Fermion emission from a Julia-Zee dyon

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A relationship is obtained between the S matrix for the charge-exchange scattering of a fermion by a Julia-Zee dyon and the flux of fermions emitted by the dyon when the mass of the fermions is sufficiently small. In the limit of a pointlike dyon, the required S-matrix elements are obtained in closed form and the corresponding fermion flux is computed explicitly.

I. INTRODUCTION

Asymptotically, the field configuration of an SU(2) Julia-Zee[1] dyon is completely equivalent to the magnetic and electric field of a Dirac magnetic monopole[2] carrying an electric charge Q/g.[3] If an SU(2) doublet of Fermi fields ψ_α is coupled to the dyon, the corresponding particles behave asymptotically as two species of fermions with particles carrying electric charge $\pm g/2$ and their corresponding antiparticles carrying charge $\mp g/2$. If the mass of these fermions is sufficiently small, the dyon becomes unstable. For $Q>0$, fermion-antifermion pairs (each particle carrying charge $+g/2$) are emitted, lowering the total dyon charge.

In a recent paper,[4] we discussed this decay process for the case of massless fermions, finding an anomalous production of chirality consistent with the axial anomaly.[5] In this paper we consider the more general case of fermions with mass. After reviewing the properties of the Julia-Zee dyon in Sec. II, we turn, in Sec. III, to the derivation of a formula expressing the spectrum of emitted fermions in terms of S-matrix elements for the charge-exchange scattering of a single Dirac particle by the Julia-Zee dyon. The resulting formula is accurate in the small-g or "one-loop" approximation in which the back-reaction of the emitted fermions on the dyon field is neglected. In Sec. IV we give an alternate derivation of the flux computed in Sec. III. This second derivation is based on a purely formal evaluation of the matrix element of the Heisenberg current operator.

The S-matrix elements required by the above formula are in general difficult to obtain.[6] However, in the limit of a pointlike dyon the non-Abelian Dirac Hamiltonian reduces to the Dirac Hamiltonian for an Abelian dyon. Although the eigenvalue equation for this Abelian problem reduces to a pair of uncoupled second-order equations,[7] the problem is complicated by a severe singularity at the origin.[8] Only if one requires unconventional, relaxed boundary conditions does a complete set of energy eigenstates exist for the Abelian Hamiltonian. The singularity is very similar to that present in the Dirac Hamiltonian[9] of an electron moving in the field of a nucleus with $Z>137$. For our two-component Abelian equation (fermionic charges $\pm g/2$) a four-parameter set of boundary conditions is possible.[10,11] However, the pointlike limit of the non-Abelian dyon determines a corresponding Abelian problem with a unique choice for these boundary conditions.[10] That choice is one which mixes the two differently charged solutions and gives nonzero S-matrix elements for charge-exchange scattering. These are computed explicitly in Sec. V and the resulting flux of fermions emitted by the pointlike dyon is obtained. The energy spectrum of these fermions shows a resonant enhancement at zero total energy (kinetic plus electrostatic). This resonance is the Jackiw-Rebbi zero-energy mode, which acquires a width when the dyon becomes unstable.

II. JULIA-ZEE DYON

In this section we summarize the properties of the Julia-Zee dyon. In the simplest static gauge, the Yang-Mills vector potential corresponding to the dyon has the form

$$A_i^a(\vec{r})=\epsilon_{aij}(\hat{r})^j[K(r)-1]\frac{1}{gr}\,,$$
$$A_0^a(\vec{r})=-(\hat{r})^a J(r)\frac{1}{g}\,. \tag{2.1}$$

Here i and 0 indicate space and time directions,[12] a is an SU(2) vector index, g is the Yang-Mills coupling constant, $r=|\vec{r}|$, and $\hat{r}=\vec{r}/r$. The Higgs field $\phi^a(\vec{r})$ has a similar structure:

$$\phi^a(\vec{r})=-(\hat{r})^a H(r)\frac{1}{g} \ . \tag{2.2}$$

These fields solve the coupled Higgs-Yang-Mills equations, are analytic at the origin, and behave for large r as

$$J(r)\sim -v+Q/r, \quad H(r)\sim h \ , \tag{2.3}$$

while $K(r)$ vanishes exponentially. We will consider the case where the constants v, Q, and h are all positive.

Asymptotically, the field configuration specified by (2.1) and (2.3) can be gauge transformed to the single-component vector potential $\mathscr{A}_\mu(\vec{r})$ of an Abelian magnetic monopole with pole strength $1/g$ which also has electric charge Q/g:

$$\mathscr{A}_\mu(\vec{r})\tau^3=U(A^a_\mu\tau^a)U^{-1}-\frac{i}{g}U\partial_\mu U^{-1} \ , \tag{2.4}$$

where the 2×2 matrix $U(\vec{r})$ is an SU(2) gauge transformation and τ^a $(a=1,2,3)$ are the usual Pauli matrices. If we choose the singular Dirac string present in $\mathscr{A}_\mu(\vec{r})$ to lie along the positive z axis, we could use

$$U(\vec{r})=\frac{1}{\sqrt{2}}\left[(1-\hat{z}\cdot\hat{r})^{1/2} + i\frac{\vec{\tau}\cdot(\hat{r}\times\hat{z})}{(1-\hat{z}\cdot\hat{r})^{1/2}}\right] \tag{2.5}$$

giving

$$\vec{\mathscr{A}}=\frac{1}{g}\frac{\vec{r}\times\hat{z}}{r^2(1-\hat{z}\cdot\hat{r})} \ , \qquad \mathscr{A}_0=-\frac{v}{g}+\frac{Q}{g}\frac{1}{r} \ , \tag{2.6}$$

with $\hat{z}$ a unit vector in the z direction.

The constant value $-v/g$ of the vector potential $\mathscr{A}_0$ at spatial infinity is a somewhat unusual feature of the Julia-Zee solution when expressed in a time-independent gauge. However, as we will see, this constant plays a central role when the production of fermions by the dyon is analyzed in such a static gauge. Physically, this asymptotic constant is a consequence of the dyon's electric charge and the topological nature of the solution ($\mathscr{A}^a_0\propto\hat{r}^a$) which requires that the potential vanish at the origin instead of the more conventional condition that it vanish at infinity. Heuristically, one may view this asymptotic constant as the difference between the electrostatic potential at infinity and at the center of the dyon: $v/g\sim(Q/g)/\langle r\rangle$ where $\langle r\rangle$ is the radius of the dyon's charge distribution.

If an SU(2)-doublet fermion field ψ is coupled to the dyon, it will obey the Dirac equation

$$[\gamma^\mu(\partial_\mu+i\frac{g}{2}\tau^a A^a_\mu)+iG\phi^a\tau^a]\psi=0 \ , \tag{2.7}$$

where we have, for simplicity, omitted a constant mass term. For large r this equation also simplifies and its solutions ψ can be written

$$\psi(\vec{r},t)\sim U^{-1}(\vec{r})\begin{bmatrix}\psi_+(\vec{r},t)\\ \psi_-(\vec{r},t)\end{bmatrix} \ , \tag{2.8}$$

where $\psi_\pm(\vec{r},t)$ is a solution of the Abelian Dirac equation for a fermion of mass $m=Gh/g$ and charge $\pm g/2$ moving in the vector potential $\mathscr{A}_\mu$ given in Eq. (2.4). Equivalently, if we begin with the non-Abelian Dirac Hamiltonian

$$H=\alpha^i\left[-i\partial_i+\frac{g}{2}A^a_i\tau^a\right]+G\phi^a\tau^a\beta + \frac{g}{2}A^a_0\tau^a \ , \tag{2.9}$$

perform the gauge transformation $H\to UHU^{-1}$, and replace the fields A^a_μ and ϕ^a by their asymptotic forms, the resulting Hamiltonian is the sum of two independent Abelian Hamiltonians

$$\mathscr{H}_\pm=\alpha^i\left[-i\partial_i\pm\frac{g}{2}\mathscr{A}_i\right]\pm m\beta\pm\frac{g}{2}\mathscr{A}_0 \tag{2.10}$$

for fermions with mass $m=Gh/g$ and charge $\pm g/2$.

III. SPECTRUM OF EMITTED FERMIONS

For $2m<v$ one expects the combined fermion-dyon system to be unstable. If a ψ_+ fermion with charge $+g/2$ and a ψ_- antifermion also with charge $+g/2$ are created near the dyon and moved to infinity, a rest mass of $2m$ must be created but an electrostatic energy of $2(\frac{1}{2}g)v/g$ can be liberated because of the constant potential v/g at infinity. Thus pair production is energetically favorable in the field of a Julia-Zee dyon when $2m<v$.

The mechanism for this fermion production by a Julia-Zee dyon can be quite easily understood by considering the non-Abelian Dirac Hamiltonian H

of Eq. (2.9) in a second-quantized form and attempting to define the ground state as one in which all the "negative-energy" levels are filled. The situation is represented pictorially in Fig. 1. If $2m < v$ then a single-particle eigenstate with energy E lying between $v/2-m$ and $-v/2+m$ has asymptotic components ψ_+ and ψ_- with both positive and negative energy. The $I_z=+\frac{1}{2}$, charge $+g/2$ component ψ_+ has a momentum

$$p_+=+[(E+v/2)^2-m^2]^{1/2} \tag{3.1}$$

and appears to be a positive-energy state whose conventional free-particle energy $+(p_+{}^2+m^2)^{1/2}$ has been shifted downward by the constant, negative electrostatic energy $-v/2$. Likewise, the $I_z=-\frac{1}{2}$, charge $-g/2$ component ψ_- has a momentum

$$p_-=+[(E-v/2)^2-m^2]^{1/2} \tag{3.2}$$

and should be interpreted as a negative-energy state whose free-particle energy $-(p_-{}^2+m^2)^{1/2}$ is shifted upward by the constant electrostatic energy $+v/2$. Clearly an observer far from the dyon would interpret ψ_+ and ψ_- as positive- and negative-energy solutions, respectively. Any energy eigenstate in the second-quantized system defined by filling some fraction of those states in regions B with $v/2-m \geq E \geq -v/2+m$ will be stable but only because of the presence of an infinite number of ψ_+ particles or ψ_- antiparticles which, by the Pauli exclusion principle, prevents the pair production described above.

A physically correct initial state must obey the following requirements: (i) For r greater than some fixed large radius R, the state should contain no incoming or outgoing particles with positive energy. Thus, if we let $\kappa=I_z$, single-particle states of the form $\exp(\pm ipr)$ with total energy $E=+(p^2+m^2)^{1/2}-\kappa v$ should be empty. (ii) Similarly, all such states with negative energy, i.e., $E=-(p^2+m^2)^{1/2}-\kappa v$ should be filled. Because asymptotic states of types (i) and (ii) are mixed by the Dirac Hamiltonian (2.9), any initial state specified in this way is necessarily time dependent.

However, the resulting particle creation is not difficult to compute. In order to specify an initial state $|i\rangle$ precisely let us introduce two complete orthonormal sets of isodoublet Dirac wave functions $\phi_{\kappa,\alpha}^{\mathscr{E},X}(\vec{r})$. The two sets are distinguished by the superscript $X=$"in" or "out" which indicates that the incoming or outgoing part of the wave carries quantum numbers specified by the subscript α. We require that asymptotically these wave functions have energy $E=\mathscr{E}-\kappa v$ and a single nonvanishing isotopic component $\psi_\pm$ for $\kappa=\pm\frac{1}{2}$. Thus for large r both the incoming and the outgoing parts of $\phi_{\kappa,\alpha}^{\mathscr{E},X}$ have charge κg and $\phi_{\kappa,\alpha}^{\mathscr{E},X}$ obeys

$$H\phi_{\kappa,\alpha}^{\mathscr{E},X}(\vec{r})=E\phi_{\kappa,\alpha}^{\mathscr{E},X}(\vec{r}), \quad r>R\ , \tag{3.3}$$

where H is the complete non-Abelian Hamiltonian. Of course, the requirement that $\phi_{\kappa,\alpha}^{\mathscr{E},X}$ have only a single isotopic component for large r implies that Eq. (3.3) cannot hold for all r: $\phi_{\kappa,\alpha}^{\mathscr{E},X}(\vec{r})$ is not an eigenstate of H. We adopt the normalization convention

$$\langle\phi_{\kappa,\alpha}^{\mathscr{E},X}\,|\,\phi_{\kappa',\alpha'}^{\mathscr{E}',X}\rangle=\delta(\mathscr{E}-\mathscr{E}')\delta_{\kappa\kappa'}\delta_{\alpha\alpha'} \tag{3.4}$$

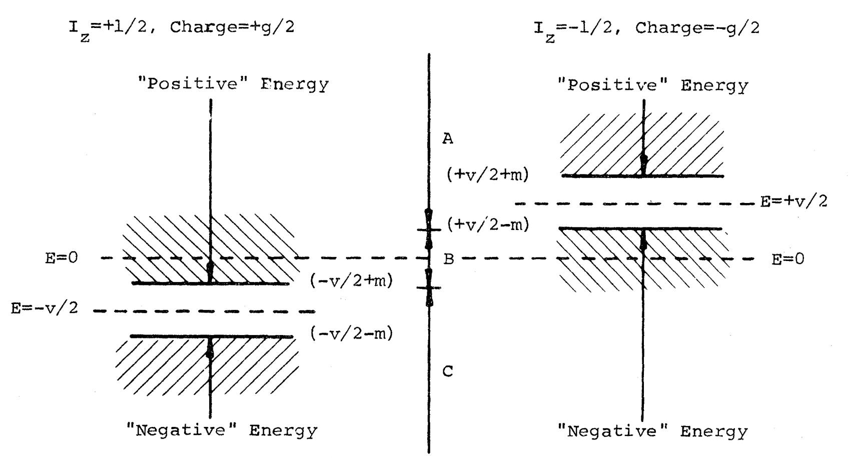

FIG. 1. Energy spectrum of particles with $I_z=\pm\frac{1}{2}$ far from the Julia-Zee dyon. Fermion emission results if the region B is not empty ($v>2m$) so that "positive"- and "negative"-energy states can mix.

for $X=$"in" or "out". Next use the "in" states to define annihilation and creation operators through the usual expansion of the Fermi field operator:

$$\psi(\vec{r})=\sum_{\kappa,\alpha}\int_0^\infty d\mathscr{E}(a^{\mathscr{E}}_{\kappa,\alpha}\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}+b^{\mathscr{E}\dagger}_{-\kappa,\bar{\alpha}}\phi^{-\mathscr{E},\text{in}}_{\kappa,\alpha}) . \quad (3.5)$$

Here $\bar{\alpha}$ specifies the quantum numbers of the antiparticle associated with the negative-energy state $\phi^{-\mathscr{E},\text{in}}_{\kappa,\alpha}$. Our initial state $|i\rangle$ is then determined by the requirement

$$a^{\mathscr{E}}_{\kappa,\alpha}|i\rangle=b^{\mathscr{E}}_{\kappa,\alpha}|i\rangle=0 . \quad (3.6)$$

In the language of second quantization, the state $|i\rangle$ is obtained by filling all the states $\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}$ whose (asymptotic total energy E) minus $(-\kappa v)$ is negative ($\mathscr{E}<0$) while leaving those states with $\mathscr{E}>0$ empty. In terms of Fig. 1 those states in region C with $I_z=+\frac{1}{2}$ or regions B and C with $I_z=-\frac{1}{2}$ are filled; those states in region A with $I_z=-\frac{1}{2}$ or regions A and B with $I_z=+\frac{1}{2}$ are empty.

Although our initial state $|i\rangle$ has been precisely specified by Eq. (3.6), its definition depends on the expansion functions $\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}$ which themselves were not uniquely determined. The constraints that we placed on the asymptotic behavior of the $\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}$ left their behavior near the origin unspecified. However, the resulting ambiguity in the state $|i\rangle$ is common to all decay problems and affects only the short-time behavior of the decaying state. As we will see, the unspecified details of the functions $\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}$ propagate off to infinity and do not affect the large-time limit of the flux of emitted particles at any fixed position.

We will now determine the flux of particles present at large fixed r in the large-time limit of the state $|i\rangle$. We make the one-loop or small-g approximation and treat the Yang-Mills dyon field as fixed, ignoring the back-reaction of the emitted fermions. Let us first examine the large-time limit of the filled states $\phi^{-\mathscr{E},\text{in}}_{-1/2,\alpha}$ with $E=-\mathscr{E}+v/2$ in region B ($m\leq\mathscr{E}\leq v-m$). The outgoing parts of these states travel off to infinity and are gone while the incoming parts continue to flow inward and are scattered from the dyon yielding new outgoing particles with $\kappa=\pm\frac{1}{2}$. We interpret the difference between the flux of particles for $t\to\infty$ and the flux present at $t=0$ as the steady-state flux of particles produced by the dyon.

Define the Dirac scattering matrix $S_{\kappa',\alpha';\kappa,\alpha}(E)$ as

$$S_{\kappa',\alpha';\kappa,\alpha}(E)\delta(E-E')=\langle\psi^{E',\text{out}}_{\kappa',\alpha'}|\psi^{E,\text{in}}_{\kappa,\alpha}\rangle , \quad (3.7)$$

where $\psi^{E,\text{in (out)}}_{\kappa,\alpha}$ is an eigenstate of the non-Abelian Dirac Hamiltonian (2.9) with eigenvalue E whose incoming (outgoing) part agrees with that of $\phi^{E+\kappa v,\text{in (out)}}_{\kappa,\alpha}$. The matrix $S_{\kappa',\alpha';\kappa,\alpha}(E)$ is unitary:

$$SS^\dagger=I . \quad (3.8)$$

In terms of S, the flux per unit energy of $\kappa=+\frac{1}{2}$, positively charged particles with "positive" energy $\mathscr{E}=E+v/2$ and quantum numbers α flowing outward from the dyon is

$$\mathscr{F}_a(\mathscr{E})=\frac{1}{2\pi}\sum_{\alpha'}|S_{+1/2,\alpha;-1/2,\alpha'}(\mathscr{E}-v/2)|^2 . \quad (3.9)$$

Here the factor $1/2\pi$ is the outgoing flux of a state satisfying the energy-δ-function normalization convention (3.4). The sum over α' in Eq. (3.9) adds the contribution of all incoming, filled negative-energy states which can scatter into the $(+\frac{1}{2},\alpha)$ state being examined.

Similarly, the outgoing flux in a $(-\frac{1}{2},\alpha)$ state with "negative" energy $E=-\mathscr{E}+v/2$ comes entirely from scattering from the filled incoming states with energy $E,\kappa=-\frac{1}{2}$ and quantum numbers α' and is given by

$$\frac{1}{2\pi}\sum_{\alpha'}|S_{-1/2,\alpha;-1/2,\alpha'}(-\mathscr{E}+v/2)|^2 . \quad (3.10)$$

If we interpret the difference between the flux $1/2\pi$ present in the "vacuum" configuration when this state is filled and the flux (3.10) present when $t\to\infty$ as the flux $\overline{\mathscr{F}}_{\bar{\alpha}}(\mathscr{E})$ of antiparticles, we obtain

$$\overline{\mathscr{F}}_{\bar{\alpha}}(\mathscr{E})=\frac{1}{2\pi}\left[1-\sum_{\alpha'}|S_{-1/2,\alpha;-1/2,\alpha'}(-\mathscr{E}+v/2)|^2\right] . \quad (3.11)$$

This equation can be put in a form similar to Eq. (3.9) by using the unitarity relation (3.8):

$$\overline{\mathscr{F}}_{\bar{\alpha}}(\mathscr{E})=\frac{1}{2\pi}\sum_{\alpha'}|S_{-1/2,\alpha;+1/2,\alpha'}(-\mathscr{E}+v/2)|^2 . \quad (3.12)$$

Finally, we observe that there are no particles emitted in the limit of large times whose energies E lie in the ranges A or C of Fig. 1. In the case of region A all the states are initially empty and remain so in the limit of large times. Of course when the filled states are time developed for a finite time and then analyzed in terms of our basis $\phi_{\kappa,\alpha}^{\mathscr{E},\mathrm{in}}$ there will be components with energy lying in the region A. These disappear in the limit of large time. Similarly, the large-time limit of the filled states in region C leaves those states filled. The finite number of states with a definite energy $E<-v/2$ are initially filled and simply rearranged among themselves in the large-time limit—unitarity requiring that they all remain filled.

The differential fluxes $\mathscr{F}_\alpha(\mathscr{E})$ and $\overline{\mathscr{F}}_\alpha(\mathscr{E})$ can be added and integrated to determine the total flux of particles and antiparticles carrying the quantum numbers α:

$$\mathscr{F}_\alpha^{\mathrm{tot}}=\frac{1}{2\pi}\sum_{\alpha'}\int_m^{v-m}d\mathscr{E}[\,|S_{1/2,\alpha;-1/2,\alpha'}(\mathscr{E}-v/2)|^2+|S_{-1/2,\bar{\alpha};1/2,\alpha'}(-\mathscr{E}+v/2)|^2]\,. \tag{3.13}$$

Likewise, repeating the steps leading to Eqs. (3.10) and (3.12), we can compute the flux of a generalized charge T carried by the emitted particles:

$$\mathscr{F}_T^{\mathrm{tot}}=\frac{1}{2\pi}\int_m^{v-m}d\mathscr{E}\,\mathrm{tr}[S^\dagger(\mathscr{E}-v/2)_{-1/2;1/2}T_{1/2;1/2}S(\mathscr{E}-v/2)_{1/2;-1/2}$$
$$-S^\dagger(-\mathscr{E}+v/2)_{1/2;-1/2}T_{-1/2;-1/2}S(-\mathscr{E}+v/2)_{-1/2;1/2}]\,. \tag{3.14}$$

Here $T_{\kappa',\alpha';\kappa,\alpha}$ is the matrix element of the operator T between single-particle states with the quantum numbers κ',α' and κ,α. The matrix operations in Eq. (3.14) refer to sums over the suppressed indices α and α' while the minus sign preceding the second term is the usual consequence of the identification of antiparticles with empty negative-energy states. The above formulas reduce the problem of calculating the flux of emitted fermions to that of finding S-matrix elements for the charge-exchange scattering of a single fermion by the dyon.

The above analysis has been carried out for a gauge in which the vector potential is time independent, with time component $\mathscr{A}_0$ approaching the constant $-v/g$ at spatial infinity. It is possible to carry out the analysis for a time-dependent gauge in which $\mathscr{A}_0$ has the more conventional behavior Q/gr for large r. In this case the particle production arises from the time dependence now present in the Dirac Hamiltonian instead of from the mixing of positive- and negative-energy states as discussed above. However, the fermion production computed in the time-dependent gauge agrees precisely with that given by Eqs. (3.9) and (3.12).

IV. OPERATOR DERIVATION OF FERMION FLUX

A somewhat different perspective on the formula (3.14) for the steady-state flux of a quantity T emitted by the dyon can be obtained by examining an alternate derivation using more conventional operator techniques. We begin by writing the flux in terms of the matrix element of the Heisenberg current operator

$$j_T^\mu(x)=\tfrac{1}{2}[\bar{\psi}(x)\gamma^\mu T,\psi(x)]\,, \tag{4.1}$$

where the commutator indicates the charge-conjugation-symmetric combination of both orderings of $\bar{\psi}$ and ψ and the matrix T is assumed to act on the Dirac and internal symmetry indices of ψ. Let us define the flux

$$F_T=\lim_{t\to\infty}\int d^2S\,\hat{r}\cdot\langle i\,|\,\vec{j}_T(\hat{r}R,t)\,|\,i\rangle\,, \tag{4.2}$$

where it is understood that the background dyon field is not affected by the emitted fermions even as $t\to\infty$.

The Heisenberg field $\psi(\vec{r},t)$ can be written in terms of the a's and b's used to define $|\,i\rangle$ if we introduce the time-developed basis functions $\phi_{\kappa,\alpha}^{\mathscr{E}}(\vec{r},t)$. These functions obey the non-Abelian, time-dependent Dirac equation (2.7) and satisfy the initial condition

$$\phi_{\kappa,\alpha}^{\mathscr{E}}(\vec{r},t)\,|_{t=0}=\phi_{\kappa,\alpha}^{\mathscr{E},\mathrm{in}}(\vec{r})\,. \tag{4.3}$$

In terms of these functions, $\psi(\vec{r},t)$ becomes

$$\psi(\vec{r},t)=\sum_{\alpha,\kappa}\int_m^\infty d\mathscr{E}[a_{\kappa,\alpha}^{\mathscr{E}}\phi_{\kappa,\alpha}^{\mathscr{E}}(\vec{r},t)+b_{-\kappa,\bar{\alpha}}^{\mathscr{E}\dagger}\phi_{\kappa,\alpha}^{-\mathscr{E}}(\vec{r},t)]\,. \tag{4.4}$$

This expression for $\psi(\vec{r},t)$ can then be used in Eq. (4.2) and the Fock-space matrix element evaluated, giving

$$F_T=\lim_{t\to\infty}\tfrac{1}{2}\sum_{\alpha,\kappa}\left[\int_{-\infty}^{-m}d\mathscr{E}-\int_{m}^{\infty}d\mathscr{E}\right]\left[\int d^2S\,\hat{r}\cdot\bar{\phi}^{\mathscr{E}}_{\kappa,\alpha}(\hat{r}R,t)\vec{\gamma}T\phi^{\mathscr{E}}_{\kappa,\alpha}(\hat{r}R,t)\right]. \quad (4.5)$$

The positive term in the commutator in Eq. (4.1) gets a nonzero contribution from the operator product $b^{\mathscr{E}'}_{\kappa',\alpha'}b^{\mathscr{E}\dagger}_{\kappa,\alpha}$ and yields the integral from $-\infty$ to $-m$ in Eq. (4.5). The integral from $+m$ to $+\infty$ comes from the product $a^{\mathscr{E}'}_{\kappa',\alpha'}a^{\mathscr{E}\dagger}_{\kappa,\alpha}$ in the negative term in the commutator in Eq. (4.1).

Next, the large-time limit in Eq. (4.5) can be evaluated if we introduce the factor $e^{-iEt}e^{+iEt}$ with $E=\mathscr{E}-\kappa v$ and use the standard formula[13]

$$\lim_{t\to\infty}|\phi^{\mathscr{E}}_{\kappa,\alpha}(t)\rangle e^{+iEt}=|\psi^{E,\text{in}}_{\kappa,\alpha}\rangle . \quad (4.6)$$

With this analysis Eq. (4.5) becomes

$$F_T=\tfrac{1}{2}\sum_{\kappa,\alpha}\left[\int_{-\infty}^{-\kappa v-m}dE-\int_{-\kappa v+m}^{\infty}dE\right]\left[\int d^2S\,\hat{r}\cdot\bar{\psi}^{E,\text{in}}_{\kappa,\alpha}(\hat{r}R)\vec{\gamma}T\psi^{E,\text{in}}_{\kappa,\alpha}(\hat{r}R)\right]. \quad (4.7)$$

The flux contributing to the integral in (4.7) comes from both the incoming and the outgoing parts of $\psi^{E,\text{in}}_{\kappa,\alpha}$ and includes not only the flux of the quantity T carried off by the emitted particles but also a possible vacuum contribution to $\langle i\,|\,\vec{j}(\vec{r},t)\,|\,i\rangle$ that may be present for large r even at $t=0$.

We can find the actual flux $\mathscr{F}^{\text{tot}}_T$ carried off by the emitted particles if we subtract from F_T the corresponding flux F^0_T computed at $t=0$:

$$\mathscr{F}^{\text{tot}}_T=F_T-F^0_T\,, \quad (4.8)$$

where

$$F^0_T=\tfrac{1}{2}\sum_{\kappa,\alpha}\left[\int_{-\infty}^{-m}d\mathscr{E}-\int_{m}^{\infty}d\mathscr{E}\right]\left[\int d^2S\,\hat{r}\cdot\bar{\phi}^{\mathscr{E},\text{in}}_{\kappa,\alpha}(\hat{r}R)\vec{\gamma}T\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}(\hat{r}R)\right]. \quad (4.9)$$

Because $\psi^{E,\text{in}}_{\kappa,\alpha}$ and $\phi^{E+\kappa v,\text{in}}_{\kappa,\alpha}$ have identical incoming parts, the incoming contribution to $\mathscr{F}^{\text{tot}}_T$ cancels between F_T and F^0_T. The remaining outgoing part can be written in terms of the S matrix of Eq. (3.7) and a similar quantity

$$\mathscr{S}_{\kappa',\alpha';\kappa,\alpha}(E)\delta(E'-E)=\langle\phi^{E'+\kappa' v,\text{out}}_{\kappa',\alpha'}\,|\,\phi^{E+\kappa v,\text{in}}_{\kappa,\alpha}\rangle \quad (4.10)$$

which specifies the outgoing parts of $\phi^{\mathscr{E},\text{in}}_{\kappa,\alpha}$. The result for $\mathscr{F}^{\text{tot}}_T$ is

$$\mathscr{F}^{\text{tot}}_T=\frac{1}{4\pi}\sum_{\kappa,\alpha}\left[\int_{-\infty}^{-\kappa v-m}dE-\int_{-\kappa v+m}^{\infty}dE\right]\langle\kappa,\alpha\,|\,(S^\dagger TS-\mathscr{S}^\dagger T\mathscr{S})\,|\,\kappa,\alpha\rangle . \quad (4.11)$$

A given energy E will contribute to both the terms $\kappa=\pm\frac{1}{2}$ with the same sign if $|E|>m+v/2$. For such energies the sum over κ and α reduces to a trace which reduces to zero:

$$\text{tr}(S^\dagger TS-\mathscr{S}^\dagger T\mathscr{S})=\text{tr}(T-T)=0 \quad (4.12)$$

using the cyclic property of the trace and the unitarity of S and $\mathscr{S}$. Thus the region $|E|>m+v/2$ can be omitted from Eq. (4.11) so that

$$\mathscr{F}^{\text{tot}}_T=\frac{1}{4\pi}\sum_{\kappa,\alpha}\left[\int_{-v/2-m}^{-\kappa v-m}dE-\int_{-\kappa v+m}^{v/2+m}dE\right]\langle\kappa,\alpha\,|\,(S^\dagger TS-\mathscr{S}^\dagger T\mathscr{S})\,|\,\kappa,\alpha\rangle , \quad (4.13)$$

an expression with no remaining ultraviolet ambiguities.[14]

Finally, we must show that this result is consistent with the formula derived in Sec. III. Toward this end we introduce P_κ, the projection operator onto the $I_z=\kappa$ subspace, and write the integrand in Eq. (4.13) as

$$\sum_{\kappa'}\text{tr}[P_\kappa(S^\dagger P_{\kappa'}TS-\mathscr{S}^\dagger P_{\kappa'}T\mathscr{S})] . \quad (4.14)$$

We assume that T commutes with P_κ; and, by definition, $\mathscr{S}$ also does. Then (4.14) can then be written

$$\text{tr}[P_\kappa(S^\dagger P_{-\kappa}TS)]+\text{tr}[P_\kappa(S^\dagger P_\kappa TS-TP_\kappa)] . \quad (4.15)$$

The first term in (4.15) is half of the integrand in Eq. (3.14) of Sec. III. The second term can be simplified further. If we replace the first P_κ by $I-P_{-\kappa}$ we obtain

$$\mathrm{tr}[(I-P_{-\kappa})(S^\dagger P_\kappa TS-TP_\kappa)]=-\mathrm{tr}(P_{-\kappa}S^\dagger P_\kappa TS)\ , \quad (4.16)$$

a quantity identical to the first term in the expression (4.15) except for the minus sign and the replacement $\kappa\to-\kappa$. However, $P_\kappa S(E)P_{-\kappa}$ vanishes unless both $E+v/2$ and $E-v/2$ are possible free-particle energies. Consequently, the combination of integrals in Eq. (4.13) in parentheses reduces to a single integral from $-v/2+m$ to $+v/2-m$ with a sign opposite to that of κ. Thus

$$\begin{aligned}\mathscr{F}_T^{\mathrm{tot}}&=\frac{1}{4\pi}\sum_\kappa\int_{-v/2+m}^{v/2-m}(-2\kappa)dE[\mathrm{tr}(P_\kappa S^\dagger P_{-\kappa}TS)-\mathrm{tr}(P_{-\kappa}S^\dagger P_\kappa TS)]\\&=\frac{1}{\pi}\sum_\kappa\int_{-v/2+m}^{+v/2-m}\kappa\,\mathrm{tr}(P_{-\kappa}S^\dagger P_{+\kappa}TS)dE\ ,\end{aligned} \quad (4.17)$$

a result identical to Eq. (3.14).

It is interesting to note how the violation of chirality required by the axial anomaly enters this formalism when $m=0$. Clearly, if the charge T being studied commutes with S and $\mathscr{S}$ then both S and $\mathscr{S}$ can be eliminated from Eq. (4.13) and the resulting T-T inside the parentheses vanishes giving zero total emitted flux for such a conserved quantity. For the case of chirality, $T=\gamma^5$ which does commute with the single-particle Dirac Hamiltonian and hence with S. However, the separation of the $\kappa=\pm\frac{1}{2}$ Dirac fields necessary for the definition of $|i\rangle$, and built into $\mathscr{S}$, violates chirality. For that case Eq. (4.13) becomes

$$\mathscr{F}_5^{\mathrm{tot}}=\frac{1}{2\pi}\sum_{\kappa,\alpha}(-\kappa)\int_{-v/2}^{v/2}dE\langle\kappa,\alpha|\gamma^5-\mathscr{S}^\dagger\gamma^5\mathscr{S}|\kappa,\alpha\rangle\ . \quad (4.18)$$

As we argued in out previous paper,[4] the work of Kazama, Yang, and Goldhaber[8,10] implies that for $\kappa=+\frac{1}{2}$ the total angular momentum $j=0$ wave has an incoming part with pure negative helicity while the outgoing part has positive helicity. Hence, for the $j=0$ wave the γ^5 term in Eq. (4.18) is -1 while $\mathscr{S}^\dagger\gamma^5\mathscr{S}$ is $+1$. For $\kappa=-\frac{1}{2}$ the signs reverse, just compensating the changing sign of the factor κ with the result

$$\mathscr{F}_5^{\mathrm{tot}}=\frac{1}{\pi}\int_{-v/2}^{+v/2}dE=\frac{v}{\pi}\ . \quad (4.19)$$

[$\mathscr{S}$ is helicity conserving for the higher partial waves which therefore do not contribute to the sum over α in Eq. (4.18).] The value of v/π for the emitted flux of chirality can also be obtained directly[4] from the anomalous divergence equation obeyed by the gauge-invariant chiral current[15]:

$$\begin{aligned}\mathscr{F}_5^{\mathrm{tot}}&=\int d^3r\,\partial_i j_5^i\\&=\frac{g^2}{16\pi^2}\int d^3r\,F_{\mu\nu}^\alpha\tilde{F}^{\alpha,\mu\nu}=\frac{v}{\pi}\ .\end{aligned} \quad (4.20)$$

V. FLUX CALCULATION FOR A POINTLIKE DYON

The single-particle S-matrix elements can be computed explicitly for the limiting case of a pointlike Julia-Zee dyon. The Dirac Hamiltonian then becomes the direct sum of two Abelian Hamiltonians:

$$H=\begin{bmatrix}\mathscr{H}_+ & 0\\ 0 & \mathscr{H}_-\end{bmatrix} \quad (5.1)$$

with

$$\begin{aligned}\mathscr{H}_\pm&=\vec{\alpha}\cdot\left[-i\vec{\nabla}\mp\frac{1}{2}\frac{\hat{r}\times\hat{z}}{r(1-\hat{z}\cdot\hat{r})}\right]\\&\quad\pm\beta m\pm\frac{1}{2}\left[\frac{Q}{r}-v\right]\ .\end{aligned} \quad (5.2)$$

We use the representation in which

$$\vec{\alpha}=\begin{bmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{bmatrix}\ ,\quad \beta=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}\ .$$

The singular $1/r$ potentials in H require the imposition of unconventional boundary conditions on the fermion wave function $\psi=\binom{\psi_+}{\psi_-}$ at the location of the point dyon ($r=0$). These conditions will be determined by the requirement that H be a well-defined Hermitian operator having the same symmetries as the nonsingular Hamiltonian for a non-Abelian dyon with spatial extent.

To find the scattering states of H, we choose the fermion wave function to be a simultaneous eigenfunction of H, $\vec{J}^2$, and J_z with corresponding eigenvalues E, $j(j+1)$, and μ.[8] Here the total angular momentum $\vec{J}=\vec{L}+\frac{1}{2}\vec{\sigma}$ with

$$\vec{L}=\vec{r}\times\left[-i\vec{\nabla}-\frac{\tau^3}{2}\frac{\hat{r}\times\hat{z}}{r(1-\hat{z}\cdot\hat{r})}\right]-\frac{\tau^3}{2}\hat{r}\,.$$

For a dyon with monopole strength $1/g$ and a fermion with electric charge $\pm g/2$, j can take on the values $0,1,2,\ldots$. For $j=0$, the eigenfunctions have the form

$$\psi_{E,j=0,\mu=0}(\vec{r})=\frac{1}{r}\begin{bmatrix} f_+(r)\eta_0^+(\theta,\phi) \\ -ig_+(r)\eta_0^+(\theta,\phi) \\ f_-(r)\eta_0^-(\theta,\phi) \\ +ig_-(r)\eta_0^-(\theta,\phi)\end{bmatrix}, \tag{5.3a}$$

where the normalized two-component spinors $\eta_0^\pm$ satisfy

$$(\vec{\sigma}\cdot\hat{r})\eta_0^\pm(\theta,\phi)=\pm\eta_0^\pm(\theta,\phi) \tag{5.3b}$$

and where the radial wave functions obey the differential equations

$$\frac{df_\pm}{dr}\mp\left[m-\frac{Q}{2r}+\frac{v}{2}\pm E\right]g_\pm=0\,,$$
$$\frac{dg_\pm}{dr}\mp\left[m+\frac{Q}{2r}-\frac{v}{2}\mp E\right]f_\pm=0\,. \tag{5.3c}$$

For $j>0$, there are two classes of eigenfunctions which can be chosen to be mutually orthogonal. Using the index $a=1,2$ to distinguish between these two classes and letting $b_a=2$ for $a=1$ and $b_a=1$ for $a=2$, these eigenfunctions can be written in the form

$$\psi_{a,E,j,\mu}(\vec{r})=\frac{1}{r}\begin{bmatrix} f_+(r)\xi_{j\mu}^{(a)+}(\theta,\phi) \\ -ig_+(r)\xi_{j\mu}^{(b_a)+}(\theta,\phi) \\ f_-(r)\xi_{j\mu}^{(a)-}(\theta,\phi) \\ +ig_-(r)\xi_{j\mu}^{(b_a)-}(\theta,\phi)\end{bmatrix}, \tag{5.4a}$$

where the normalized two-component spinors $\xi_{j\mu}^{(1,2)\pm}$ satisfy

$$(\vec{\sigma}\cdot\hat{r})\xi_{j\mu}^{(1)\pm}(\theta,\phi)=-\xi_{j\mu}^{(2)\pm}(\theta,\phi) \tag{5.4b}$$

and where the radial wave functions obey the differential equations

$$\left[\frac{d}{dr}-\frac{(b_a-a)[j(j+1)]^{1/2}}{r}\right]f_\pm+\left[m-\frac{Q}{2r}+\frac{v}{2}\pm E\right]g_\pm=0\,,$$
$$\left[\frac{d}{dr}+\frac{(b_a-a)[j(j+1)]^{1/2}}{r}\right]g_\pm+\left[m+\frac{Q}{2r}-\frac{v}{2}\mp E\right]f_\pm=0\,. \tag{5.4c}$$

The solutions of Eqs. (5.3c) and (5.4c) for small r are

$$f,g\approx r^{\pm[j(j+1)-Q^2/4]^{1/2}} \tag{5.5}$$

for any $j\geq 0$.

For simplicity, let us consider a dyon whose electric charge $Q/g<\sqrt{8}/g$. Then for $j\neq 0$, the only acceptable solutions are those which have the positive exponent. This means that energy eigenfunctions with nonzero angular momentum satisfy the conventional boundary conditions

$$f_\pm=g_\pm=0 \quad \text{at } r=0 \tag{5.6}$$

and do not give rise to any charge mixing. On the other hand, both $j=0$ solutions ($r^{\pm iQ/2}$) are square integrable in the neighborhood of the origin and no nontrivial linear combination of them vanishes as $r\to 0$. The appropriate boundary conditions for these waves can be determined from considerations of Hermiticity and symmetry.[10] For the Hamiltonian H to be Hermitian, the surface integral generated at small r by integrating a term like $\psi_{E'}^\dagger\vec{\alpha}\cdot\vec{p}\psi_E$ by parts must vanish:

$$r^2\int d\Omega\,\psi_{E'}^\dagger\vec{\alpha}\cdot\hat{r}\psi_E=0\,. \tag{5.7}$$

This gives

$$f_+'^*g_+-g_+'^*f_++f_-'^*g_--g_-'^*f_-=0 \tag{5.8}$$

which may be written as

$$\begin{bmatrix} f'_+ + ig'_+ \\ f'_- + ig'_- \end{bmatrix}^\dagger \begin{bmatrix} f_+ + ig_+ \\ f_- + ig_- \end{bmatrix} = \begin{bmatrix} f'_+ - ig'_+ \\ f'_- - ig'_- \end{bmatrix}^\dagger \begin{bmatrix} f_+ - ig_+ \\ f_- - ig_- \end{bmatrix} . \quad (5.9)$$

The general solution is

$$\begin{bmatrix} f_+ + ig_+ \\ f_- + ig_- \end{bmatrix} = \mathscr{U} \begin{bmatrix} f_+ - ig_+ \\ f_- - ig_- \end{bmatrix} , \quad (5.10)$$

where $\mathscr{U}$ is any two-by-two unitary matrix.[16] It may depend on r but must be independent of the energy E. For each choice of the four real parameters needed to specify $\mathscr{U}$, one has a set of boundary conditions which makes H Hermitian. By taking the pointlike limit of a spatially extended non-Abelian dyon, these parameters can be determined.

As a first step in determining the appropriate matrix $\mathscr{U}$, observe that the Hermiticity of $\vec{\sigma}\cdot\vec{p}$ for the extended dyon requires that

$$r^2 \int d\Omega\, \psi_{E'}^\dagger \vec{\sigma}\cdot\hat{r}\psi_E = 0 \quad (5.11)$$

for small r when the pointlike dyon limit is taken. This gives

$$f'^*_+ f_+ + g'^*_+ g_+ - f'^*_- f_- - g'^*_- g_- = 0 . \quad (5.12)$$

Letting $f'_\pm = f_\pm$ and $g'_\pm = g_\pm$ in Eqs. (5.8) and (5.12), we obtain

$$|f_+ + ig_+|^2 + |f_- + ig_-|^2 = |f_+ - ig_+|^2 + |f_- - ig_-|^2 \quad (5.13)$$

and

$$|f_+ + ig_+|^2 - |f_- + ig_-|^2 = -|f_+ - ig_+|^2 + |f_- - ig_-|^2 . \quad (5.14)$$

The sum and difference of these equations require $|f_+ + ig_+| = |f_- - ig_-|$ and $|f_- + ig_-| = |f_+ - ig_+|$ which imply that $\mathscr{U}$ must have the form

$$\mathscr{U} = \begin{bmatrix} 0 & e^{i\delta_1} \\ e^{i\delta_2} & 0 \end{bmatrix} \quad (5.15)$$

with δ_1 and δ_2 real. Next, we note that $\beta\sigma^2\tau^2 K$, with K denoting the operation of complex conjugation, anticommutes with the non-Abelian Dirac Hamiltonian and therefore transforms an eigenfunction with energy E into one with energy $-E$. For this to remain valid in the pointlike limit, both ψ_E and $\beta\sigma^2\tau^2\psi_E^*$ must satisfy the same boundary conditions. To compute $\beta\sigma^2\tau^2\psi_E^*$, we need the fact that

$$\sigma^2\eta_0^+(\theta,\phi)^* = \eta_0^-(\theta,\phi) , \quad (5.16)$$

which follows from $\sigma^2 K\vec{J} = -(\tau^1\vec{J}\tau^1)\sigma^2 K$ and an appropriate choice of phases for $\eta_0^\pm$. Thus,

$$\beta\sigma^2\tau^2 K \begin{bmatrix} f_+\eta_0^+ \\ -ig_+\eta_0^+ \\ f_-\eta_0^- \\ +ig_-\eta_0^- \end{bmatrix} = i \begin{bmatrix} f_-^*\eta_0^+ \\ +ig_-^*\eta_0^+ \\ f_+^*\eta_0^- \\ -ig_+^*\eta_0^- \end{bmatrix} . \quad (5.17)$$

Imposing the boundary condition (5.10) on the wave function $\beta\sigma^2\tau^2\psi_E^*$ gives

$$\begin{bmatrix} f_-^* - ig_-^* \\ f_+^* - ig_+^* \end{bmatrix} = \mathscr{U} \begin{bmatrix} f_-^* + ig_-^* \\ f_+^* + ig_+^* \end{bmatrix} , \quad (5.18)$$

from which we conclude that $\tau^1 K\mathscr{U} = \mathscr{U}\tau^1 K$. This implies that $\delta_2 = -\delta_1$. The appropriate boundary conditions for energy eigenfunctions with zero angular momentum therefore take the final form[17]

$$f_+ = e^{i\delta} f_- \text{ and } g_+ = -e^{i\delta} g_- \quad \text{at } r=0 \quad (5.19)$$

and give rise to charge mixing in the $j=0$ channel. (The value of the phase factor $e^{i\delta}$ will not be needed for the calculation which follows.)

To compute the charge-exchange S-matrix elements, we must solve the differential equations (5.3c) subject to the boundary conditions (5.19). Let

$$\mathscr{E}_\pm = E \pm \frac{v}{2} ,$$
$$p_\pm = +(\mathscr{E}_\pm^{\,2} - m^2)^{1/2} , \quad (5.20)$$
$$\beta_\pm = \frac{p_\pm}{\pm\mathscr{E}_\pm} ,$$

so that $\mathscr{E}_+ \geq m$ and $\mathscr{E}_- \leq -m$ for the range of integration in (3.13). We now make the substitutions[18]

$$f_\pm(r) = (\mathscr{E}_\pm \pm m)^{1/2} e^{ip_\pm r} r^{\pm iQ/2} [\mu_{1\pm}(r) + \mu_{2\pm}(r)] ,$$
$$g_\pm(r) = +i(\mathscr{E}_\pm \mp m)^{1/2} e^{ip_\pm r} r^{\pm iQ/2} [\mu_{1\pm}(r) - \mu_{2\pm}(r)] , \quad (5.21)$$

with $\arg[(\mathscr{E}_- \pm m)^{1/2}] = \mp\pi/2$, and find that $\mu_{1\pm}$ obeys a confluent hypergeometric equation

$$\rho\frac{d^2\mu_{1\pm}}{d\rho^2}+(1\pm iQ-\rho)\frac{d\mu_{1\pm}}{d\rho}\mp i\frac{Q}{2}\left[1+\frac{\mathscr{E}_\pm}{p_\pm}\right]\mu_{1\pm}=0 \tag{5.22}$$

and that

$$\mu_{2\pm}=\mp\frac{(p_\pm+\mathscr{E}_\pm)}{m}\mu_{1\pm}+\frac{2ip_\pm}{mQ}\rho\frac{d\mu_{1\pm}}{d\rho} \tag{5.23}$$

with $\rho=-2ip_\pm r$. The solution for $\mu_{1\pm}$ is

$$\mu_{1\pm}=C_\pm F\left[\pm i\frac{Q}{2}\left[1+\frac{\mathscr{E}_\pm}{p_\pm}\right],1\pm iQ,\rho\right]+D_\pm\rho^{\mp iQ}F\left[\pm\frac{iQ}{2}\left[-1+\frac{\mathscr{E}_\pm}{p_\pm}\right],1\mp iQ,\rho\right]. \tag{5.24}$$

$C_\pm$ and $D_\pm$ are arbitrary constants and $F(\alpha,\gamma,\rho)$ denotes the confluent hypergeometric function

$$F(\alpha,\gamma,\rho)=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}\,. \tag{5.25}$$

The small-ρ and large-ρ behaviors of this function are given by

$$\begin{aligned}&F(\alpha,\gamma,\rho)\to 1+\frac{\alpha}{\gamma}\rho \quad \text{as } |\rho|\to 0\,,\\ &F(\alpha,\gamma,\rho)\to\frac{\Gamma(\gamma)}{\Gamma(\alpha)}(\rho)^{\alpha-\gamma}e^{\rho}+\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-\rho)^{-\alpha} \quad \text{as } |\rho|\to\infty\end{aligned} \tag{5.26}$$

with $\arg(\pm\rho)=\mp\pi/2$. These limits imply that the upper-component radial wave functions $f_\pm$ have the forms

$$\begin{aligned}&f_\pm\to a_\pm r^{\pm iQ/2}+b_\pm r^{\mp iQ/2} \quad \text{as } r\to 0\,,\\ &f_\pm\to A_\pm r^{-iQ/2\beta_\pm}e^{+ip_\pm r}+B_\pm r^{+iQ/2\beta_\pm}e^{-ip_\pm r} \quad \text{as } r\to\infty\end{aligned} \tag{5.27}$$

with the constants $a_\pm$, $b_\pm$, $A_\pm$, and $B_\pm$ related by

$$\begin{aligned}A_\pm&=2p_\pm\left[\frac{a_\pm\omega_{2\pm}}{\pm m+p_\pm-\mathscr{E}_\pm}-\frac{b_\pm\omega_{1\pm}^*}{\pm m-p_\pm-\mathscr{E}_\pm}\right],\\ B_\pm&=2p_\pm\left[\frac{b_\pm\omega_{2\pm}^*}{\pm m+p_\pm-\mathscr{E}_\pm}-\frac{a_\pm\omega_{1\pm}}{\pm m-p_\pm-\mathscr{E}_\pm}\right],\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}\omega_{1\pm}&=\frac{\Gamma(\pm iQ)}{\Gamma(\pm(iQ/2)(1\pm 1/\beta_\pm))}(2p_\pm)^{\mp i(Q/2)(1\mp 1/\beta_\pm)}e^{\pm(\pi Q/4)(-1\pm 1/\beta_\pm)}\,,\\ \omega_{2\pm}&=\frac{\Gamma(\pm iQ)}{\Gamma(\pm(iQ/2)(1\mp 1/\beta_\pm))}(2p_\pm)^{\mp(iQ/2)(1\pm 1/\beta_\pm)}e^{\pm(\pi Q/4)(1\pm 1/\beta_\pm)}\,.\end{aligned} \tag{5.29}$$

The lower-component radial wave functions $g_\pm$ can be obtained directly from $f_\pm$ by using the first equation in (5.3c).

The non-Abelian boundary condition, $f_+ = e^{i\delta}f_-$ near the origin, requires that

$$a_+=e^{i\delta}b_- \text{ and } b_+=e^{i\delta}a_-\,, \tag{5.30}$$

so that A_+ and B_+ are related to A_- and B_-. This mixing of the two isospin components yields the charge-exchange S-matrix elements. An incoming wave with only an $I_z=+\frac{1}{2}$ component ("positive" energy $\mathscr{E}_+$) will scatter from the dyon and produce an outgoing wave with both $I_z=\pm\frac{1}{2}$ components ("positive" and "negative" energies $\mathscr{E}_\pm$):

$$A_-=0,\quad B_+=\left[\frac{m+\mathscr{E}_+}{4\pi p_+}\right]^{1/2} \tag{5.31a}$$

for the incoming wave with flux $1/2\pi$, and

$$A_+ = S_{+1/2;+1/2}\left[\frac{m+\mathscr{E}_+}{4\pi p_+}\right]^{1/2}, \quad B_- = S_{-1/2;+1/2}\left[\frac{m-\mathscr{E}_-}{4\pi p_-}\right]^{1/2} \tag{5.31b}$$

for the outgoing wave. Using Eqs. (5.28), (5.30), and (5.31a), we can determine A_+ and B_-. In particular, we find

$$S_{-1/2;+1/2} = \frac{e^{-i\delta}}{2(\beta_+\beta_-)^{1/2}\mathscr{D}}, \tag{5.32}$$

where

$$\mathscr{D} = \frac{\omega_{2-}\omega_{1+}}{[(1+\beta_+)(1+\beta_-)]^{1/2}} - \frac{\omega_{2+}^*\omega_{1-}^*}{[(1-\beta_+)(1-\beta_-)]^{1/2}}. \tag{5.33}$$

Similarly, an incoming wave with only an $I_z = -\frac{1}{2}$ component ("negative" energy $\mathscr{E}_-$) will scatter from the dyon and produce an outgoing wave with both $I_z = \mp\frac{1}{2}$ components ("negative" and "positive" energies $\mathscr{E}_\mp$):

$$B_+ = 0, \quad A_- = \left[\frac{m-\mathscr{E}_-}{4\pi p_-}\right]^{1/2} \tag{5.34a}$$

for the incoming wave with flux $1/2\pi$, and

$$B_- = S_{-1/2;-1/2}\left[\frac{m-\mathscr{E}_-}{4\pi p_-}\right]^{1/2}, \quad A_+ = S_{+1/2;-1/2}\left[\frac{m+\mathscr{E}_+}{4\pi p_+}\right]^{1/2} \tag{5.34b}$$

for the outgoing wave. Using Eqs. (5.28), (5.30), and (5.34a), we can determine B_- and A_+. In particular, we find

$$S_{+1/2;-1/2} = \frac{e^{+i\delta}}{2(\beta_+\beta_-)^{1/2}\mathscr{D}}. \tag{5.35}$$

The differential flux of fermions emitted by the decaying dyon is now given by Eqs. (3.9) and (3.12):

$$\mathscr{F} + \overline{\mathscr{F}} = \frac{1}{4\pi\beta_+\beta_- |\mathscr{D}|^2}, \tag{5.36}$$

where β_+ is the velocity of the emitted particles (with charge $+g/2$) and β_- is the velocity of the emitted antiparticles (also with charge $+g/2$). The expression (5.33) simplifies considerably when the two velocities β_+ and β_- are equal:

$$|\mathscr{D}|^2 = \left[\frac{1}{4\beta_+\beta_-}\right]\Bigg|_{\beta_+=\beta_-}, \tag{5.37}$$

so that

$$(\mathscr{F} + \overline{\mathscr{F}})\,|_{\beta_+=\beta_-} = \frac{1}{\pi}. \tag{5.38}$$

In particular, for massless fermions ($\beta_+ = \beta_- = 1$), this result is in agreement with (4.19), the flux of particles plus antiparticles being equal to the flux of chiral charge in this case.

We can also evaluate expression (5.33) for $|\mathscr{D}|^2$ in the nonrelativistic limit ($\beta_+,\beta_- << 1$) by using Stirling's asymptotic formula for the Γ function:

$$|\mathscr{D}|^2 \approx \frac{1}{144}\frac{(Q+1/Q)^2}{\beta_+\beta_-\sinh^2(\pi Q)}(\beta_+{}^2 - \beta_-{}^2)^2 \times e^{+\pi Q(1/\beta_+ + 1/\beta_-)}. \tag{5.39}$$

If this result is compared with Eq. (5.37), we see that we must add the nonleading term $1/(4\beta_+\beta_-)$ for Eq. (5.39) to be correct when $\beta_+ = \beta_-$. The resulting formula for $|\mathscr{D}|^2$ then gives the fermion flux

$$\mathscr{F} + \overline{\mathscr{F}} = \frac{1}{\pi}\frac{\Gamma^2/4}{(|\mathscr{E}_+| - |\mathscr{E}_-|)^2 + \Gamma^2/4}, \tag{5.40}$$

where the function $\Gamma(\beta_+,\beta_-)$ is given by

$$\Gamma(\beta_+,\beta_-) = \frac{6m\sinh(\pi Q)\exp[-(\pi Q/2)(1/\beta_+ + 1/\beta_-)]}{(Q+1/Q)}. \tag{5.41}$$

Thus, in the limit where $v - 2m$ is much smaller than both the inverse dyon radius (the pointlike limit) and the fermion mass m (the nonrelativistic limit), we find a resonant enhancement in the emitted fermion spectrum when $|\mathscr{E}_+| = |\mathscr{E}_-|$. Since the total energy (kinetic plus electrostatic) is

$$E = |\mathscr{E}_+| - \frac{v}{2} = -|\mathscr{E}_-| + \frac{v}{2},$$

this resonance has $|\mathscr{E}_\pm| = v/2$, i.e., $E = 0$. We conclude that the zero-energy bound state of Jackiw and Rebbi[19] acquires a width $\Gamma(\beta,\beta)_{\beta=(1-4m^2/v^2)^{1/2}}$ when $v-2m$ becomes positive and appears as a resonance in the particle production spectrum.

VI. CONCLUSION

We have shown that to leading order in the Yang-Mills coupling constant g the spectrum of fermions with $2m<v$ emitted by a Julia-Zee dyon can be computed in terms of the S matrix for the single-particle Dirac equation in the dyon background. Let us discuss further the physics of our semiclassical, small-g approximation.

In the limit of small g, the dyon's mass and charge become much larger than the mass and charge of the individual fermion quanta. Furthermore, our result for the fermion flux, Eq. (3.13), is independent of the Yang-Mills coupling g. Thus, as g becomes small, the rate of fermion emission is fixed so that the resulting flux of energy is also constant while the flux of charge is proportional to g. Consequently, as $g\to 0$ the radiation of energy and charge must have a smaller and smaller effect on the dyon mass (M/g^2) and charge (Q/g) so that our treatment of the dyon background as constant becomes more and more accurate. If g is sufficiently small, there will exist a range of times $t_1<<t<<t_2$ within which the fermion emission is correctly described as a steady-state flux of particles emanating from the dyon. Here t_1 is the time scale required for the transient effects associated with the arbitrariness in our definition of the initial state $|i\rangle$ to dissipate: t_1^{-1} should be a typical fermion energy $\sim v$. The time t_2 might be thought of as the dyon lifetime, a time over which the fermion emission has a significant effect on the dyon's properties: $t_2 vg\sim Q/g$ or $t_2v^2\sim M/g^2$. The first estimate compares the charge emitted in the time t_2 with the dyon charge, the second compares the emitted energy with the dyon mass. If g^2 is sufficiently small, then $t_2/t_1\sim 1/g^2>>1$ and the regime required by our computational scheme exists.

Finally, we should observe that our analysis cannot be extended to describe the entire dyon decay by allowing a slow decrease of the parameters Q and v in the classical solution. The long-term effects of the fermion emission on the dyon background will be more complex than a simple lowering of Q and v so that for times of the order of t_2 boson radiation should also become important.

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[14]It should be emphasized that the cancellation for $|E|>m+v/2$ between the S and $\mathscr{S}$ terms would occur even if we had introduced a regularization scheme to make the operator $j_F^{\#}$ better defined. For example, if we had defined $j_F^{\#}$ using the usual point-separation technique, the $S^\dagger TS-\mathscr{S}^\dagger T\mathscr{S}$ term in Eq. (4.11) becomes $S^\dagger T_R(E)S-\mathscr{S}^\dagger T_R(E)\mathscr{S}$, where the

modified matrix $T_R(E)$ regulates the integral over E but does not prevent the use of unitarity in Eq. (4.12) and the subsequent cancellation for $|E| > m + v/2$.

[15]Equations (4.19) and (4.20) must agree for the gauge-invariant chiral current since for that choice the "vacuum" flux F_5^0 vanishes and the emitted particles carry the only flux of chirality present at infinity. For a different definition of the chiral current the flux of chirality $\mathscr{F}_5^{\rm tot}$ carried by the emitted particles should remain unchanged. Any change in the divergence $\partial_\mu j_5^\mu$ will be compensated by a corresponding change in F_5^0.

[16]Von Neumann's method for constructing the self-adjoint extensions of a symmetric operator yields the same result. See, for example, N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Ungar, New York, 1961), Vol. 2.

[17]This form can also be obtained by examining the pointlike limit of the zero-energy eigenstates of the non-Abelian Hamiltonian. These zero-energy scattering states can be obtained quite easily by solving the non-Abelian Dirac equation decomposed in angular momentum eigenstates: R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).

[18]This analysis of the Dirac equation in the field of an Abelian dyon was given by Ju-Fei Tang, Ref. 7.

[19]R. Jackiw and C. Rebbi, Ref. 17.