Quantum fluctuations around topological singularities and the large-N limit in the CP^{N-1} model

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To analyze topological effects in quantum field theory, we propose a new scheme in which quantum fluctuations are considered around topological singularities rather than classical solutions. We study the (1 + 1)-dimensional CP^{N-1} model in detail, where topological singularities are labeled by the root lattice of SU(N). In this scheme, once topological singularities are extracted explicitly from field configurations, it is possible to calculate quantum effects by applying Feynman graph techniques. We obtain the following conclusions. (i) Ultraviolet as well as infrared divergences automatically disappear. (ii) Topological effects remain finite in the large-N limit. (iii) Topological effects do not lead to the confinement of fundamental representation charges. (iv) The long-distance behavior of the effective action is described by the SU(N) sine-Gordon model. All of these results are direct consequences of the group-theoretical structure of topological singularities, that is, their charges are on the root lattice of SU(N).

I. INTRODUCTION

An entirely new insight has been obtained in quantum field theories since topological excitations such as solitons and instantons were discovered.[1–3] The structure of a Lagrangian system would be modified significantly from the perturbative one if vacuum fluctuations are dominated by virtual creations of topological excitations. Such effects have been calculated by making use of semiclassical approximations in the Euclidean functional integral. However, there are at least two obstacles against these methods. First, the knowledge of classical solutions is indispensable in applying them. Thus, topological effects cannot be considered unless there are corresponding classical solutions. Second, the set of field configurations generated by quantum fluctuations around classical solutions might occupy a negligible portion of the field manifold over which we are to integrate in the generating functional. The primary aim of this paper is to present a new formalism in which we are able to calculate quantum effects of all kinds of topological excitations.

In this new formalism, we evaluate quantum fluctuations around *topological singularities* instead of classical solutions. These singularities are extracted from the field manifold by way of a change of integration variables, which is known as a singular gauge transformation in gauge theories.[4–6] Under such a change of variables, the matter field acquires a phase which is a multivalued function. We have used the terminology *topological singularities* to stress this specific feature, that is, the phase field becomes multivalued with branch-singularity points, lines, or sheets in the Euclidean space-time.

Actually this scheme has already been proposed and applied to various models.[7–11] However, no explicit calculations of quantum fluctuations have yet been performed. In this paper, we present a full detail of a quantum field theory of topological singularities in the (1 + 1)-dimensional CP^{N-1} model.[12–16] In particular, we carefully examine cancellation mechanisms of various divergences in this model.

A salient feature of this method is that, once we have extracted topological singularities which are pointlike objects, we are able to evaluate quantum corrections by applying Feynman graph techniques. Thus, calculations are considerably simplified as compared with standard semiclassical methods.[15,16]

We shall use the $1/N$ expansion to calculate quantum fluctuations around topological singularities. Because there are controversies on the relation of topological excitations and the $1/N$ expansion,[13–18] we wish to clarify the points as far as our formalism is concerned. The first problem is if the $1/N$ expansion produces topological effects.

The second problem is if topological effects survive the large-N limit.

The answer to the first one would be obvious when we recognize that the $1/N$ expansion is a technique to sum certain Feynman graphs in perturbation theory and that topological excitations are not generated as vacuum fluctuations within perturbation theory. Thus, by doing a simple $1/N$ expansion, we certainly miss all topological effects.[13] However, once we extract topological singularities by way of a change of integration variables, we are free to carry out the $1/N$ expansion to collect important Feynman graphs involving topological singularities.

In order to answer the second problem, we need to analyze the group-theoretical structure of topological singularities. First of all we emphasize that we are not dealing with instantons but topological objects in SU(N). It is essential to recognize that our topological singularities are labeled by the root lattice of SU(N) in the CP^{N-1} model, just as magnetic monopoles are labeled by the root lattice of SU(N) in SU(N) gauge theories.[19] Namely, each topological singularity carries charge vectors which are on the root lattice of SU(N). Then, it is quite easy to prove that the contribution of each topological singularity to the generating functional is exp[−(self-energy)] and remains finite in the large-N limit. (This self-energy is ultraviolet divergent but it is absorbed into the bare fugacity.) We wish to remark that our result is not in contradiction with that of Witten,[13] who has argued that instantons are suppressed since they behave as exp[$-N$(instanton action)]. This N dependence follows because instantons carry the U(1) charge and not the SU(N) charge. Therefore, instanton effects disappear but our topological effects remain in the large-N limit.

Although topological effects remain finite in the large-N limit, they have no connection with charge confinement in the CP^{N-1} model. The reason is as follows. This model possesses a local U(1) gauge symmetry as well as a global SU(N) gauge symmetry. The symmetry structure should not be altered by topological effects. Therefore, if charge confinement is to occur, it must occur by a long-range force associated with the local U(1) gauge symmetry. However, our topological singularities do not carry the U(1) charge, as we have stressed.

For completeness, we should remark that charge confinement is possible in models which have local SU(N) gauge symmetry, such as the Georgi-Glashow model,[8] the Higgs model,[11] and the Yang-Mills theory. Just as in the CP^{N-1} model, topological effects must remain finite in the large-N limit. However, in contrast with the CP^{N-1} model, these models possess local SU(N) gauge symmetries. For instance, in the $(1+1)$-dimensional SU(N) Higgs model, topological effects recover the local SU(N) gauge symmetry and the associated long-range Coulomb force confines fundamental-representation charges. In higher dimensions, it has generally been recognized that color flux would be squeezed into vortices and confine charges by topological excitations.

We now discuss some technical points. In our formalism, it is necessary to deal with two kinds of ultraviolet divergences. One is of a dynamical origin and the other is of a kinematical origin. Dynamical divergences refer to those which arise due to loop integrals. We shall show that such divergences are eventually canceled out because topological singularities are characterized by the root lattice of SU(N). Kinematical divergences arise because topological singularities are pointlike objects. They are separated as the self-energy term in the effective action. Then, these divergences are shown to be canceled by the corresponding divergences in the Jacobian induced by the change of integration variables. As a result, our results are free from any divergences.

We summarize our conclusions. First, topological effects remain finite in the large-N limit. Second, the effective action of topological singularities corresponds to a statistical system of particles bearing charges labeled by the root lattice of SU(N). The long-distance behavior is identical with that of a Coulomb gas. Hence, it is equivalent to the SU(N) sine-Gordon model. Our result agrees with that of instanton calculations for SU(2).[15,16] On the other hand, the short-distance behavior is considerably different from that of a naive Coulomb gas. When plus and minus charges approach each other, a repulsive Yukawa force becomes dominant over the attractive Coulomb force.

This paper is composed as follows. In Sec. II we review topological singularities and characterize them by the root lattice of SU(N). In Sec. III, expanding the effective action in a power series of $1/\sqrt{N}$, we show that topological effects contribute as an $O(1)$ term. Sections IV and V are devoted to explicit evaluations of the $O(1)$ term of the effective action. We use the proper-time regularization scheme[20] and obtain an exact formula which determines the long-distance structure. Then, we perform a perturbative calculation and analyze the short-distance behavior. In Sec. VI we derive an effective Lagrangian of the system. We also discuss the problem of charge confinement. Finally, in Sec. VII we summarize our conclusions.

II. TOPOLOGICAL SINGULARITIES

It is well known that topological charges are defined as a global concept.[4,5] This is essential when we treat topological solitons, because for a soliton to be a stable physical object it must carry a superselection-rule charge.[6] However, this definition is not adequate when we treat instantons or any other excitations in order to analyze the vacuum structure, because in so doing we need to integrate over all field configurations in the generating functional. Thus, even if the functional space is restricted to the topologically trivial sector, each field configuration may contain "topological" excitations as local lumps. Furthermore, the essential property of topological excitations in this context is not that they carry superselection-rule charges but that they are not created as vacuum fluctuations within perturbative field theory.

Let us call a field configuration singular when it is not obtainable within perturbative field theory. Then, it would be convenient to introduce a concept of *topological singularities* so that a field configuration is singular provided that it contains these singularities. In gauge theories these singularities are generated by performing singular gauge transformations.

Let us consider the (1 + 1)-dimensional Abelian Higgs model as the simplest example.[6] The model is defined by

$$\mathscr{L}=\tfrac{1}{4}F_{\mu\nu}{}^2+|(\partial_\mu+iA_\mu)\phi|^2+V(\phi)\ ,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ , \tag{2.1}$$

where the topological charge is

$$Q=\frac{1}{2\pi}\int d^2x\,\epsilon_{\mu\nu}\partial_\mu A_\nu\ . \tag{2.2}$$

It is convenient to parametrize

$$\phi=\rho e^{i\chi},\ \ A_\mu=U_\mu-\partial_\mu\chi\ . \tag{2.3}$$

Because ρ and U_μ become massive, it is obvious that the phase field χ carries the topological charge. It is to be emphasized that field χ is a multivalued function when it carries a nonzero topological charge. The branch-singularity point is identified with the vortex center. It is easy to convince oneself that topological singularities are introduced as branch-singularity points of phase field χ in this model. Then, we are able to construct a field configuration containing topological singularities in an arbitrary manner by requiring

$$\epsilon_{\mu\nu}\partial_\mu\partial_\nu\chi=2\pi\sum_j q_j\delta^{(2)}(x-a_j)\ , \tag{2.4}$$

with q_j being integers.

In quantum field theory we calculate the generating functional

$$Z=\int[d\phi][d\bar{\phi}][dA_\mu]\exp\left[-\int\mathscr{L}\right] \tag{2.5a}$$

$$=\int[\rho\,d\rho][d\chi][dU_\mu]\exp\left[-\int\mathscr{L}\right]. \tag{2.5b}$$

In this formula we may recall that a field configuration containing topological singularities is singular because a multivalued field χ cannot be approximated by Gaussian fluctuations around the vacuum ($\chi=0$). In order to analyze (2.5) within perturbation theory, it is necessary to extract topological singularities explicitly and then to evaluate quantum fluctuations around them. Namely, we decompose

$$\chi=\chi^r+\chi^{\rm cl} \tag{2.6}$$

in (2.5b), or set

$$\phi=\phi^r\exp(i\chi^{\rm cl})\ ,\qquad A_\mu=A_\mu^r-\partial_\mu\chi^{\rm cl}\ , \tag{2.7}$$

in (2.5a). Here, $\chi^{\rm cl}$ represents a particular solution to (2.4) such that

$$\chi^{\rm cl}=\sum_j q_j\theta_j\ , \tag{2.8}$$

θ_j being an azimuthal angle around singularity point a_j. We remark that the change of variables (2.7) is precisely known as a singular gauge transformation. It should be emphasized that in our formalism the introduction of topological singularities does not depend on the existence of classical solutions.

We have analyzed the Abelian Higgs model in detail, although what we have described is common knowledge of physicists. However, the generalization to the CP^{N-1} model is much less trivial. We now discuss this case.

The Lagrangian is given by[12–14]

$$\mathscr{L}=\frac{N}{2f}\sum_{\alpha=1}^{N}|(\partial_\mu+iA_\mu)w_\alpha|^2\ , \tag{2.9}$$

together with the constraint

$$\sum_{\alpha=1}^{N} |w_\alpha|^2=1 . \tag{2.10}$$

Field A_μ is regarded as an auxiliary variable. This Lagrangian possesses a local U(1) gauge symmetry,

$$\begin{aligned} &w_\alpha \to e^{ig} w_\alpha , \\ &A_\mu \to A_\mu - \partial_\mu g , \end{aligned} \tag{2.11}$$

in addition to a global SU(N) symmetry. The topological charge is

$$Q=\frac{1}{2\pi}\int d^2x\, \epsilon_{\mu\nu}\partial_\mu A_\nu , \tag{2.12}$$

as in the Abelian Higgs model.

It is to be noted that the topological charge is defined only with respect to the local U(1) gauge symmetry; it is entirely blind to the global SU(N) charges. In the case of the Abelian Higgs model, the introduction of topological singularities is not conceptually new because they carry the Abelian charge after all. However, this is not true in the CP^{N-1} model.

The generating functional is given by

$$Z=\prod_\alpha \int [dw_\alpha][d\bar{w}_\alpha][dA_\mu]\exp\left[-\int \mathscr{L}\right] \tag{2.13a}$$

$$=\prod_\alpha \int [\rho_\alpha d\rho_\alpha][d\chi_\alpha][dU_\mu]\exp\left[-\int \mathscr{L}\right] , \tag{2.13b}$$

where we have parametrized

$$\begin{aligned} &w_\alpha=\rho_\alpha \exp(i\chi_\alpha) , \\ &A_\mu=U_\mu-\frac{1}{N}\sum_\alpha \partial_\mu \chi_\alpha . \end{aligned} \tag{2.14}$$

As in the Abelian Higgs model, topological singularities may be incorporated by allowing χ_α to be multivalued:

$$\epsilon_{\mu\nu}\partial_\mu\partial_\nu\chi_\alpha=2\pi\sum_j q_j^\alpha \delta^{(2)}(x-a_j^\alpha) , \tag{2.15}$$

q_j^α being integers. In order to calculate quantum fluctuations around them, we decompose

$$\begin{aligned} &\chi_\alpha=\chi_\alpha^r+\chi_\alpha^{\mathrm{cl}} , \\ &A_\mu=A_\mu^r-\frac{1}{N}\sum_\alpha \partial_\mu \chi_\alpha^{\mathrm{cl}} , \end{aligned} \tag{2.16}$$

in (2.13b), or make a change of variables

$$\begin{aligned} &w_\alpha=w_\alpha^r \exp(i\chi_\alpha^{\mathrm{cl}}) , \\ &A_\mu=A_\mu^r-\frac{1}{N}\sum_\alpha \partial_\mu \chi_\alpha^{\mathrm{cl}} , \end{aligned} \tag{2.17}$$

in (2.13a), where

$$\chi_\alpha^{\mathrm{cl}}=\sum_j q_j^\alpha \theta_j^\alpha , \tag{2.18}$$

θ_j^α being an azimuthal angle around singularity point a_j^α. Then, we perform Gaussian integrations over w_α^r. We note that topological singularities act effectively as point sources accompanied by "colored" potentials $A_\mu^{\alpha,\mathrm{ext}}$:

$$\begin{aligned} A_\mu^{\alpha,\mathrm{ext}} &= \sum_j q_j^\alpha \partial_\mu \theta_j^\alpha \\ &= \sum_j q_j^\alpha \epsilon_{\mu\nu}\frac{(x-a_j^\alpha)_\nu}{(x-a_j^\alpha)^2} . \end{aligned} \tag{2.19}$$

Thus, the whole problem is reduced to the evaluation of quantum fluctuations in the presence of external potentials (2.19).

A comment is in order. When $\chi_\alpha^{\mathrm{cl}}$ does not depend on indices α, the change of variables (2.17) is merely a local gauge transformation (2.11). It is easy to check that the overall phase χ^{cl} is canceled out completely from Lagrangian (2.9). Note that the CP^{N-1} model is very different on this point from the Abelian Higgs model, where such a change of variables gives rise to topological singularities. The reason is very simple: In the CP^{N-1} model, the kinetic term for the gauge potential is absent which is to acquire a nonzero contribution from a singular gauge transformation. Therefore, we should only consider topological singularities such that

$$\sum_{\alpha=1}^{N} \chi_\alpha^{\mathrm{cl}}=0 \tag{2.20a}$$

or

$$\sum_{\alpha=1}^{N} q_j^\alpha=0 . \tag{2.20b}$$

As we shall show in succeeding sections, this property removes ultraviolet divergences from the system of topological singularities.

We now discuss the group-theoretical structure of colored charges q_j^α carried by topological singularities. Taking one singularity point at the origin, we express (2.17) as

$$w=\exp(i\theta\Omega)w^r , \tag{2.21}$$

$$\Omega=\mathrm{diag}(q^1,\ldots,q^\alpha,\ldots,q^N),\quad \sum_{\alpha=1}^{N} q^\alpha=0 , \tag{2.22}$$

in matrix notation. It is possible to write (2.22) in terms of diagonal Gell-Mann matrices:

$$\Omega=\sum_{H=1}^{N-1} \eta^H \lambda^H , \tag{2.23}$$

where

$$\lambda^H=\left[\frac{2}{H(H+1)}\right]^{1/2}\text{diag}(1,\ldots,1,-H,0,\ldots,0)\,, \tag{2.24}$$

with 1 appearing H times. The quantization condition for charge q^α may be reformulated as

$$\exp(i2\pi\Omega)=1\,, \tag{2.25}$$

with (2.22). Thus, the condition for charges η^H is given by (2.25) with (2.23). The same problem has been analyzed in detail for the classification of monopoles in SU(N) gauge theories.[19] We only cite the result.[11] It follows that an $(N-1)$-dimensional vector $\vec{\eta}=(\eta^1,\ldots,\eta^{N-1})$ is on the root lattice of group SU(N). All these vectors are constructed from $N-1$-independent elementary root vectors $\vec{r}_s$:

$$\vec{\eta}=\sum_{s=1}^{N-1} n_s\vec{r}_s\,, \tag{2.26}$$

with n_s being integers. For instance, we may choose

$$\vec{r}_1=(e_1,e_2,\ldots,Ne_{N-1})\,,$$

$$\vec{r}_s=(0,\ldots,0,(1-s)e_{s-1},e_s,\ldots,Ne_{N-1})$$

(with $s-2$ zeros),

$$\vec{r}_{N-1}=(0,\ldots,0,(2-N)e_{N-2},Ne_{N-1})\,, \tag{2.27}$$

with

$$e_s=1/[2s(s+1)]^{1/2}\quad(1\le s\le N-1)\,. \tag{2.28}$$

Note that

$$\vec{r}_s\cdot\vec{r}_t=\begin{cases}1 & \text{for } s=t\,,\\ \frac{1}{2} & \text{for } s\neq t\,.\end{cases} \tag{2.29}$$

Thus, topological singularities are characterized by the root lattice of SU(N).

In actual calculations, however, it is more convenient to use $\vec{q}$ charges rather than $\vec{\eta}$ charges, where $\vec{q}$ is an N-component vector $\vec{q}=(q^1,\ldots,q^\alpha,\ldots,q^N)$. Corresponding to $N-1$-independent elementary root vectors (2.27), we may construct $N-1$-independent vectors $\vec{q}_s$ by way of (2.22) and (2.23):

$$\vec{q}_1=(1,0,\ldots,0,-1),$$

$$\vec{q}_s=(0,\ldots,0,1,0,\ldots,0,-1) \tag{2.30}$$

(with $s-1$ zeros preceding the 1),

$$\vec{q}_{N-1}=(0,\ldots,0,1,-1)\,.$$

Note that

$$\vec{q}_s\cdot\vec{q}_t=\begin{cases}2 & \text{for } s=t\,,\\ 1 & \text{for } s\neq t\,.\end{cases} \tag{2.31}$$

An arbitrary vector $\vec{q}$ is written as

$$\vec{q}=\sum_{s=1}^{N-1} n_s\vec{q}_s\,.$$

Identifying $\vec{q}$ with $\vec{\eta}$, we also call $\vec{q}$ the root vector of SU(N).

We conclude this section by summarizing that the charge of a topological singularity is labeled by the root lattice of SU(N). Therefore, the independent degrees of freedom in the generating functional (2.13) are given by field variables w^r and A_μ^r as well as locations a_j^q and charges $\vec{q}$ of topological singularities.

III. 1/N EXPANSIONS

The generating functional for the CP^{N-1} model is given by

$$Z=\int[dw][d\bar{w}][dA_\mu]\delta\left(|w|^2-\frac{N}{2f}\right)e^{-S} \tag{3.1}$$

with

$$S=\int d^2x\,|D_\mu w|^2,\quad D_\mu=\partial_\mu+\frac{i}{\sqrt{N}}A_\mu\,. \tag{3.2}$$

We have rescaled w_α by a factor $(N/2f)^{1/2}$ and A_μ by $N^{-1/2}$. We rewrite (3.1) as

$$Z=\int[dw][d\bar{w}][d\sigma][dA_\mu]e^{-S'}\,, \tag{3.3}$$

with

$$S'=\int d^2x\left[-\bar{w}D_\mu D_\mu w+\left[m^2-\frac{i}{\sqrt{N}}\sigma\right]\bar{w}\cdot w+\frac{i\sqrt{N}}{2f}\sigma\right], \tag{3.4}$$

where we have introduced a mass parameter $m^2>0$ which is irrelevant at this stage.

In order to separate topological singularities, we make a change of variables (2.17), or

$$\begin{aligned} w_\alpha &= w_\alpha^r \exp\left[i\sum_j q_j^\alpha \theta_j^\alpha\right], \\ A_\mu &= A_\mu^r, \end{aligned} \tag{3.5}$$

where $\vec{q}_j=(q_j{}^1,\ldots,q_j{}^N)$ is a root vector of SU(N). Note that topological singularities do not contribute to the gauge potential A_μ owing to (2.20a). When we rewrite (3.4) in terms of a new integration variable w_α^r, special care is needed for the phase factor in (3.5). To define it unambiguously at the branch-singularity point a_j^q, we must use the complex z plane, where $\partial_x=\partial_z+\partial_{\bar{z}}$, $\partial_y=i(\partial_z-\partial_{\bar{z}})$, and $\partial_\mu{}^2=4\partial_z\partial_{\bar{z}}$. Then,

$$i\theta_j^q=\tfrac{1}{2}\ln[(z-a_j^q)/(\bar{z}-\bar{a}_j^q)] \tag{3.6a}$$

or

$$\exp(i\theta_j^q)=[(z-a_j^q)/(\bar{z}-\bar{a}_j^q)]^{1/2}. \tag{3.6b}$$

Now, we obtain

$$\begin{aligned} S'=\int d^2x\Bigg[&4\sum_{\alpha=1}^N \bar{w}_\alpha^r D_+^\alpha D_-^\alpha w_\alpha^r \\ &+\left[m^2-\frac{i}{\sqrt{N}}\sigma-\frac{1}{\sqrt{N}}\epsilon_{\mu\nu}\partial_\mu A_\nu\right] \\ &\times \bar{w}^r\cdot w^r+\frac{i\sqrt{N}}{2f}\sigma\Bigg], \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} D_+^\alpha &= I_+^\alpha+\frac{i}{\sqrt{N}}A_+, \\ D_-^\alpha &= I_-^\alpha-\frac{i}{\sqrt{N}}A_-, \\ A_\pm &= (A_x\mp iA_y)/2, \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} I_+^\alpha &= \exp\left[-i\sum_j q_j^\alpha\theta_j^\alpha\right]\partial_z \exp\left[i\sum_j q_j^\alpha\theta_j^\alpha\right] \\ &= \partial_z+\sum_j \frac{q_j^\alpha}{2}\left[\frac{1}{z-a_j^q}+\pi(\bar{z}-\bar{a}_j^q)\delta^{(2)}(x-a_j^q)\right], \\ I_-^\alpha &= (I_+^\alpha)^\dagger. \end{aligned} \tag{3.9}$$

A comment is in order. The last term in the square brackets of (3.9) has been derived by making use of the formula $\partial_z(1/\bar{z})=\pi\delta^{(2)}(x)$. We cannot *a priori* neglect this term because it gives rise to a contact term $q_i^\alpha q_j^\alpha\delta_{ij}\delta^{(2)}(x-a_j^q)$ in $I_+^\alpha I_-^\alpha$. When we carry out a perturbative calculation in q_j^α, it plays an important role in removing kinematical divergences caused by the pointlike character of topological singularities.

We next examine the Jacobian induced by the change of integration variable (3.5). The independent degrees of freedom are given by field variables w^r and A_μ^r as well as location a_j^q and charge $\vec{q}$ of topological singularities. Because this change of variables involves only a linear transformation as is obvious in the form of (2.13b) with (2.16), the associated Jacobian is just a c number.[9,10] Hence,it follows that

$$\int[dw_\alpha][d\bar{w}_\alpha]=\not{\Sigma}\int[dw_\alpha^r][d\bar{w}_\alpha^r], \tag{3.10}$$

where $\not{\Sigma}$ indicates the integration over locations a_j^q and the summation over charges $\vec{q}$; the vector $\vec{q}$ runs over all points on the root lattice of SU(N). Thus,

$$\not{\Sigma}=\sum_{n=0}^\infty\sum_{\vec{q}}\left[\prod\frac{1}{N_q!}\right]\int\prod_j^n(\lambda_0 d^2a_j^q), \tag{3.11}$$

where λ_0 is a bare fugacity. It has dimensions [(mass)^2] and must be there on dimensional grounds. Since the topological singularity is a pointlike object, we have $\lambda_0\sim\delta^{(2)}(0)$.[9] By regularizing a singularity by

$$\delta^{(2)}(x)=\int_{|k|<\Lambda}\frac{d^2k}{(2\pi)^2}\exp(ikx), \tag{3.12}$$

we obtain $\lambda_0=c\Lambda^2$, where c is an inessential numerical constant.

We go on to perform a Gaussian integration over w_α^r. We obtain

$$Z=\not{\Sigma}\int[d\sigma][dA_\mu]\exp(-S_{\rm eff}), \tag{3.13}$$

where

$$\begin{aligned} S_{\rm eff}=\sum_\alpha {\rm Tr}\ln\Bigg[&4D_+^\alpha D_-^\alpha \\ &+m^2-\frac{i}{\sqrt{N}}\sigma-\frac{1}{\sqrt{N}}\epsilon_{\mu\nu}\partial_\mu A_\nu\Bigg] \\ &+i\frac{\sqrt{N}}{2f}\int d^2x\,\sigma(x). \end{aligned} \tag{3.14}$$

The effective action can be expanded in a power series of $1/\sqrt{N}$:

$$S_{\rm eff}=\sum_{\nu=1}^{\infty} N^{1-\nu/2}S^{(\nu)} . \quad (3.15)$$

It should be emphasized that topological singularities are labeled by the root lattice of SU(N) and that charges q_j^α assume values in a special set (2.32). Therefore, the contribution of each topological singularity to the effective action is $O(1)$. We note that, if topological singularities carried the U(1) charge, the contribution had been $O(N)$. Hence, as in the absence of topological singularities,[13,14] the $O(\sqrt{N})$ term is given by

$$S^{(1)}=\frac{i}{2f}\int d^2x\, \sigma(x) - i\,\mathrm{Tr}(\sigma G_{xy}) , \quad (3.16)$$

where

$$G_{xy}=\int \frac{d^2p}{(2\pi)^2}\frac{e^{ip(x-y)}}{p^2+m^2} =\frac{1}{2\pi}K_0(m\,|x-y\,|) \quad (3.17)$$

is a modified Bessel function. For $x=y$, it reads

$$G_{xx}=\frac{1}{4\pi}\ln(\Lambda^2/m^2) , \quad (3.18)$$

Λ being a momentum cutoff. Then, the saddle-point condition $S^{(1)}=0$ may be imposed, which is equivalent to choosing the arbitrary parameter m^2 such that

$$m^2=\Lambda^2\exp\left[-\frac{2\pi}{f}\right] . \quad (3.19)$$

Thus, m is the renormalization-group-invariant mass of w particles. It is to be remarked that the mass of w particles is not modified by the existence of topological excitations.

The $O(1)$ terms are found to be

$$S^{(2)}=S_0^{(2)}+\sum_\alpha \Gamma^\alpha , \quad (3.20)$$

where $S_0^{(2)}$ is the contribution in the absence of topological singularities, and

$$\Gamma^\alpha=\mathrm{Tr}\ln(4I_+^\alpha I_-^\alpha+m^2) . \quad (3.21)$$

An explicit expression of $S_0^{(2)}$ has been given in Refs. 13 and 14. Our major task is the calculation of (3.21), which is the topic of the following two sections.

As is obvious in (3.20), there are no interactions between topological singularities and gauge potential A_μ in the leading order of $1/N$. Their interactions appear in $O(1/\sqrt{N})$. We shall postpone the analysis of $O(1/\sqrt{N})$ terms to a future paper.

We summarize that the contribution of topological singularities to the effective action is $O(1)$ and given by formula (3.21) in the large-N limit.

IV. CALCULATION OF DETERMINANTS *A*

In this section we analyze (3.21) by making use of the proper-time regularization.[20] Let us start with a brief review of this method, because it is not yet a common tool of physicists.

The first problem is to extract the ultraviolet divergences from $\mathrm{Tr}\ln Q$, where Q is an elliptic operator[20]:

$$Q=-g^{\mu\nu}\partial_\mu\partial_\nu+a^\mu\partial_\mu+b . \quad (4.1)$$

Let λ_i be a nonzero eigenvalue of operator Q. We note that

$$\int_\epsilon^\infty \frac{dt}{t}\exp(-t\lambda_i)=-\ln\lambda_i-\ln\epsilon e^{-\gamma} +O(\epsilon^s),\ \ s>0 , \quad (4.2)$$

with $\epsilon=\Lambda^{-2}$, Λ being understood as an ultraviolet cutoff of the theory. Therefore, we may write

$$\mathrm{Tr}\ln Q=-\int_\epsilon^\infty \frac{dt}{t}\mathrm{Tr}(e^{-tQ}-\hat{\Pi})-(\text{ultraviolet-divergent terms}) , \quad (4.3)$$

where $\hat{\Pi}$ is the operator subtracting the zero modes ($\lambda_i=0$). It is possible to define the regularized $\mathrm{Tr}\ln Q$ by formula (4.3) in the limit $\epsilon\to 0$. This scheme is known as the proper-time regularization.

In order to determine the ultraviolet-divergent terms, we may use the Seeley expansion[21] which is valid for $t\approx+0$;

$$e^{-tQ}=\sum_{n=0}^{\infty}E_n(Q)t^{(n-d)/2} , \quad (4.4)$$

where d (space-time dimension)$=2$ in our model. The coefficients $E_n(Q)$ have been calculated in the literature.[21,22] By substituting (4.4) into (4.3), the ultraviolet terms are explicitly fixed. Then, subtracting these terms, we obtain the regularized $\operatorname{Tr}\ln Q$ as

$$\operatorname{Tr}\ln Q = -\int_\epsilon^\infty \frac{dt}{t}\operatorname{Tr}(e^{-Qt}-\hat{\Pi}) - \sum_{n=0}^{d-1}\frac{2}{n-d}a_n\epsilon^{(n-d)/2} - (a_d - p)\ln\epsilon \;, \tag{4.5}$$

where $a_n = \operatorname{Tr}E_n$ and p is the number of the zero modes.

When the operator Q is of the form

$$Q = -D_\mu D_\mu + X \;, \tag{4.6}$$

D_μ being a covariant derivative in flat space-time, the Seeley coefficients are particularly simple[21,22]:

$$\begin{aligned} &E_0(Q)=(4\pi)^{-d/2}\;,\quad E_{2n+1}(Q)=0\;,\\ &E_2(Q)=-(4\pi)^{-d/2}X\;,\\ &E_4(Q)=(4\pi)^{-d/2}\left(\tfrac{1}{2}X^2-\tfrac{1}{12}F_{\mu\nu}{}^2+\tfrac{1}{6}\partial^2X\right)\;,\end{aligned} \tag{4.7}$$

etc., where $F_{\mu\nu}$ is the curvature.

We proceed to use the above method for the evaluation of (3.21), that is, of $\Gamma^\alpha = \operatorname{Tr}\ln Q^\alpha$ with

$$Q^\alpha = 4I_+^\alpha I_-^\alpha + m^2 \;. \tag{4.8}$$

In the proper-time regularization, however, operator Q is required not to contain singular functions. We need to regularize singular quantities. It is quite easy to show that the net effect is just to drop off the contact term in $I_+^\alpha I_-^\alpha$. Therefore, we introduce operators

$$\begin{aligned} &J_+^\alpha = \partial_z + \tfrac{1}{2}\sum_j q_j^\alpha (z-a_j^q)^{-1}\;,\\ &J_-^\alpha = (J_+^\alpha)^\dagger\;,\end{aligned} \tag{4.9}$$

omitting the last term in (3.9). Then,

$$\begin{aligned} J_+^\alpha J_-^\alpha &= I_+^\alpha I_-^\alpha - \frac{\pi}{2}\sum_j (q_j^\alpha)^2\delta^{(2)}(x-a_j^q)\\ &= -\frac{1}{4}\left[\partial_\mu - i\sum_j q_j^\alpha \frac{\epsilon_{\mu\nu}(x-a_j^q)_\nu}{(x-a_j^q)^2}\right] + \frac{\pi}{2}\sum_j q_j^\alpha \delta^{(2)}(x-a_j^q)\;,\end{aligned} \tag{4.10}$$

and

$$[J_+^\alpha, J_-^\alpha] = \pi\sum_j q_j^\alpha \delta^{(2)}(x-a_j^q)\;. \tag{4.11}$$

We may use $J_\pm^\alpha$ as the regularized expression of $I_\pm^\alpha$. Hence, we find that operator Q^α is formally of the type (4.6) with

$$X = m^2 + 2\pi\sum_j q_j^\alpha \delta^{(2)}(x-a_j^q)\;. \tag{4.12}$$

In formulas (4.9), (4.11), and (4.12), it is understood that $(z-a_j^q)^{-1}$ and $\delta^{(2)}(x-a_j^q)$ have been smeared out.

Now, the Seeley coefficients (4.7) may be explicitly calculated. As a result, we obtain

$$\Gamma^\alpha = -\int_\epsilon^\infty \frac{dt}{t}\operatorname{Tr}(e^{-tQ^\alpha}) + \frac{1}{4\pi}(\Lambda^2 - m^2\ln\Lambda^2)\left[\int d^2x\right] - \sum_j q_j^\alpha \ln\Lambda\;, \tag{4.13}$$

where we have set $\hat{\Pi}=0$ because operator Q^α has no zero modes ($Q^\alpha > 0$). When singular functions are smeared out as we have stated, there are no other divergences but those displayed in (4.13). Here, the second term corresponds to $\operatorname{Tr}\ln(-\partial^2+m^2)$ and should be canceled by normalization. On the other hand, the third term is proportional to the total topological charges $\sum_j q_j^\alpha$. However, for each topological singularity, the net contribution is exactly zero because $\sum_\alpha q_j^\alpha = 0$, as we have stressed in Sec. II. Thus, quantum fluctuations around topological singularities do not produce any ultraviolet divergences.

The next problem is to analyze the finite part of $\operatorname{Tr}\ln Q^\alpha$. Although it is impossible to calculate it explicitly, we are able to determine the dependence of $\operatorname{Tr}\ln Q^\alpha$ on locations a_j^q of topological singularities. Following Ref. 15, we evaluate

$$\delta\operatorname{Tr}\ln Q^\alpha = -\delta\int_0^\infty \frac{dt}{t}\operatorname{Tr}(e^{-tQ^\alpha})\;, \tag{4.14}$$

where δ denotes a variation of parameters a_j^q. We have set $\epsilon=0$ in (4.14) because $\mathrm{Tr}E_0$ and $\mathrm{Tr}E_2$ do not depend on parameters a_j^q.

It is convenient to rewrite (4.9) as

$$J_+^\alpha=\exp(-\sigma^\alpha)\partial_z \exp(\sigma^\alpha) , \tag{4.15}$$

where

$$\sigma^\alpha=\tfrac{1}{2}\sum_j q_j^\alpha \ln(z-a_j^q) . \tag{4.16}$$

Note that

$$\partial^2(\sigma^\alpha+\bar{\sigma}^\alpha)=2\pi\sum_j q_j^\alpha \delta^{(2)}(x-a_j^q) . \tag{4.17}$$

Then, by setting

$$R_+^\alpha(t)=\exp(-4tJ_+^\alpha J_-^\alpha) ,$$
$$R_-^\alpha(t)=\exp(-4tJ_-^\alpha J_+^\alpha) , \tag{4.18}$$

for notational simplicity, it follows from (4.14) that

$$\begin{aligned}\delta\,\mathrm{Tr}\ln Q^\alpha&=4\int_0^\infty dt\,\mathrm{Tr}[\delta(J_+^\alpha J_-^\alpha)e^{-tQ^\alpha}]\\&=\int_0^\infty dt\,e^{-tm^2}\frac{\partial}{\partial t}\mathrm{Tr}\{\,\delta(\sigma^\alpha+\bar{\sigma}^\alpha)[R_+^\alpha(t)-R_-^\alpha(t)]\,\} .\end{aligned} \tag{4.19}$$

Let us assume that the mass parameter m^2 is very large, since we are mainly interested in the infrared behavior of $\mathrm{Tr}\ln Q^\alpha$. Then, the integration region $t\approx 0$ is important in (4.19). Therefore, we can use the Seeley expansion of $R^\alpha(t)$:

$$R_+^\alpha(t)-R_-^\alpha(t)=\sum_{n=0}^\infty t^{(n-d)/2}[E_n(4J_+^\alpha J_-^\alpha)-E_n(4J_-^\alpha J_+^\alpha)] . \tag{4.20}$$

Inserting (4.20) into (4.19), and integrating over t, we obtain

$$\delta\,\mathrm{Tr}\ln Q^\alpha=\sum_{n=2}^\infty \Gamma\left[\frac{n-d}{2}+1\right]m^{-(n-d)}\mathrm{Tr}\{\,[E_n(4J_+^\alpha J_-^\alpha)-E_n(4J_-^\alpha J_+^\alpha)]\delta(\sigma+\bar{\sigma})\,\} . \tag{4.21}$$

Here, all the terms with $n>d$ vanish in the limit $m\to\infty$.

Then, keeping only the first term in (4.21), we obtain

$$\delta\,\mathrm{Tr}\ln Q^\alpha\,\hat{=}\,-\int d^2x\sum_j q_j^\alpha\delta^{(2)}(x-a_j^q)\delta(\sigma^\alpha+\bar{\sigma}^\alpha) , \tag{4.22}$$

where we have used formula (4.7) together with

$$X=2\pi\sum_j q_j^\alpha\delta^{(2)}(x-a_j^q)$$

and relation (4.11). The symbol $\hat{=}$ indicates the equality of the infrared structure. Finally, inserting (4.17) into (4.22), we obtain

$$\begin{aligned}\delta\,\mathrm{Tr}\ln Q^\alpha&\,\hat{=}\,\tfrac{1}{4}\delta\int d^2x\,[\partial_\mu(\sigma^\alpha+\bar{\sigma}^\alpha)]^2\\&=\delta\Big[-\tfrac{1}{4}\sum_{i,j}q_i^\alpha q_j^\alpha\ln(a_i^q-a_j^q)^2\Big]\end{aligned} \tag{4.23}$$

or

$$\mathrm{Tr}\ln Q^\alpha\,\hat{=}\,-\tfrac{1}{2}\sum_{i,j}q_i^\alpha q_j^\alpha\ln|a_i^q-a_j^q|+(\text{terms independent of } a\text{'s}) . \tag{4.24}$$

This is the main result obtained in this section. It gives the exact formula which determines the long-distance structure of topological singularities. We note that it is not surprising to have a logarithmic interaction since our topological singularities are quite similar to those in the XY model. The short-distance structure cannot be obtained by this method. We shall solve this problem in the next section.

V. CALCULATION OF DETERMINANTS B

In this section we calculate (3.21) in a power series of q_j^α. Before evaluating Feynman integrals explicitly, we make some comments. As we have stressed in the previous section, we would not have any ultraviolet divergences due to quantum effects except the one which is linear in q_j^α [see (4.13)]. Therefore, provided that this term is subtracted out, no specific regularization methods would be necessary such as the dimensional or the Pauli-Villars methods. On the other hand, we will get ultraviolet divergences which have a kinematical origin because topological singularities are point-like objects. To deal with them, we represent these objects consistently in the momentum space and use a momentum cutoff as a transient means. In actual calculations, keeping track of all δ functions, we find that cancellations between various would-be divergences actually occur.

Let us start calculations. First, inserting (3.9) into (3.21), we obtain

$$\Gamma^\alpha=\mathrm{Tr}\ln\left\{-\partial^2+m^2+\sum_j q_j^\alpha\left[2\pi\delta^{(2)}(x-a_j^q)-2i\epsilon_{\mu\nu}\frac{(x-a_j^q)_\mu\partial_\nu}{(x-a_j^q)^2}\right]+\sum_{i,j}q_i^\alpha q_j^\alpha\left[2\pi\delta_{ij}\delta^{(2)}(x-a_j^q)+\frac{(x-a_j^q)_\mu(x-a_i^q)_\mu}{(x-a_j^q)^2(x-a_i^q)^2}\right]\right\}. \tag{5.1}$$

The contact term $2\pi\delta_{ij}\delta^{(2)}(x-a_j^q)$ in the last set of square brackets has arisen as a crossing term between the last two terms in (3.9). The existence of this term is crucial in removing ultraviolet divergences of the kinematical origin.

The first-order term in q_j^α is easily extracted from (5.1), which reads

$$\Gamma_1^\alpha=\mathrm{Tr}\left[2\pi\sum_j q_j^\alpha\delta^{(2)}(x-a_j^q)G_{xy}\right]=\sum_j q_j^\alpha\ln(\Lambda/m)\,. \tag{5.2}$$

This is precisely the last term in (4.13) that has been derived in the proper-time formulation. As we have noted there, this divergence is harmless since we have $\sum_\alpha\Gamma_1^\alpha=0$ because of the property (2.20b).

We now calculate the second-order term in q_j^α. It is necessary to analyze the terms which are proportional to $(q_j^\alpha)^2$ and $q_i^\alpha q_j^\alpha$ $(i\neq j)$ separately, because there is a contact term $2\pi\delta_{ij}\delta^{(2)}(x-a_j^q)$ in (5.1). First, we consider the term $(q_j^\alpha)^2$. Without loss of generality we may place a singularity point at $x=0$. The relevant terms are collected from

$$\mathrm{Tr}\ln\left[1_{xy}-2iq\epsilon_{\mu\nu}\frac{x_\mu\partial_\nu}{x^2}G_{xy}+q^2\frac{1}{x^2}G_{xy}+2\pi q\delta^{(2)}(x)G_{xy}+2\pi q^2\delta^{(2)}(x)G_{xy}\right], \tag{5.3}$$

where G_{xy} is given by (3.17) and $q=q_j^\alpha$. We evaluate each term in momentum space. For instance, we use

$$\frac{x^\mu}{x^2}=-2\pi\int\frac{d^2k}{(2\pi)^2}\frac{ik^\mu}{k^2}e^{ikx}\,. \tag{5.4}$$

Then,

$$\begin{aligned}&\mathrm{Tr}(x^{-2}G_{xy})=\ln(\Lambda/\mu)\ln(\Lambda/m)\,,\\&\mathrm{Tr}[2\pi\delta^{(2)}(x)G_{xy}]=\ln(\Lambda/m)\,,\\&\mathrm{Tr}[2\pi\delta^{(2)}(x)G_{xz}2\pi\delta^{(2)}(z)G_{zy}]=[\ln(\Lambda/m)]^2\,,\\&\mathrm{Tr}[(2x^{-2}\epsilon_{\mu\nu}x_\mu\partial_\nu G_{xy})^2]=-[\ln(\Lambda/m)]^2+\ln(e\Lambda/\mu)[2\ln(\Lambda/m)-1]\,,\end{aligned} \tag{5.5}$$

where μ is an infrared cutoff:

$$\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}=\frac{1}{2\pi}\ln(\Lambda/e\mu)\,. \tag{5.6}$$

As a result, we obtain

$$\Gamma^{\alpha}_{i=j}=\tfrac{1}{2}\sum_j (q_j^{\alpha})^2 \ln(\Lambda/\mu)\ . \tag{5.7}$$

These divergences originate in the fact that a topological singularity is a pointlike charged object. Indeed, $\Gamma^{\alpha}_{i=j}$ is nothing but the "classical energy" of topological singularities:

$$\sum_j (q_j^{\alpha})^2 \int d^2x\ |\partial_\mu \theta_j^q|^2\ .$$

We proceed to evaluate the term proportional to $q_i^{\alpha}q_j^{\alpha}$, $i\neq j$. It is easy to extract relevant terms from (5.1) and to summarize them as

$$\Gamma^{\alpha}_{i\neq j}=-\tfrac{1}{2}\sum_{i\neq j} q_i^{\alpha}q_j^{\alpha}\left\{K_0(m\ |a_i^q-a_j^q|\ )^2+2\pi\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}\exp[ik(a_i^q-a_j^q)]\right.$$
$$\left.\times\left[1-\left[1+\frac{4m^2}{k^2}\right]^{1/2}\ln\left[\frac{(4m^2+k^2)^{1/2}+k^2}{(4m^2+k^2)^{1/2}-k^2}\right]\right]\right\}. \tag{5.8}$$

Explicit integrations are tedious. We only give the result:

$$\Gamma^{\alpha}_{i\neq j}=-\tfrac{1}{2}\sum_{i\neq j} q_i^{\alpha}q_j^{\alpha}\ln|a_i^q-a_j^q|\mu_0$$
$$+\sum_{i\neq j} q_i^{\alpha}q_j^{\alpha}[m^2(a_i^q-a_j^q)^2(K_1{}^2-K_0{}^2)-m\ |a_i^q-a_j^q|K_0K_1]\ , \tag{5.9}$$

where $K_0=K_0(m\ |a_i^q-a_j^q|\ )$ and $K_1=-K_0'$ are modified Bessel functions, while μ_0 is an infrared cutoff introduced by

$$\int\frac{d^2k}{(2\pi)^2}\frac{e^{ikx}}{k^2}=\frac{1}{2\pi}\ln(16\mu_0|x|e^{-1-8\gamma})\ , \tag{5.10}$$

γ being the Euler constant.

Here we make a comment. In calculations (5.5)–(5.8), we confront a finite renormalization ambiguity, which appears as indefiniteness of the coefficient ($\frac{1}{2}$) in (5.7) and of the constant (1) in the last square brackets of (5.8). We have determined them so that the effective action (5.9) agrees with that of the previous section, Eq. (4.24), in the infrared behavior.

Combining (5.7) and (5.9) we obtain

$$\sum_\alpha \Gamma_2^{\alpha}=-\tfrac{1}{2}\sum_\alpha\sum_{i\neq j} q_i^{\alpha}q_j^{\alpha}\ln|a_i^q-a_j^q|\Lambda$$
$$+\sum_\alpha\sum_{i\neq j} q_i^{\alpha}q_j^{\alpha}[m^2(a_i^q-a_j^q)^2(K_1{}^2-K_0{}^2)-m\ |a_i^q-a_j^q|K_0K_1]+\tfrac{1}{2}\sum_\alpha\left[\sum_j q_j^{\alpha}\right]^2\ln(\Lambda/\mu)\ , \tag{5.11}$$

where we have chosen $\mu=\mu_0$. This is the main result obtained in this section. We remark upon some prominent features therein. First, the infrared divergences are canceled out in the charge neutral sector $\sum_j q_j^{\alpha}=0$. Hereafter, we consider only this case because the charged sectors do not contribute to the generating functional. Second, each topological singularity is accompanied with a cloud of w_α particles. The structure of this cloud is transparent in (5.11). We shall come back to this point in the next section.

Although it is possible to continue perturbative calculations in q_j^{α}, it would not be interesting. This is so because they do not change either the ultraviolet or the infrared structure of the theory; they could only modify the detailed structure of the w cloud around a topological singularity.

VI. EFFECTIVE LAGRANGIANS

In the preceding sections we have calculated effective action (3.14) to the leading order of $1/N$. We have shown that

$$S'_{\text{eff}}=S_0^{(2)}+S_2^{(2)}+O(1/\sqrt{N}) , \tag{6.1}$$

with $S_0^{(2)}$ being the term independent of topological singularities, and

$$S_2^{(2)}=-\frac{1}{2}\sum_{i\neq j}\vec{q}_i\cdot\vec{q}_j \ln|a_i^q-a_j^q|\Lambda + \sum_{i\neq j}\vec{q}_i\cdot\vec{q}_j[m^2(a_i^q-a_j^q)^2(K_1{}^2-K_0{}^2)-m\,|a_i^q-a_j^q|K_0K_1] , \tag{6.2}$$

where $\vec{q}_j$ is a root vector of SU(N) defined by (2.32). Let us extract the self-energy of each topological excitation. For this purpose, we take a pair of charges $\vec{q}_i=-\vec{q}_j$ and set $|a_i^q-a_j^q|=\Lambda^{-1}$ in (6.2). The result must be twice the self-energy of a topological singularity carrying charge $\vec{q}_j$. It follows that

$$S_{\text{self}}^{(2)}=\frac{1}{2}\sum_j \vec{q}_j{}^2\ln(\Lambda^2/m^2) . \tag{6.3}$$

This term has originated entirely from the cloud of w particles. Hence, the self-energy includes a dynamical effect. The ultraviolet divergence is merely due to the pointlike character of topological singularities. By separating the self-energy terms, we may represent (6.2) as

$$S_2^{(2)}=S_{\text{self}}^{(2)}+S_R^{(2)} , \tag{6.4}$$

where

$$S_R^{(2)}=:\left\{-\frac{1}{2}\sum_{i\neq j}\vec{q}_i\cdot\vec{q}_j\ln|a_i^q-a_j^q| + \sum_{i\neq j}\vec{q}_i\cdot\vec{q}_j[m^2(a_i^q-a_j^q)^2(K_1{}^2-K_0{}^2)-m\,|a_i^q-a_j^q|K_0K_1]\right\}: . \tag{6.5}$$

Here, the colon indicates that the self-energies have been removed, or equivalently that the normal ordering has been made.

The generating functional is given by

$$Z=Z_0Z_2 , \tag{6.6}$$

where

$$Z_0=\int[d\sigma][dA_\mu]\exp(-S_0^{(2)}) \tag{6.7}$$

and

$$Z_2=\sum\!\!\!\!\!\!\int \exp(-S_2^{(2)}) . \tag{6.8}$$

The symbol $\sum\!\!\!\!\!\!\int$ has been defined by (3.11). It is remarkable that, if and only if $\vec{q}_j{}^2=2$, the divergence in the self-energy term (6.3) is precisely canceled by the bare fugacity λ_0, $\lambda_0=c\Lambda^2$. When $\vec{q}_j{}^2>2$, the cancellation is imperfect and the generating functional Z_2 vanishes. As is obvious from (2.30) and (2.32), this fact implies that charge vectors $\vec{q}$ are given by elementary root vectors of SU(N). Therefore, (6.8) is equivalent to

$$Z_2=\sum\!\!\!\!\!\!\int_R \exp(-S_R^{(2)}) \tag{6.9}$$

with

$$\sum\!\!\!\!\!\!\int_R=\sum_{n=0}^{\infty}\sum_{\vec{q}}{}'\left[\prod\frac{1}{N_q!}\right]\int\prod_j^n(\lambda_R d^2a_j^q) , \tag{6.10}$$

where λ_R is a renormalized fugacity; $\lambda_R=cm^2$. Here, the symbol $\sum'_{\vec{q}}$ indicates that charge vector $\vec{q}_j$ runs over only a set of elementary root vectors.

Formula (6.9) together with (6.5) gives the grand partition function of topological excitations. As far as the infrared structure is concerned, it describes a system of Coulomb gas bearing charge vectors labeled by the root lattice of SU(N). On the other hand, the ultraviolet structure is significantly different from that of a naive Coulomb gas. It is well known that the naive Coulomb gas suffers an ultraviolet divergence when plus and minus charges approach each other. However, there are no such divergences in (6.5) because the Yukawa-type repulsive force dominates over the Coulomb-type attractive force for $|a_i^q-a_j^q|<m^{-1}$.

In what follows we shall mainly analyze the infrared structure of effective action (6.5):

$$S_R^{(2)}\hat{=}:\left[-\frac{1}{2}\sum_{i\neq j}\vec{q}_i\cdot\vec{q}_j\ln|a_i^q-a_j^q|\right]: . \tag{6.11}$$

It is interesting to examine the CP^1 model by extrapolating this formula to $N=2$. In this case, there are only two elementary root vectors: $\vec{q}=\pm(1,-1)$. Hence, (6.11) reads

$$S_R^{(2)} \hat{=} :\left[-\sum_{i\neq j} q_i q_j \ln|a_i^q - a_j^q|\right]: , \qquad (6.12)$$

where $q_i=\pm 1$. This result describes effectively an Abelian Coulomb gas at temperature $T=1$, and agrees exactly with the result of instanton calculations.[15,16]

It is well known that effective action (6.12) amounts to the sine-Gordon model. By the same reasoning, it is quite easy to prove that (6.11) gives rise to the SU(N) sine-Gordon model. Namely, we obtain

$$Z_2 = \sum\!\!\!\!\!\!\int_R \exp(-S_R^{(2)}) = \int [d\vec{U}] \exp\left[-\int d^2x\, \mathscr{L}_{\text{eff}}(\vec{U})\right] , \qquad (6.13)$$

where

$$\mathscr{L}_{\text{eff}} = \tfrac{1}{2}(\partial_\mu \vec{U})^2 + \frac{c}{2} m^2 \sum_{p=1}^{N(N-1)} :\cos(2\sqrt{\pi}\vec{U}\cdot\vec{\eta}_p): , \qquad (6.14)$$

with $\vec{U}(x)$ being an $(N-1)$-dimensional vector field. Note that we have used $(N-1)$-component root vectors $\vec{\eta}$ instead of N-component root vectors $\vec{q}$. In this formula, $\vec{\eta}_p$ are nontrivial elementary root vectors defined by (2.26) with $n_s=\pm 1$ therein; there are $N(N-1)$ such vectors in total.

We proceed to analyze the problem of charge confinement based on an effective Lagrangian (6.14). Before so doing, we recall that the CP^{N-1} model possesses a local U(1) gauge symmetry as well as a global SU(N) gauge symmetry. Thus, it is necessary to consider the confinement of U(1) charges and SU(N) charges separately. As to the U(1) charge, we have nothing to add to the well-known result obtained in the naive $1/N$ expansion,[13] because topological singularities do not bear the U(1) charge. Namely, also in our formalism, U(1) charges are confined in the order of $1/\sqrt{N}$ due to interactions with the U(1) gauge potential A_μ.

We next study the possibility of confinement of fundamental representation charges of SU(N). We expect *a priori* that this is impossible because topological effects cannot create a new symmetry, that is, the local SU(N) gauge symmetry in the CP^{N-1} model. In fact, an effective Lagrangian (6.14) has a global SU(N) symmetry[23] but not a local one. (See note added in proof.) We are able to check our observation by introducing a test charge into the system.

Let us add a source term

$$S_J = ip^\alpha \int d^2x\, \partial_\mu \chi^\alpha J_\mu \qquad (6.15)$$

to the action in (2.13), where J_μ denotes the current of a test particle,

$$J_\mu = \int d\tau\, \dot{y}_\mu(\tau)\delta^{(2)}(x-y(\tau)) , \qquad (6.16)$$

and p^α represents a fundamental-representation charge of SU(N). Namely, $\vec{p}=(p^1,\ldots,p^\alpha,\ldots,p^N)$ is an N-component charge vector on the weight lattice of SU(N).[11] Let us define $J(x)$ by

$$J_\mu = \epsilon_{\mu\nu}\partial_\nu J . \qquad (6.17)$$

When J_μ stands for a static current, i.e., $J_\mu=\delta_{\mu 0}\delta^{(1)}(x)$, we obtain $J=\theta(x)$. Because (6.15) is now rewritten as

$$S_J = 2\pi i \sum_j \vec{q}_j\cdot\vec{p}_j J(a_j^\alpha) , \qquad (6.18)$$

where use was made of (2.15), the net effect is only to add this term to (6.11). Then, it is quite easy to show that (6.14) is modified as

$$\mathscr{L}^J_{\text{eff}} = \tfrac{1}{2}(\partial_\mu \vec{U})^2 + \frac{c}{2} m^2 \sum_{p=1}^{N(N-1)} :\cos(2\sqrt{\pi}\vec{U}\cdot\vec{\eta}_p - 2\pi\vec{\eta}_p\cdot\vec{\epsilon}J): , \qquad (6.19)$$

where $\vec{\epsilon}$ is an $(N-1)$-dimensional charge vector[11] on the weight lattice of SU(N). By making a change of variable, $\vec{V}=\vec{U}-\sqrt{\pi}\vec{\epsilon}J$, we find that the external source $\vec{\epsilon}J$ is affected only by a short-range effect. Therefore, we conclude that fundamental-representation charges are not confined in the CP^{N-1} model.

For the sake of completeness, we wish to review why fractional charges are confined in the Abelian Higgs model.[7,9,24] In this case, we add a source term

$$S_J = i\epsilon \int d^2x\, \partial_\mu \chi J_\mu \qquad (6.20)$$

to the action (2.1). Then, we may derive an effective Lagrangian

$$\mathscr{L}^J_{\text{eff}} = \tfrac{1}{2}(\partial_\mu U)^2 - \tfrac{1}{2} m_V{}^2 U^2 + cm^2 :\cos(2\pi v U - 2\pi\epsilon J): , \qquad (6.21)$$

where $\langle \phi \rangle = v$, ϕ being the Higgs field. It follows from this formula that the external source ϵJ is affected by a long-range effect when ϵ is a fractional number. The essential difference of (6.21) from (6.19) is given by the existence of the mass term $m_V{}^2 U^2$. The mass term signals the presence of an underlying local gauge symmetry in 1 + 1 dimensions. As is well known, the sine-Gordon model is equivalent to the Thirring model, but the massive sine-Gordon model is equivalent to a gauged Thirring model such as the massive Schwinger model.

We conclude that charge confinement does not occur as a topological effect in the CP^{N-1} model for the following reasons. First, topological excitations do not carry the U(1) charge. Second, the CP^{N-1} model does not contain a local SU(N) gauge symmetry. Therefore, the CP^{N-1} model is not a good model to simulate the confinement mechanism of quarks in Yang-Mills theory.

VII. CONCLUSIONS

In this paper we have proposed to analyze topological effects in a field theory by evaluating quantum fluctuations around topological singularities rather than classical solutions. We have emphasized that topological singularities are characterized by the root lattice of SU(N) in the CP^{N-1} model. Therefore, their charges are entirely different from the naive topological charge which is an Abelian object. Because of this fact, quantum effects due to topological singularities remain finite, though instanton effects vanish,[13] in the large-N limit.

We have explicitly calculated quantum fluctuations around topological singularities to the nontrivial leading order of $1/N$. An approximate effective Lagrangian is given by the SU(N) sine-Gordon model. Based on this result, we have argued that the topological effects do not confine fundamental-representation charges in the CP^{N-1} model. This is not surprising because the local SU(N) gauge symmetry cannot be generated from the global SU(N) gauge symmetry by topological excitations.

We have shown that each topological singularity gets dressed with a cloud of constituent particles via quantum corrections. This cloud of constituent particles softens the short-distance behavior of topological excitations. When plus and minus charges approach each other, a repulsive Yukawa force dominates over the system. Therefore, the grand partition function of topological excitations describes an ideal system of Coulomb gas whose ultraviolet structure is regularized.

In the CP^{N-1} model, it may not be interesting to analyze topological singularities, since they have no connection with charge confinement. However, this model is an ideal laboratory in which cancellation mechanisms of various divergences are carefully examined. There are two types of ultraviolet divergences; one is of a dynamical origin and the other is of a kinematical origin. The dynamical divergence is due to loop integrals in Feynman graphs. We have shown that this type of ultraviolet divergence disappears automatically thanks to the fact that topological singularities are labeled by the root lattice of SU(N). On the other hand, the kinematical divergence arises from the pointlike character of topological singularities. Keeping track of all singular quantities, we have shown that this type of ultraviolet divergence also disappears automatically.

It is very interesting to apply our calculational method to various models in higher dimensions, such as Georgi-Glashow models, Higgs models, and eventually Yang-Mills theories. In these models, topological singularities trace world lines or world sheets, which generate magnetic monopoles[4] or magnetic vortices.[5] We recall that magnetic monopoles are labeled by the root lattice of SU(N) in SU(N) gauge theories.[19] As in the CP^{N-1} model, their effect would remain finite in the large-N limit. However, in contrast with the CP^{N-1} model, topological excitations would be essential agents that lead to quark confinement in these models. This is so because these models involve local SU(N) gauge symmetries and because the associated non-Abelian flux is expected to be squeezed into electric vortices by topological excitations. We hope to report on a generalization of our scheme to include these models in forthcoming papers.

Note added in proof. We remark that the SU(N) sine-Gordon Lagrangian (6.14) is equivalent to the following SU(N) Thirring Lagrangian:

$$L_{\mathrm{eff}} = i\bar{\psi}\partial\psi - \pi \sum_{a=1}^{N^2-1} \left[\bar{\psi}\gamma_\mu \frac{\lambda^a}{2}\psi \right]^2 ,$$

ψ being the SU(N) fermion field, where the global SU(N) symmetry is manifest.

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