_\lambda{}^2+\tfrac{1}{9}tb(t)\ , \tag{4.7b}$$

and

$$\langle U\rangle_\lambda=\tfrac{1}{3}a+2\langle U(r_{13})\rangle_\lambda =\tfrac{1}{3}a+\tfrac{1}{6}ta(t)+\alpha^2 tb(t)/3\alpha_\lambda{}^2\ , \tag{4.7c}$$

$$t=4\alpha_\lambda{}^2/(\alpha_\lambda{}^2+3\alpha^2)\ . \tag{4.8}$$

Hence, aside from hyperfine effects, the difference in mass between $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ is given by

$$(\Lambda\tfrac{5}{2}^- - \Sigma\tfrac{5}{2}^-)_0=\omega\left[1-\left[\frac{2x+1}{3}\right]^{1/2}\right] +\langle U\rangle_\rho-\langle U\rangle_\lambda\ . \tag{4.9}$$

The hyperfine interaction gives rise to an energy difference[2]:

$$(\Lambda\tfrac{5}{2}^- - \Sigma\tfrac{5}{2}^-)_{\rm hyp}=\delta/4[-16/15+(xy^2-x/3)f -xg/5+3xh/5]\ , \tag{4.10}$$

where

$$y=[(2x+1)/3]^{1/4},\quad f=32(3y^2+1)^{-5/2}\ ,$$

$$g=32y^2(3y^2+1)^{-3/2}(y^2+3)^{-1}(1+y^2)^{-1}(4-2y^2)$$

$$h=64y^2(3y^2+1)^{-3/2}(y^2+3)^{-1}/(1+y^2)\ ,$$

and δ is the parameter introduced in Eq. (3.3). The total mass difference between $\Lambda\frac{5}{2}^-$ and $\Sigma\frac{5}{2}^-$ is given by the sum of Eqs. (4.9) and (4.10) and involves the six parameters a, b, c, x, ω, and δ. (These parameters plus the parameter m_s are sufficient to determine the mass of all the ground-state, negative-parity, and positive-parity baryons). To obtain these values we use as inputs the masses of N, Δ, Σ, Λ, $\Sigma\frac{5}{2}^-(1765)$, $N(1670)$, $\Delta(1950)$. The latter two are used because they are "stretched" states and are not mixed by the hyperfine interaction with other states.

V. CALCULATION OF THE PARAMETERS AND DISCUSSION OF RESULTS

First consider the ground-state sector. As noted earlier, Isgur and Karl[1] showed that to fit the masses of these particles properly the mixing between the ground-state and the positive-parity baryons caused by the hyperfine interaction must be included. The relevant matrix elements for the $S=0$ sector are contained in paper I. For the $S=-1$ sector we have included the $\alpha_\lambda \neq \alpha_\rho$ effects neglected there. The corresponding matrix elements are given in Appendix B. We can immediately deduce the parameters x and δ from the ground-state baryon masses. In doing this we have chosen to use the following values for the masses of the ground-state baryons in the $S=0$ and $S=-1$ sectors[7]:

$$N=938.9,\quad \Lambda=1115.6,\quad \Sigma=1193.4,$$

$$\Delta=1232,\quad \Sigma^*=1385\ .$$

Only the value of Δ is significantly different from that of Isgur and Karl[1] ($\Delta=1240$ in their paper). The mass matrices for Δ and N can be parametrized in terms of δ and

$$E_0=3xm_s+3\omega+a\ . \tag{5.1}$$

By use of the experimental values of Δ and N, values for both of the parameters are obtained. Since Eqs. (4.4)–(4.9) and (B1)–(B8) only affect the $S=-1$ sector, our values of these parameters differ from those of Isgur and Karl only by the different choice of the mass of the Δ. Thus Isgur and Karl obtain $\delta=260$ MeV and we use $\delta=253$ MeV. Similarly the masses of Λ, Σ, and Σ^* are obtained in terms of x and

$$M_0(\Lambda,\Sigma)=(2x+1)m_s+(\tfrac{3}{2})\omega\{1+[(2x+1)/3]^{1/2}\}+\langle U\rangle_s\ , \qquad (5.2)$$

where $\langle U\rangle_s$ is given by Eq. (4.7a).

It is of course possible to use the experimental values of the masses of two of Λ, Σ, and Σ^* to obtain the parameters x and M_0 and predict the mass of the third particle. In practice we used a brute force method of inputting values and diagonalizing the mass matrices for all three particles simultaneously. The results for some selected values of M_0 and x are shown in Table I. Since agreement can be obtained down to the level of electromagnetic corrections there is some confidence in the parameters. It is suggested by the results that the value of x can be chosen to be between 0.50 and 0.60.

Equations (5.1) and (5.2) provide two of the required equations to evaluate the parameters ω, m_s, a, b, and c. In terms of the values of x and δ three more equations are obtained for the masses of $\Sigma\frac{5}{2}^-(1765)$, $N(1670)$, and $\Delta(1950)$. The procedure used was to input values of the masses of $N(1670)$, $\Delta(1950)$, and $\Sigma\frac{5}{2}^-(1765)$, solve for m_s, ω, a, b, and c, and substitute the value into Eqs. (4.9) and (4.10).

Results are given in Table II. We note that for $x=0.50$ a value of the $\Lambda\frac{5}{2}^-$-$\Sigma\frac{5}{2}^-$ difference consistent with experiment can be obtained using values of the masses of $N\frac{5}{2}^-$, $\Delta\frac{7}{2}^+$, and $\Sigma\frac{5}{2}^-$. For this value of x, the mass of the ground-state baryons (Table I) is also in very good agreement. The effective oscillator spacing from Table II is close to the oscillator frequency obtained for the p-wave baryons and thus this parameter set should yield the same general results as Isgur and Karl obtained in II. A full calculation of the masses of all the negative-parity and positive-parity baryons is planned to make sure that these indications work out in detail.

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TABLE I. Values of the mass of Λ, Σ, Σ^* for various values of the parameters x, M_0.

x	M_0 (MeV)	Particle	Mass (GeV)	% deviation from experiment
0.70	1295	Λ	1114.1	0.13
		Σ	1165.7	2.32
		Σ^*	1389.5	0.33
0.60	1295	Λ	1115.8	0.02
		Σ	1186.1	0.61
		Σ^*	1384.6	0.02
0.57	1295	Λ	1116.4	0.07
		Σ	1192.5	0.08
		Σ^*	1383.1	0.13
0.55	1297	Λ	1116.7	0.10
		Σ	1196.8	0.28
		Σ^*	1382.0	0.21
0.50	1295	Λ	1117.6	0.18
		Σ	1207.8	1.2
		Σ^*	1379.1	0.42

TABLE II. Fitting the $\Lambda\frac{5}{2}^- - \Sigma\frac{5}{2}^-$ mass difference

x	E_0[a] (MeV)	Ω[a] (MeV)	Δ[a] (MeV)	Mass (MeV) $N\frac{5}{2}^-(1650-1685)$[b]	$\Delta\frac{7}{2}^+(1910-1950)$[b]	$\Sigma\frac{5}{2}^-(1774\pm7)$[b]	$\Lambda\frac{5}{2}^- - \Sigma\frac{5}{2}^-(30-63)$[b]
0.55	1127	472	331	1650	1960	1783	30
	1127	483	331	1660	1980	1783	37
	1127	473	281	1650	1980	1763	42
	1127	473	306	1650	1970	1763	42
0.50	1127	478	381	1655	1950	1782	36
	1127	483	406	1660	1950	1782	39
	1127	473	306	1650	1970	1763	44
	1127	479	337	1655	1970	1763	48

[a]These parameters are defined by Eq. (3.2) of Sec. III.
[b]The values in parentheses are the experimental values. Values in the body of the table for $N\frac{5}{2}^-$, $\Delta\frac{7}{2}^+$, and $\Sigma\frac{5}{2}^-$ are input values.

APPENDIX A

The normalized harmonic-oscillator ground state is

$$\Psi_s = \psi_0(\alpha,\rho)\psi_0(\alpha_\lambda,\lambda) , \quad \text{(A1)}$$

where $\psi_0(\alpha,r) = \alpha^{3/2}\pi^{-3/4}\exp(-\alpha^2 r^2/2)$. Consider a change of coordinates as follows:

$$\vec{\rho} = a\vec{r}_{13} + b\vec{x} ,$$
$$\vec{\lambda} = c\vec{r}_{13} + d\vec{x} . \quad \text{(A2)}$$

Then

$$|\Psi_s|^2 = \alpha^3\alpha_\lambda^3\pi^{-3}\exp[-\alpha^2(a^2 r_{13}^2 + b^2 x^2) - \alpha_\lambda^2(c^2 r_{13}^2 + d^2 x^2) - 2\vec{x}\cdot\vec{r}_{13}(\alpha^2 ab + cd\alpha^2\lambda)] .$$

If we require that the coefficient of $\vec{x}\cdot\vec{r}_{13}$ vanish and that the Jacobian of the transformation Eq. (A2) be unity, then

$$\begin{bmatrix}\vec{\rho}\\ \vec{\lambda}\end{bmatrix} = \begin{bmatrix} t/(2\sqrt{2}) & -(\frac{3}{2})^{1/2} \\ t(\frac{3}{2})^{1/2}(\alpha^2/\alpha_\lambda^2) & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\vec{r}_{13}\\ \vec{x}\end{bmatrix} . \quad \text{(A3)}$$

In this coordinate system, the integrals for $\langle U(r_{13})\rangle_s$, $\langle U(r_{13})\rangle_\rho$, and $\langle U(r_{13})\rangle_\lambda$ are easy to do and the results [Eqs. (4.7)] follow.

APPENDIX B

The relevant hyperfine matrix elements are

$$\langle \Lambda^2 S_{\lambda\lambda}\tfrac{1}{2}^+ | H_{\rm hyp} | \Lambda^2 S\tfrac{1}{2}^+\rangle = 0 , \quad \text{(B1)}$$

$$\langle \Lambda^2 S_{\rho\lambda}\tfrac{1}{2}^+ | H_{\rm hyp} | \Lambda^2 S\tfrac{1}{2}^+\rangle = (16\sqrt{3})\delta q , \quad \text{(B2)}$$

$$\langle \Lambda^2 S_{\rho\rho}\tfrac{1}{2}^+ | H_{\rm hyp} | \Lambda^2 S\tfrac{1}{2}^+\rangle = \frac{\sqrt{6}}{4}\delta , \quad \text{(B3)}$$

$$\langle \Sigma^2 S_{\rho\rho}\tfrac{1}{2}^+ | H_{\rm hyp} | \Sigma^2 S\tfrac{1}{2}^+\rangle = -\frac{\sqrt{6}}{12}\delta + 8(\tfrac{2}{3})^{1/2}\delta r , \quad \text{(B4)}$$

$$\langle \Sigma^2 S_{\lambda\lambda}\tfrac{1}{2}^+ | H_{\rm hyp} | \Sigma^2 S\tfrac{1}{2}^+\rangle = 8\sqrt{6}\delta y^2 r , \quad \text{(B5)}$$

$$\langle \Sigma^2 S_{\rho\lambda}\tfrac{1}{2}^+ | H_{\rm hyp} | \Sigma^2 S\tfrac{1}{2}^+\rangle = 16\sqrt{3}\delta q , \quad \text{(B6)}$$

$$\langle \Sigma^4 S_{\lambda\lambda}\tfrac{3}{2}^+ | H_{\rm hyp} | \Sigma^4 S\tfrac{3}{2}^+\rangle = -4\sqrt{6}\delta y^2 r , \quad \text{(B7)}$$

$$\langle \Sigma^4 S_{\rho\rho}\tfrac{3}{2}^+ | H_{\rm hyp} | \Sigma^4 S\tfrac{3}{2}^+\rangle = -\frac{\sqrt{6}}{12}\delta - 4(\tfrac{2}{3})^{1/2}\delta r , \quad \text{(B8)}$$

where

$$q = xy^2(5+5y^4-6y^2)^{-1/2}(1+3y^2)^{-5/2} ,$$
$$r = x[1+3y^2]^{-5/2} ,$$

and

$$y = [(2x+1)/3]^{1/4} .$$

[1]N. Isgur and G. Karl, Phys. Rev. D 21, 3175 (1980).
[2]N. Isgur and G. Karl, Phys. Lett. 72B, 109 (1977); 74B, 353 (1978); Phys. Rev. D 18, 4187 (1978).
[3]N. Isgur and G. Karl, Phys. Rev. D 19, 2653 (1979).
[4]O. W. Greenberg, Ann. Rev. Nucl. Part. Sci. 28, 327 (1978); A. J. G. Hey, invited talk at the IVth General Conference of the European Physical Society, York, England, 1978, Southampton University Report No. THEP 77/8-27 (unpublished).
[5]C. S. Kalman, R. L. Hall, and S. K. Misra, Phys. Rev. D 21, 1908 (1980).
[6]R. L. Hall and B. Schwesinger, J. Math. Phys. 20, 2481 (1979).
[7]We choose $M(N)=\frac{1}{2}[M(n)+M(p)]$ and $M(\Sigma)=\frac{1}{2}[M(\Sigma^+)+M(\Sigma^-)]$. Δ is given to be between 1230 and 1236 in Particle Data Group, Phys. Lett. 75B, 1 (1978). The value for Δ is an educated guess.