Reduction formulas for quantum electrodynamics

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The problem of deriving the Lehmann-Symanzik-Zimmermann reduction formulas for manifestly covariant quantum electrodynamics is reexamined. The problem of defining states which satisfy the Gupta-Bleuler condition in the infrared limit is resolved by applying a nonlocal instantaneous pseudounitary transformation to the direct-product Fock space before applying the infrared transformation. These states are reduced to find that the interpolating Heisenberg spinor fields pick up an operator-valued phase which makes them manifestly gauge invariant. The time-ordered products associated with the scattering amplitudes are shown to have perturbative representations consistent with the form of asymptotic limits selected for the interpolating fields. The phase on the spinor fields causes the *S* matrix to have the usual Coulomb-gauge Feynman rules, while the infrared problem is resolved by developing the infrared form for the asymptotic Coulomb-gauge fields. Strict satisfaction of the Gupta-Bleuler condition is seen to be unnecessary in a charge-conserving theory.

I. INTRODUCTION

It has long been known that the Lehmann-Symanzik-Zimmermann (LSZ) reduction technique[1] encounters difficulties when a manifestly covariant formulation of quantum electrodynamics (QED) is attempted. There are several factors which conspire to complicate such a program for QED, each of which has a long history of analysis in the literature. For the sake of placing the work presented here in its proper perspective these aspects of the problem will be discussed in the first part of this section.

The first difficulty is created by the presence of the long-range Coulomb interaction which precludes use of the noninteracting adiabatic switching (hereafter NAS) assumption. Stated simply, this prevents the renormalized interacting Heisenberg fields, which obey the full nonlinear equations of motion, from asymptotically approaching the respective free Heisenberg fields. In this respect it was first pointed out by Bloch and Nordsieck,[2] and subsequently discussed by other authors,[3] that a state corresponding to a physical charged particle must necessarily contain an infinite number of coherent photons. Dollard[4] showed for the nonrelativistic case that careful examination of the interaction leads to selection of a more complicated form for the asymptotic Hamiltonian, which, however, can still be diagonalized. In generalizing this result to the relativistic case Kulish and Faddeev[5] showed that the fields whose time developments were given by this Hamiltonian could be obtained from the free fields by a time-dependent pseudounitary transformation. These are the asymptotic limits of the Heisenberg fields when an interacting adiabatic switching (hereafter IAS) assumption is made. When applied to the original Fock space of the theory this pseudounitary transformation generates a space of coherent states which is orthogonal to the Fock space. These states are referred to as infraparticles. In essence, IAS allows a new interaction-picture representation of the *S* matrix in terms of operators manifesting distortion by the tail of the Coulomb interaction. The advantage obtained from this improved *S* matrix is the absence of infrared divergences, although ultraviolet divergences remain.

It is important to note that the standard treatment in textbooks[6] defines the *S* matrix for QED in the Coulomb gauge and forces NAS on the theory. It is this assumption which causes the infrared divergences in the perturbative representation of the *S* matrix. However, these divergences can be canceled in a consistent way, as shown by Yennie, Frautschi, and Suura,[7] although it requires two steps. In intermediate stages of calculation the photon is given a finite mass. A physical process is then defined by summing appropriate self-energy and soft bremstrahlung graphs to cancel infrareddivergent pieces, and the photon mass is then restored to zero. Forcing NAS on the theory has not destroyed it, but has created the need for a rather tedious redefinition of physical processes. Nevertheless, many practitioners are content with such a formulation of QED.

A second difficulty associated with the NAS assumption in manifestly covariant QED was first uncovered by Källén,[8] who showed that if the asymptotic vector field satisfies the standard free-field commutator and the interacting field satisfies the manifestly covariant Yang-Feldman equation,[9] then causality is violated in the interacting region in the sense that the interpolating field fails to commute at spacelike distances. This difficulty is not serious to the physical interpretation of the theory since the components of the Maxwell tensor, which are the physical observables, do commute at spacelike distances. Källén showed that this problem is alleviated if the commutator for the asymptotic vector field carries a vestige of the interaction in the form of a constant. In later work Nakanishi[10] used this commutator to develop a manifestly covariant LSZ reduction formula for scalar electrodynamics in order to examine spontaneous breakdown of symmetry. Hammer and DeFacio[11] have pointed out certain inconsistencies in Nakanishi's work. However, any attempt to construct an LSZ reduction formula for Lorentz-gauge QED particle states must contend with Källén's results.

A third source of difficulty in manifestly covariant QED stems from the need to quantize the theory in an indefinite-metric space.[12] In order to maintain a probabilistic interpretation the Gupta-Bleuler (hereafter GB) condition[13] is used to select a subspace with positive-definite seminorm. This procedure is unambiguous and clear for the noninteracting case, but the generalization to the interacting case has not been trivial. Evans and Fulton[14] showed that NAS is inconsistent with the free GB condition, indicating a nonmanifestly covariant formulation of QED is necessary. Several authors[15] have showed, under very general assumptions, that there exist no localized charged states which satisfy the GB condition. Recently Fröhlich, Morchio, and Strocchi[16] have shown that the charged sector of scattering states in QED is neither localized nor boost covariant. For these reasons it is tempting to abandon the GB condition in favor of alternative constraints which maintain locality and Lorentz covariance. However, the viewpoint of this paper is that the GB condition is necessary to define correctly the charged sector. This viewpoint is supported by the work of Haller[17] who has shown that the interaction-picture states consistent with NAS which satisfy the GB condition form a coherent subspace obtainable from the subspace that satisfies the free GB condition by a pseudounitary transformation. In agreement with rigorous results the charged states are no longer local and do not transform covariantly because the pseudounitary operator contains the instantaneous Coulomb potential. However, when the particle states are defined as these transformed free-particle states, the S matrix for their scattering derived from the NAS assumption is term-by-term equivalent to the standard Coulomb-gauge S matrix. Furthermore, any allowed zero-norm state contributes nothing to scattering. As a result, the GB condition forces manifestly covariant formulations of QED to be dynamically equivalent to the Coulomb gauge, at least for the assumption of NAS.

It is logical to suppose that the infraparticle states of Kulish and Faddeev could be used as the asymptotic particle states of QED. These could then be reduced by the LSZ technique to obtain formulas consistent with the infrared structure of the theory. Such reduction formulas should lead to an S matrix free of infrared divergences. This program has been examined in a series of papers by Zwanziger.[18] Zwanziger's formulation employs asymptotic states which do not satisfy the asymptotic form of the GB condition. Instead, alternative measures are taken to insure positivity. A second difficulty with the results arises from the fact that the asymptotic vector fields obey the free-field commutation relations while the interpolating field satisfies the same Yang-Feldman equation as that assumed by Källén. As a result, the problem with causality discussed earlier in this section surfaces.

It is the purpose of this paper to reexamine the LSZ formulation of QED in both the NAS and IAS cases. It will be seen that it is possible to construct states which obey the asymptotic form of the GB condition consistent with the form of adiabatic switching selected, either NAS of IAS. In both cases these asymptotic particle states are represented by coherent states obtained by operating on the subspace of the Fock space which satisfies the free GB condition with a pseudounitary transformation. In the NAS case the pseudounitary transformation is the one used by Haller, while in the IAS case it is a combination of the one used by Haller and the one found by Kulish and Faddeev. In either case the scattering amplitude takes the same form when written in terms of interpolating fields. These reduction formulas are obtained by emptying the original Fock particle state and leaving the transformation in place on the Fock vacuum since it leaves the Fock vacuum unchanged. It will be shown that use of the NAS assumption leads directly from the reduction formula to the standard Coulomb-gauge S matrix with its well-known Feynman rules, while using the IAS assumption leads to an S matrix with very similar rules but which uses different propagators for the charged particles. In either case the argument made by Källén is circumvented because the Yang-Feldman equations for the interpolating fields differ from the standard forms.

Throughout this work familiarity with the LSZ program at the textbook level[19] will be assumed, along with the results of Kulish and Faddeev, although a brief sketch of their work is presented in Sec. II. The work presented will be heuristic in nature because of the manipulation of unbounded operators. This paper is not meant to be a rigorous analysis of scattering in QED. Instead, motivation for the steps taken will be made, where possible, by appealing to formal arguments which have their origin in perturbation theory. However, nowhere in this paper will the renormalizability or convergence of any perturbation series derived be discussed.

The remainder of this paper can now be outlined. In Sec. II the notation is established and the results of Kulish and Faddeev are reviewed. Section III develops the states which satisfy the asymptotic GB condition consistent with either the NAS or IAS assumption. In Sec. IV these states are reduced for both cases and shown to take the same manifestly gauge-invariant form in terms of interpolating Heisenberg fields. Section V derives the S matrix for both cases, verifying that NAS leads to the Coulomb-gauge Feynman rules with their infrared divergences, while IAS gives an S matrix which is free of such problems but is considerably more difficult to calculate with. Section VI contains conclusions as well as a discussion as to why strict satisfaction of the GB condition is not necessary in a charge-conserving theory such as QED.

II. PRELIMINARIES

In this section the notation and basic assumptions will be established and a sketch of the derivation of the infrared coherent states will be presented. The reason for including this is to refresh the reader's memory regarding several key points in the derivation which will be important later. For simplicity attention will be restricted to the Feynman gauge, although generalization is straightforward.

A. Notation

The vector field A_μ and the bispinor field ψ are assumed to obey the equations of motion

$$\Box A_\mu = e\bar{\psi}\gamma_\mu\psi \equiv eJ_\mu \ , \quad (2.1a)$$

$$(i\gamma^\mu\partial_\mu - m)\psi = eA_\mu\gamma^\mu\psi \ . \quad (2.1b)$$

It follows from current conservation that

$$\Box\partial_\mu A^\mu = 0 \ . \quad (2.2)$$

It is assumed that (2.1) is derived from an action which can also be used to find the Hamiltonian and the momenta canonically conjugate to the fields. The part of the Hamiltonian described by the interaction is the standard coupling

$$H_I = \int d^3x\,[eA_\mu\bar{\psi}\gamma^\mu\psi] \ . \quad (2.3)$$

It is relevant to review briefly the quantization of the free vector field a_μ and the bispinor field ϕ.[20] The bispinor field is decomposed according to

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\left[\frac{m}{\epsilon_{\vec{p}}}\right]^{1/2}\sum_{s=1}^{2}(b^s_{\vec{p}}u^s_{\vec{p}}e^{-ipx} + d^{s\dagger}_{\vec{p}}v^s_{\vec{p}}e^{ipx}) \ , \quad (2.4a)$$

where

$$p_0 = \epsilon_{\vec{p}} = (\vec{p}^{\,2} + m^2)^{1/2} \ . \quad (2.4b)$$

The field is quantized with the anticommutation relations

$$\{ b^{s\dagger}_{\vec{p}}, b^{s'}_{\vec{k}} \} = \{ d^{s\dagger}_{\vec{p}}, d^{s'}_{\vec{k}} \} = \delta_{ss'}\delta^3(\vec{p} - \vec{k}) \ . \quad (2.5)$$

The vector field has the decomposition

$$a_\mu(x)=\frac{1}{(2\pi)^{3/2}}\int d^3k(2\omega_{\vec{k}})^{-1/2}[a_\mu(\vec{k})e^{-ikx}+a_\mu^\dagger(\vec{k})e^{ikx}]\ , \tag{2.6a}$$

where

$$k_0=\omega_{\vec{k}}=|\vec{k}| \tag{2.6b}$$

and

$$a_\mu(\vec{k})=\epsilon_\mu^\lambda(\vec{k})\alpha_{\vec{k}}^\lambda\ . \tag{2.6c}$$

In (2.6c) the ϵ_μ are the polarization vectors and the sum is over all four polarizations. The field is quantized in an indefinite-metric space by the commutation relation

$$[\alpha_{\vec{p}}^\lambda,\alpha_{\vec{k}}^{\lambda'\dagger}]=g^{\lambda\lambda'}\delta^3(\vec{p}-\vec{k})\ , \tag{2.7}$$

where $g^{\lambda\lambda'}$ is the metric tensor with $g^{00}=-1$.

The need to identify $\alpha_{\vec{k}}^{0\dagger}$ as a creation operator introduces negative-norm states into the theory and these threaten the positive-definiteness of the Hamiltonian. This problem is avoided by defining the physical subspace as the one which satisfies the Gupta-Bleuler condition.[13] The time-independent operator $a[f]$ is defined as

$$a[f]=\int d^3x\, f^{(+)}(\vec{x},t)\overleftrightarrow{\partial}_t\partial_\mu a^\mu(\vec{x},t)\ , \tag{2.8a}$$

where $f^{(+)}$ is a positive-frequency wave packet satisfying

$$\Box f^{(+)}=0\ , \tag{2.8b}$$

and is assumed to be well behaved at spatial infinity. The GB condition restricts the physical states of the theory to those Fock states which satisfy

$$a[f]\,|\,\text{physical}\rangle_{\text{Fock}}=0\ . \tag{2.9}$$

Using (2.6a) shows that this is equivalent to demanding that

$$(\alpha_{\vec{k}}^3+\alpha_{\vec{k}}^0)\,|\,\text{physical}\rangle_{\text{Fock}}=0\ ,\quad \forall\vec{k}\ . \tag{2.10}$$

Because of the indefinite metric it follows that

$$[\alpha_{\vec{k}}^3+\alpha_{\vec{k}}^0,\alpha_{\vec{p}}^{3\dagger}+\alpha_{\vec{p}}^{0\dagger}]=0\ , \tag{2.11}$$

so that the physical subspace decomposes into the direct sum of two subspaces, V_T, the set of all transverse photon states of positive norm, and V_0, a subspace of zero-norm ghost states obtained by operating on the states of V_T with arbitrary products of $(\alpha_{\vec{k}}^{3\dagger}+\alpha_{\vec{k}}^{0\dagger})$. The unphysical subspace V_u is obtained by operating on the states of $V_{\text{phys}}=V_T\oplus V_0$ with operators of the form $(\alpha_{\vec{k}}^{3\dagger}-\alpha_{\vec{k}}^{0\dagger})$.

B. Infrared asymptotic fields

The form of the asymptotic condition is critical to the LSZ reduction program. For the purposes of this paper the IAS form of the asymptotic fields will be developed using an argument borrowed from perturbation theory. The argument is heuristic in nature, but the results allow immediate implementation in perturbative representations of the S matrix.

The interpolating fields are related to their free counterparts by the time-dependent unitary transformation $Z(t)$ in the manner

$$\begin{aligned}Z(t)A_\mu(\vec{x},t)Z^{-1}(t)&=a_\mu(\vec{x},t)\ ,\\ Z(t)\psi(\vec{x},t)Z^{-1}(t)&=\phi(\vec{x},t)\ .\end{aligned} \tag{2.12}$$

Such a transformation has been shown not to exist in the infinite-volume limit.[21] Nevertheless, this assumption is crucial to the development of an interaction-picture representation in operator formalisms. It follows from (2.12) that

$$Z(t)\dot{A}_\mu(\vec{x},t)Z^{-1}(t)=\dot{a}_\mu(\vec{x},t)-[\dot{Z}(t)Z^{-1}(t),a_\mu(\vec{x},t)]\ , \tag{2.13}$$

so that, for Feynman-gauge QED,

$$\dot{Z}(t)Z^{-1}(t)=-i\int d^3x[e\bar{\phi}\gamma^\mu\phi a_\mu]+i\epsilon_0(t)\equiv -iH_I(t)+i\epsilon_0(t) \tag{2.14}$$

with the time of the operators in (2.14) coinciding with t, and $\epsilon_0(t)$ is some indeterminate time-dependent c number.

Care must be taken to state the asymptotic limit of the interpolating fields in terms of a smeared form with the weak limit understood. The assumption is that

$$\underset{t\to t_{\text{as}}}{\text{w-lim}}\int d^3x\, f(\vec{x},t)A_\mu(\vec{x},t)=\underset{t\to t_{\text{as}}}{\text{w-lim}}\int d^3x\, f(\vec{x},t)a_\mu^{\text{as}}(\vec{x},t)\ , \tag{2.15}$$

where $f(x,t)$ is a wave packet satisfying

$$\Box f(\vec{x},t)=0\ , \tag{2.16}$$

and it is understood that t_{as} is arbitrarily far in the future or past. Similar expressions are assumed for ψ smeared with a free bispinor function. The assumption to be used in this paper is that the asymptotic fields are derivable from the free fields by a pseudounitary transformation $U(t)$, so that

$$\begin{aligned} a_\mu^{as}(\vec{x},t)&=U(t)a_\mu(\vec{x},t)U^{-1}(t)\ ,\\ \phi^{as}(\vec{x},t)&=U(t)\phi(\vec{x},t)U^{-1}(t)\ . \end{aligned} \tag{2.17}$$

Clearly, (2.17) is not the most general form for the asymptotic field. However, it allows an immediate connection to standard perturbative approaches and thus the infrared form of the fields may be found. Furthermore, substituting (2.12) into (2.15) and using (2.17) shows that

$$\operatorname*{w-lim}_{t\to t_{as}} U(t)Z(t)=\lambda^{\pm}\ , \tag{2.18}$$

where $\lambda^{\pm}$ is a phase factor.

The noninteracting-adiabatic-switching assumption posits that $U(t)$ is a constant, so that the fields totally decouple at large times. If $U(t)$ is assumed to be nontrivial, as in the interacting-adiabatic-switching case, relations (2.18) and (2.13) place severe restrictions on its form, and indeed allow a perturbative calculation of $U(t)$. What follows is a brief recapitulation of the derivation of $U(t)$.

The basic tactic is to find a time-ordered representation for $U(t)U^{-1}(t')$ which can be separated to obtain $U(t)$. From (2.12) and (2.17) it is clear that

$$\begin{aligned} &\operatorname*{w-lim}_{t\to t_{as}} \int d^3x\, f(\vec{x},t)\{\, Z(t)\dot{A}_\mu(\vec{x},t)Z^{-1}(t)+[\dot{Z}(t)Z^{-1}(t),a_\mu(\vec{x},t)]\,\}\\ &\qquad =\operatorname*{w-lim}_{t\to t_{as}} \int d^3x\, f(\vec{x},t)\{\, U^{-1}(t)\dot{a}_\mu^{as}(\vec{x},t)U(t)+[\dot{U}^{-1}(t)U(t),a_\mu(\vec{x},t)]\,\}\ . \end{aligned} \tag{2.19}$$

As t approaches t_{as} the first terms on each side cancel because of (2.15) and (2.18). Thus $-i\dot{U}^{-1}(t)U(t)$ must be related to the large-time behavior of (2.14). Inserting the standard plane-wave decompositions (2.4a) and (2.6a) into (2.14), and using the large-time form

$$\exp[i(\omega_{\vec{k}}-\epsilon_{\vec{p}}+\epsilon_{\vec{p}+\vec{k}})t_{as}]=\exp(ik_\mu p^\mu t_{as}/\epsilon_{\vec{p}})\ ,\quad k_\mu\approx 0 \tag{2.20}$$

and dropping all terms which oscillate rapidly in this limit gives

$$\dot{U}^{-1}(t)U(t)=-iH_I^{as}(t)=-i\int d^3x j_\mu^{as}(\vec{x},t)a^\mu(\vec{x},t)\ , \tag{2.21}$$

where

$$j_\mu^{as}(\vec{x},t)=e\int \frac{d^3p}{(2\pi)^{3/2}}\rho(\vec{p})\frac{p_\mu}{\epsilon_{\vec{p}}}\delta\left[\vec{x}-\frac{\vec{p}}{\epsilon_{\vec{p}}}t\right] \tag{2.22}$$

and

$$\rho(\vec{p})=\sum_{s=1}^{2}(b_{\vec{p}}^{s\dagger}b_{\vec{p}}^{s}-d_{\vec{p}}^{s\dagger}d_{\vec{p}}^{s})\ . \tag{2.23}$$

That (2.21) is nonvanishing is a result of the masslessness of the photon which allows condition (2.20) to be met.

The iteration of expression (2.21) gives

$$U(t_2)U^{-1}(t_1)=T\left\{\exp\left[i\int_{t_1}^{t_2}dt\, H_I^{as}(t)\right]\right\}\ . \tag{2.24}$$

The time-ordered product may be evaluated explicitly to obtain

$$U(t_2)U^{-1}(t_1)=\exp\left[i\int_{t_1}^{t_2}dt\, H_I^{as}(t)\right]\exp\left[\tfrac{1}{2}i\int d^4x\, d^4x' j_\mu^{as}(x)\Delta_R^{\mu\nu}(x-x')j_\nu^{as}(x')\right]\ , \tag{2.25}$$

where $\Delta_R^{\mu\nu}$ is the retarded Green's function which has the representation

$$\Delta_R^{\mu\nu}(x-x')=-i\theta(t-t')[a^\mu(x),a^\nu(x')]=-i\theta(t-t')\int\frac{d^3k}{(2\pi)^3}(2\omega_{\vec{k}})^{-1}[e^{ik(x-x')}-e^{-ik(x-x')}]\,, \tag{2.26}$$

with θ the standard step function. It is straightforward to find

$$\exp\left[i\int_{t_1}^{t_2}dt\,H_I^{\rm as}(t)\right]=\exp\{-[R(t_2)-R(t_1)]\}\,, \tag{2.27}$$

where

$$R(t)=\frac{e}{(2\pi)^{3/2}}\int d^3p\,d^3k(2\omega_{\vec{k}})^{1/2}\frac{\rho(\vec{p})p_\mu}{(pk)}[a^{\mu\dagger}(\vec{k})e^{ikpt/\epsilon_{\vec{p}}}-a^\mu(\vec{k})e^{-ikpt/\epsilon_{\vec{p}}}]\,. \tag{2.28}$$

Using the Baker-Campbell-Hausdorf relation shows that expression (2.27) becomes

$$\exp\{-[R(t_2)-R(t_1)]\}=\exp[-R(t_2)]\exp[R(t_1)]\exp[iC(t_2,t_1)]\,, \tag{2.29}$$

where

$$C(t_2,t_1)=\frac{e^2}{(2\pi)^3}\int d^3p\,d^3kd^3q\rho(\vec{p})\rho(\vec{q})(pq)[2\omega_{\vec{k}}(pk)(qk)]^{-1}\sin\left[\frac{kpt_2}{\epsilon_{\vec{p}}}-\frac{kqt_1}{\epsilon_{\vec{p}}}\right]. \tag{2.30}$$

Clearly $C(t_2,t_1)$ vanishes only when $t_2=t_1$, and is not separable in t_2 and t_1. Fortunately, it can be canceled against a piece of the other term in (2.25). It is not difficult to show that

$$\exp\left[\tfrac{1}{2}i\int d^4x\,d^4x'j_\mu^{\rm as}(x)\Delta_R^{\mu\nu}(x-x')j_\nu^{\rm as}(x')\right]=\exp[-i\beta(t_2)+i\beta(t_1)-iC(t_2,t_1)]\,, \tag{2.31}$$

where

$$\beta(t)=\frac{e^2}{(2\pi)^3}\int d^3k\,d^3p\,d^3q\,\rho(\vec{p})\rho(\vec{q})(pq)\left[2\omega_{\vec{k}}\epsilon_{\vec{p}}(kq)\left[\frac{kp}{\epsilon_{\vec{p}}}-\frac{kq}{\epsilon_{\vec{q}}}\right]\right]^{-1}\sin\left[\left[\frac{kp}{\epsilon_{\vec{p}}}-\frac{kq}{\epsilon_{\vec{q}}}\right]t\right]. \tag{2.32}$$

Because C, R, and β commute at all times the C term is canceled, and the form for $U(t)$ is then given by

$$U(t)=\exp[-R(t)]\exp[-i\beta(t)]\,. \tag{2.33}$$

This is very similar to forms given elsewhere. The difference lies in the time dependence of $\beta(t)$. This arises from the assumption in other derivations[22] that there is a time t_0 for which $U(t_0)=1$, and this leads to an artificial introduction of t_0 into the expression for U. Since such a time t_0 clearly does not exist, the dependence of U on t_0 is usually discarded. The derivation presented here has the advantage that this problem never appears, and this is the primary reason it was presented in such detail. The secondary reason is that this derivation will be repeated in Sec. V for a different interaction. As a final note it is easy to show that $\beta(t)$ is of the form (sgn$t$) multiplied by a time-independent operator, and that U does satisfy (2.21). The expression normally presented in the literature does not, although it is derived by assuming that it does.

When the asymptotic fields are evaluated using (2.33) in (2.17) it shows that the asymptotic vector field is the free Feynman gauge field (2.6a) with a modification due to the Lienard-Wiechert potential of any charges present. The charged particles pick up an eikonal phase which represents a distortion of the plane wave due to the Lienard-Wiechert potentials of the other charged particles present. The standard definition of the infraparticle spectrum is

$$|\,{\rm as}\rangle=U(t_{\rm as})\,|\,{\rm physical}\rangle_{\rm Fock}\,. \tag{2.34}$$

It will be seen in the next section that this definition does not satisfy the asymptotic form of the GB condition in the interacting theory for either the NAS or IAS case.

III. THE GUPTA-BLEULER CONDITION

Because the Lorentz scalar $\partial_\mu A^\mu$ satisfies the free field equation (2.2), it is possible to define the time-independent annihilation operator

$$A[f]=\int d^3x\,f^{(+)}(\vec{x},t)\overleftrightarrow{\partial}_t\partial_\mu A^\mu(\vec{x},t)\,, \tag{3.1}$$

where $f^{(+)}$ is the positive-frequency wave packet which satisfies (2.8b). The Gupta-Bleuler (GB) condition for the interacting theory demands that the physical states satisfy the analog of (2.9), so that

$$A[f]\,|\,\text{physical}\rangle=0\ . \tag{3.2}$$

Using assumption (2.12) gives

$$A[f]=Z^{-1}(t)\left[\int d^3x\, f^{(+)}\overleftrightarrow{\partial}_t\partial_\mu a^\mu\right]Z(t)-\int d^3x\, f^{(+)}[\dot{Z}^{-1}(t)Z(t),Z^{-1}(t)\partial_\mu a^\mu Z(t)]\ . \tag{3.3}$$

It is straightforward to show that

$$[\dot{Z}^{-1}Z,Z^{-1}\partial_\mu a^\mu Z]=-Z^{-1}[\dot{Z}Z^{-1},\partial_\mu a^\mu]Z\ , \tag{3.4}$$

so that, from (2.14) and the commutation relations,

$$A[f]=Z^{-1}a[f]Z+Z^{-1}\left[\int d^3x\, f^{(+)}j_0\right]Z\ , \tag{3.5}$$

where

$$j_0\equiv e\bar{\phi}\gamma_0\phi\ . \tag{3.6}$$

In the large-time limit the first term in (3.5) depends upon the form of the asymptotic condition selected. The second term does not. To see that it does not, it is only necessary to recall the photon-like time dependence of $f^{(+)}$ and the large-time limit (2.20) to find, for t large,

$$Z^{-1}\left[\int d^3x\, f^{(+)}j_0\right]Z=\int d^3x\, f^{(+)}j_0^{\text{as}}\ , \tag{3.7}$$

regardless of the asymptotic condition selected. To be sure that the order of limit and commutator in (3.7) is irrelevant it need only be noted that for large t

$$e\int d^3x\, f^{(+)}\bar{\phi}^{\text{as}}\gamma_0\phi^{\text{as}}=\int d^3x\, f^{(+)}j_0^{\text{as}}\ . \tag{3.8}$$

Thus, the asymptotic form of the operator (3.3) is

$$A^{\text{as}}[f]=Ua[f]U^{-1}+\int d^3x\, f^{(+)}j_0^{\text{as}}\ . \tag{3.9}$$

The difficulty in implementing the GB condition (3.2) is now apparent. If the asymptotic states are defined by (2.34) it follows that

$$A^{\text{as}}[f]\,|\,\text{as}\rangle=U(t_{\text{as}})\int d^3x\, f^{(+)}(\vec{x},t_{\text{as}})j_0^{\text{as}}(\vec{x},t_{\text{as}})\,|\,\text{physical}\rangle_{\text{Fock}}\ , \tag{3.10}$$

which does not vanish if there is net charge in the Fock state. The solution to the dilemma posed by (3.10) requires the abandonment of locality and manifest Lorentz covariance in the charged sector by preparing the physical Fock space with the pseudounitary transformation[17]

$$V^{\text{as}}(t_{\text{as}})=\exp\left[\tfrac{1}{2}i\int d^3x\, d^3x'[\vec{\nabla}\cdot\vec{a}(\vec{x},t_{\text{as}})-\dot{a}_0(\vec{x},t_{\text{as}})]G(\vec{x}-\vec{x}\,')j_0^{\text{as}}(\vec{x},t_{\text{as}})\right]\ , \tag{3.11}$$

where $G(\vec{x}-\vec{x}\,')$ is the instantaneous Coulomb Green's function which satisfies

$$\nabla^2G(\vec{x}-\vec{x}\,')=\delta^3(\vec{x}-\vec{x}\,')\ . \tag{3.12}$$

The photon operators appearing in (3.11) are the zero norm ghosts of the form $(\alpha_{\vec{k}}^{3\dagger}-\alpha_{\vec{k}}^{0\dagger})$ excluded from the Fock space by the free GB condition (2.9). If the asymptotic particle states are defined as

$$|\,\widetilde{\text{as}}\rangle=U(t_{\text{as}})V(t_{\text{as}})\,|\,\text{physical}\rangle_{\text{Fock}}\ , \tag{3.13}$$

it is straightforward to show that

$$A^{\text{as}}[f]\,|\,\widetilde{\text{as}}\rangle=0\ , \tag{3.14}$$

where the commutators

$$\tfrac{1}{2}[\vec{\nabla}\cdot\vec{a}(\vec{x},t)-\dot{a}_0(\vec{x},t),\partial_\mu a^\mu(\vec{x}\,',t)]=0 \tag{3.15a}$$

and

$$-\tfrac{1}{2}[\vec{\nabla}\cdot\vec{a}(\vec{x},t)-\dot{a}_0(\vec{x},t),\partial_\mu\dot{a}^\mu(\vec{x}\,',t)]=i\nabla^2\delta^3(\vec{x}-\vec{x}\,') \tag{3.15b}$$

are useful.

The necessity of introducing V can be seen from another argument based on examination of the asymptotic form of Maxwell's equations written in terms of the vector potential. For consistency it must obtain that

$$\underset{t\to t_{\rm as}}{\text{w-lim}}\int d^3x\, f(\vec{x},t)[\vec{\nabla}\times\vec{B}(\vec{x},t)+\dot{\vec{E}}(\vec{x},t)-\vec{J}(\vec{x},t)]=0 \tag{3.16a}$$

and

$$\underset{t\to t_{\rm as}}{\text{w-lim}}\int d^3x\, f(\vec{x},t)[\vec{\nabla}\cdot\vec{E}(\vec{x},t)-J_0(\vec{x},t)]=0\ , \tag{3.16b}$$

where f is defined as a wave packet satisfying (2.16). The other two Maxwell equations hold trivially when written in terms of the vector potential and need not be considered. It is left as an exercise to show that

$$\underset{t\to t_{\rm as}}{\text{w-lim}}\int d^3x\, f(\vec{\nabla}\times\vec{B}+\dot{\vec{E}}-\vec{J})=U(t_{\rm as})\left[\int d^3x\, f(\vec{x},t_{\rm as})\vec{\nabla}\partial_\mu a^\mu(\vec{x},t_{\rm as})\right]U^{-1}(t_{\rm as})\ , \tag{3.17}$$

and

$$\underset{t\to t_{\rm as}}{\text{w-lim}}\int d^3x\, f(\vec{\nabla}\cdot\vec{E}-J_0)=U(t_{\rm as})\left[\int d^3x\, f(\vec{x},t_{\rm as})\partial_\mu\dot{a}^\mu(\vec{x},t_{\rm as})\right]U^{-1}(t_{\rm as})-\int d^3x\, f j_0^{\rm as}\ . \tag{3.18}$$

When placed between states of the form (2.34) relation (3.18) fails to vanish. However, when placed between states of the form (3.13), hereafter referred to as infra-Gaussian states, both relations vanish. The reader will recall that the breakdown of covariance and locality in the charged sector, discussed in more rigorous analyses,[15,16] is deduced by careful examination of the implications of Gauss's law (3.16b). Such results are in support of the manipulations of this section.

IV. THE REDUCTION FORMULAS

In this section the LSZ reduction program will be applied to the infra-Gaussian states defined by (3.13). The Fock state will be emptied by moving the operators past the UV product. Once the state is emptied the UV product will be left in place since

$$U(t)V^{\rm as}(t)\,|\,0\rangle=|\,0\rangle\ . \tag{4.1}$$

All the zero-norm coherent states in the spectrum defined by (3.13) are excluded from contributing to scattering by construction, as the reader may readily verify, and thus only states with transverse photons and charged particles in the Fock state will be reduced.

The transverse photon can be reduced very simply because the transverse photon operator commutes with the V operator. It follows then that an in state with a transverse photon of momentum k_μ and helicity λ ($\lambda=1,2$) can be written

$$|\,\tilde{k},\tilde{\lambda},A\,\rangle_{\rm in}=U(t_{\rm in})V^{\rm as}(t_{\rm in})\alpha_{\vec{k}}^{\lambda\dagger}\,|\,B\,\rangle_{\rm Fock}=U(t_{\rm in})\left[-i\int d^3x\, f_\mu^{(-)\lambda}(k,x)\overleftrightarrow{\partial}_t a^\mu(x)\right]\Big|_{t=t_{\rm in}}V^{\rm as}(t_{\rm in})\,|\,B\,\rangle_{\rm Fock}\ , \tag{4.2}$$

where $f_\mu^{(-)\lambda}$ is a negative-frequency wave packet satisfying

$$\Box f_\mu^{(-)\lambda}=\partial^\mu f_\mu^{(-)\lambda}=f_0^{(-)\lambda}=0\ . \tag{4.3}$$

It follows that

$$U\dot{a}_\mu=\left[\frac{\partial}{\partial t}(Ua_\mu U^{-1})\right]U-[\dot{U}U^{-1},a_\mu^{\rm in}]U\ . \tag{4.4}$$

Using the form (2.33) shows that the second term in (4.4) vanishes. As a result

$$|\tilde{k},\tilde{\lambda},B\rangle_{\rm in}=i\int d^3x\, f_\mu^{(-)\lambda}(k,x)\overleftrightarrow{\partial}_t a^{\mu\,{\rm as}}(x)\Big|_{t=t_{\rm in}}|\tilde{B}\rangle_{\rm in}\,. \tag{4.5}$$

From the asymptotic condition (2.15) relation (4.5) can be written

$$|\tilde{k},\tilde{\lambda},B\rangle_{\rm in}=\underset{t\to t_{\rm in}}{\text{w-lim}}\, i\int d^3x\, f_\mu^{(-)\lambda}(k,x)\overleftrightarrow{\partial}_t A^\mu(x)\,|\tilde{B}\rangle_{\rm in}\,. \tag{4.6}$$

Using the standard replacement

$$\lim_{t\to t_{\rm in}}=-\int_{t_{\rm in}}^{t_{\rm out}}dt\frac{\partial}{\partial t}+\lim_{t\to t_{\rm out}}\,, \tag{4.7}$$

the definition of time ordering, the absence of forward scattering, and the equation of motion (4.3), it can be shown

$${}_{\rm out}\langle\tilde{A}\,|\,T\{\,\cdots\,\}\,|\tilde{k},\tilde{\lambda},B\rangle_{\rm in}=i\int d^4x\, f_\mu^{(-)\lambda}(k,x)\overrightarrow{\Box}_x\, {}_{\rm out}\langle\tilde{A}\,|\,T\{\,\cdots A^\mu(x)\}\,|\tilde{B}\rangle_{\rm in}\,. \tag{4.8}$$

The complex conjugate of (4.8) occurs when a photon is reduced out of the out state.

The reduction formulas for the charged particles are affected by the presence of the V operator. It is easy to show

$$V^{\rm as}(t)b^{s\dagger}_{\vec{p}}=b^{s\dagger}_{\vec{p}}\exp\left\{-\tfrac{1}{2}ie\int d^3x\, G\left[\vec{x}-\frac{\vec{p}}{\epsilon_{\vec{p}}}t\right][\vec{\nabla}\cdot\vec{a}(\vec{x},t)-\dot{a}_0(\vec{x},t)]\right\}V^{\rm as}(t)\,, \tag{4.9a}$$

while

$$V^{\rm as}(t)d^{s\dagger}_{\vec{p}}=d^{s\dagger}_{\vec{p}}\exp\left\{\tfrac{1}{2}ie\int d^3x\, G\left[\vec{x}-\frac{\vec{p}}{\epsilon_{\vec{p}}}t\right][\vec{\nabla}\cdot\vec{a}(\vec{x},t)-\dot{a}_0(\vec{x},t)]\right\}V^{\rm as}(t)\,. \tag{4.9b}$$

In the large-time limit these equations have the equivalent representations

$$V^{\rm as}(t)d^{s\dagger}_{\vec{p}}=\int d^3x\, v^\dagger_{\vec{p}s}(x)\phi(x)e^{iC(x)}V^{\rm as}(t)\,, \tag{4.10a}$$

$$V^{\rm as}(t)b^{s\dagger}_{\vec{p}}=\int d^3x\,\phi^\dagger(x)u_{\vec{p},s}(x)e^{-iC(x)}V^{\rm as}(t)\,, \tag{4.10b}$$

where u and v are free spinor functions satisfying

$$(\lambda^\mu\overrightarrow{\partial}_\mu+im)u_{\vec{p},s}=\bar{v}_{\vec{p},s}(\gamma^\mu\overleftarrow{\partial}_\mu-im)=0\,, \tag{4.11}$$

and the phase of the spinor fields is given by

$$C(\vec{x},t)=\tfrac{1}{2}e\int d^3x'\, G(\vec{x}-\vec{x}\,')[\vec{\nabla}\cdot\vec{a}(\vec{x}\,',t)-\dot{a}_0(\vec{x}\,',t)]\,. \tag{4.12}$$

To see that relations (4.9) and (4.10) coincide in the large-time limit the exponential phase is expanded in a power series and the integration is performed using the plane-wave representations (2.4a) and (2.6a) for the free fields. The large-time limit (2.20) is then used to find the asymptotic form. A simple calculation is illustrative since this is a critical point. For convenience the operator $\vec{\nabla}\cdot\vec{a}-\dot{a}_0$ will be given the decomposition

$$\vec{\nabla}\cdot\vec{a}-\dot{a}_0=\int\frac{d^3k}{(2\pi)^{3/2}}(B^*_{\vec{k}}e^{ikx}+B_{\vec{k}}e^{-ikx})\,, \tag{4.13}$$

while the Coulomb Green's function (3.12) has the standard representation

$$G(\vec{x}-\vec{x}\,')=-\int\frac{d^3k}{(2\pi)^3}|\vec{k}|^{-2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\,. \tag{4.14}$$

The first nontrivial term in the expansion of (4.10b) is given by

$$-i\int d^3x\,\phi^\dagger(\vec{x},t)u_{\vec{p},s}(\vec{x},t)C(\vec{x},t)=\tfrac{1}{2}ie\int d^3p\,d^3k\left[\frac{m^2}{\epsilon_{\vec{p}}\epsilon_{\vec{p}+\vec{k}}}\right]^{1/2}u^\dagger_{\vec{p}+\vec{k},s}u_{\vec{p},s}b^{s\dagger}_{\vec{p}+\vec{k}}$$
$$\times|\vec{k}|^{-2}(B^*_{\vec{k}}e^{ikpt/\epsilon_{\vec{p}}}+B_{-\vec{k}}e^{-ikpt/\epsilon_{\vec{p}}})\,. \tag{4.15}$$

In the limit $|t|$ becomes large all but vanishingly small values of k_μ are suppressed in the integral, so that (4.15) becomes

$$-b^{s\dagger}_{\vec{p}}\tfrac{1}{2}ie\int d^3x\, G\left[\vec{x}-\frac{\vec{p}}{\epsilon_{\vec{p}}}t\right][\vec{\nabla}\cdot\vec{a}(\vec{x},t)-\dot{a}_0(\vec{x},t)]\ , \tag{4.16}$$

which is the first nontrivial term in the expansion of (4.9a)

Another derivation of the form (4.10) follows from the fact that, in the large-time limit,

$$\lim_{t\to t_{as}} V(t)\equiv \lim_{t\to t_{as}} \exp\left\{-\tfrac{1}{2}\int d^3x\, d^3x'[\vec{\nabla}\cdot\vec{a}(\vec{x},t)-\dot{a}_0(\vec{x},t)G(\vec{x}-\vec{x}\,')j_0(\vec{x}\,',t)\right\}=V^{as}(t_{as})\ , \tag{4.17}$$

where j_0 is given by (3.6). The reason that (4.17) holds is that the time dependence of the operator V is identical to that of the interaction-picture Hamiltonian (2.14), and so the limit (2.20) is once again appropriate. Using the commutators

$$\begin{aligned}&[j_0(\vec{x},t),\phi^\dagger(\vec{x}\,',t)]=e\phi^\dagger(\vec{x}\,',t)\delta^3(\vec{x}-\vec{x}\,')\ ,\\ &[j_0(\vec{x},t),\phi(\vec{x}\,',t)]=-e\phi(\vec{x}\,',t)\delta^3(\vec{x}-\vec{x}\,')\end{aligned} \tag{4.18}$$

quickly verifies the relations of (4.10), valid in the large-time limit.

The reduction formulas may now be developed. Reducing a charged particle of helicity s and momentum k_μ from the in state gives

$$\begin{aligned}{}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \}\,|\,\vec{k},\vec{s},B\rangle_{\rm in}&={}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \}U(t_{\rm in})V(t_{\rm in})b^{s\dagger}_{\vec{k}}\,|\,B\rangle_{\rm Fock}\\ &={}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \}\int d^3x\phi^{\dagger as}(x)e^{-iC^{\rm in}(x)}u_{\vec{k},s}(x)\Big|_{t=t_{\rm in}}|\,\widetilde{B}\rangle_{\rm in}\ ,\end{aligned} \tag{4.19}$$

where $C^{\rm in}$ is C with the free fields replaced by the asymptotic fields at time $t_{\rm in}$. Using the form (4.7), the definition of time ordering, and the absence of forward scattering, gives

$${}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \}\,|\,\vec{k},\vec{s},B\rangle_{\rm in}=-i\int d^4x\,{}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \bar{\psi}(x)e^{-iC_H(x)}\}\,|\,\widetilde{B}\rangle_{\rm in}(-i\gamma^\mu\overleftarrow{\partial}_\mu-m)u_{\vec{k},s}(x)\ , \tag{4.20}$$

where

$$C_H(x)=\tfrac{1}{2}e\int d^3x\, G(\vec{x}-\vec{x}\,')[\vec{\nabla}\cdot\vec{A}(\vec{x}\,',t)-\dot{A}_0(\vec{x}\,',t)]\ . \tag{4.21}$$

An antiparticle is reduced similarly, giving

$${}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \}\,|\,\tilde{\vec{k}},\tilde{\vec{s}},B\rangle_{\rm in}=i\int d^4x\bar{v}_{\vec{k},s}(x)(i\gamma^\mu\overrightarrow{\partial}_\mu-m)\,{}_{\rm out}\langle\widetilde{A}\,|\,T\{\ \cdots\ \psi(x)e^{iC_H(x)}\}\,|\,\widetilde{B}\rangle_{\rm in}\ . \tag{4.22}$$

Related formulas are obtained for reducing particles out of the out state.

While the appearance of the nonlocal instantaneous phase C_H in the expressions for the charged particles might seem unusual, it is possible to view it as a logical outgrowth of the gauge invariance of QED. The generator of gauge transformations for the Heisenberg theory in the Feynman gauge is

$$Q(t)=\int d^3x\, \Lambda(\vec{x},t)\overleftrightarrow{\partial}_t\partial_\mu A^\mu(\vec{x},t)\ , \tag{4.23}$$

where

$$\Box\Lambda=0\ . \tag{4.24}$$

Because of (4.24) the Q operator breaks into two parts,

$$Q(t)=A^\dagger[\Lambda]+A[\Lambda]\ , \tag{4.25}$$

where $A[\Lambda]$ is given by (3.1), so that the operator Q vanishes when evaluated between states which satisfy the GB condition. Clearly, the infra-Gaussian states satisfy this condition asymptotically by explicit construction, and thus the reduction formulas should be manifestly gauge invariant. It is apparent that under a gauge transformation which satisfies (4.24)

$$[Q(t),C_H(x,t)]=e\Lambda(\vec{x},t) \tag{4.26}$$

so that the combination ψe^{iC_H} is manifestly gauge invariant as promised. This will be discussed further in the Conclusions.

V. THE S MATRIX

While the reduction formulas derived in the previous section have an intrinsic value by exhibiting the scattering amplitude as an exact expression in terms of interpolating fields, most calculations resort to a perturbative representation of the time-ordered fields. In this section the perturbative representation consistent with the adiabatic switching chosen will be derived.

The starting point is the Green's function associated with a totally reduced amplitude. This takes the form of a set of time-ordered fields such as (4.8), (4.20), and (4.22) between the Fock vacuum. This Green's function will be denoted

$$F_\alpha(x_1,x_2,x_3,\dots)=\langle 0\,|\,T\{\,A_\mu(x_1)\psi(x_2)e^{iC_H(x_2)}\bar{\psi}(x_3)e^{-iC_H(x_3)}\cdots\,\}\,|\,0\rangle\;, \tag{5.1}$$

where α is the set of Lorentz and spinor indices associated with the fields in (5.1). It follows from (2.12) and (4.17) that

$$A_\mu(\vec{x},t)=Z^{-1}(t)V^{-1}(t)a_\mu(\vec{x},t)V(t)Z(t)\;, \tag{5.2a}$$

$$\psi(\vec{x},t)e^{iC_H(\vec{x},t)}=Z^{-1}(t)V^{-1}(t)\phi(\vec{x},t)V(t)Z(t)\;. \tag{5.2b}$$

The fact that A_μ is integrated against a wave packet that satisfies (4.3) allows (5.2a) to hold, while it is necessary to use V, as opposed to $V^{\rm as}$, because the fields are in the interpolating region where $|\,t\,|$ is not necessarily large.

The asymptotic fields are introduced by the relation

$$a_\mu=\tilde{U}^{-1}\tilde{a}_\mu^{\rm as}\tilde{U}\;, \tag{5.3a}$$

$$\phi=\tilde{U}^{-1}\tilde{\phi}^{\rm as}\tilde{U}\;. \tag{5.3b}$$

The NAS assumption is the $\tilde{U}$ is a constant. Of course, careful examination of the S matrix will make such an assumption obviously incorrect. Nevertheless, for the completeness of this paper the form of the S matrix consistent with such an assumption will be derived. For the IAS case it must be noted that $\tilde{U}$ may not coincide with the U of (2.24), and there is no *a priori* reason to expect that it should. This is due to the presence of the V operator. The form for U derived in Sec. II was developed by examining the asymptotic limit of (2.14) which is related through (2.19) only to the Z operator. With the necessity of introducing the V operator by the relations (5.2), the effective interaction has been changed, and a form for $\tilde{U}$ must be found which allows this interaction to switch adiabatically off at asymptotic times. As a preview, $\tilde{U}$ will differ from U in the exclusion of ghost operators and the inclusion of the instantaneous Coulomb interaction. Note that the asymptotic relation (2.17) for the Feynman gauge fields still holds, but the presence of V forces a redefinition of U in the S matrix.

In order to find U the representation of the S matrix consistent with assumptions (5.2) and (5.3) must be found. The amplitude (5.1) can now be written

$$F_\alpha(x_1,x_2,x_3,\dots)=\langle 0\,|\,Z_0{}^{-1}V_0{}^{-1}\tilde{U}_0{}^{-1}T\{\,\tilde{U}_0V_0Z_0Z_1{}^{-1}V_1{}^{-1}\tilde{U}_1\tilde{a}_\mu^{\rm as}(x_1)\tilde{U}_1V_1Z_1\cdots Z_i{}^{-1}\times V_i{}^{-1}\tilde{U}_i{}^{-1}\,\}\tilde{U}_iV_iZ_i\,|\,0\rangle\;, \tag{5.4}$$

where the subscripts refer to the time of the operator, 0 being $t_{\rm out}$ and i being $t_{\rm in}$. Using the fact that asymptotically ZU goes into a constant and V goes into $V^{\rm as}$ which leaves the Fock vacuum unchanged, and the nature of time ordering, it can be shown that

$$F_\alpha(x_1,\dots)=\lambda^+\lambda^-\langle 0\,|\,\tilde{U}_0{}^{-1}T\{\,\tilde{a}_\mu^{\rm as}(x_1)\tilde{\phi}^{\rm as}(x_2)\tilde{\bar{\phi}}^{\rm as}(x_3)\cdots\tilde{U}_0V_0Z_0Z_i{}^{-1}V_i{}^{-1}\tilde{U}_i{}^{-1}\,\}\tilde{U}_i\,|\,0\rangle\;. \tag{5.5}$$

Relation (5.5) shows that the condition

$$\tilde{U}(t_{\rm as})\,|\,0\rangle=|\,0\rangle \tag{5.6}$$

must be met in order that the Dyson-Wick contraction method can be employed.

Defining $W(t)$ by

$$W(t)=\tilde{U}(t)V(t)Z(t)\ , \tag{5.7}$$

and $\tilde{H}(t)$ by

$$\dot{W}(t)W^{-1}(t)=-i\tilde{H}(t)\ , \tag{5.8}$$

the evolution operator $W(t)W^{-1}(t')$ is found to satisfy the differential equation

$$\frac{\partial}{\partial t}[W(t)W^{-1}(t')]=-i\tilde{H}(t)W(t)W^{-1}(t)\ . \tag{5.9}$$

Iteration of this equation[23] leads to the representation

$$W(t_{\text{out}})W^{-1}(t_{\text{in}})=T\left\{\exp\left[-i\int_{t_{\text{in}}}^{t_{\text{out}}}dt\tilde{H}(t)\right]\right\}. \tag{5.10}$$

By definition

$$-i\tilde{H}(t)=\dot{\tilde{U}}\tilde{U}^{-1}+\tilde{U}(\dot{V}V^{-1}+V\dot{Z}Z^{-1}V^{-1})\tilde{U}^{-1}\ . \tag{5.11}$$

From the forms (2.14) and (4.17) it follows that

$$\begin{aligned}V\dot{Z}Z^{-1}V^{-1}+\dot{V}V^{-1}=&-i\int d^3x\, j_i(\vec{x},t)a_i^T(\vec{x},t)\\&+i\tfrac{1}{2}\int d^3x\,d^3x'\{\,\partial_\mu j^\mu(\vec{x},t)G(\vec{x}-\vec{x}\,')[\vec{\nabla}\cdot\vec{a}(\vec{x}\,',t)-\dot{a}_0(\vec{x}\,',t)]\\&\qquad+\vec{\nabla}\cdot\vec{j}(\vec{x},t)G(\vec{x}-\vec{x}\,')\partial_\mu a^\mu(\vec{x}\,',t)+j_0(\vec{x},t)G(\vec{x}-\vec{x}\,')\partial_\mu\dot{a}^\mu(\vec{x}\,',t)\\&\qquad-j_0(\vec{x},t)G(\vec{x}-\vec{x}\,')j_0(\vec{x}\,',t)\,\}\ ,\end{aligned} \tag{5.12}$$

where a_i^T refers to the transverse part of a_i, and is given by

$$a_i^T(\vec{x},t)=a_i(\vec{x},t)-\nabla_i\int d^3x'G(\vec{x}-\vec{x}\,')\vec{\nabla}\cdot\vec{a}(\vec{x}\,',t)\ . \tag{5.13}$$

Using conservation of the free current j_μ simplifies (5.12) to the form

$$\begin{aligned}V\dot{Z}Z^{-1}V^{-1}+\dot{V}V^{-1}=&-i\int d^3x\ \vec{j}(\vec{x},t)\cdot a^T(\vec{x},t)\\&-\frac{i}{2}\int d^3x\,d^3x'\left\{j_0(\vec{x},t)G(\vec{x}-\vec{x}\,')j_0(\vec{x}\,',t)+2[j_0(\vec{x},t)G(\vec{x}-\vec{x}\,')\partial_\mu\dot{a}^\mu(\vec{x}\,',t)]\right\}.\end{aligned} \tag{5.14}$$

At this point the NAS version of the S matrix may be found by setting $\tilde{U}$ equal to some constant and inserting (5.14) into (5.10) and (5.5) to obtain

$$\begin{aligned}F_\alpha(x_1,x_2,x_3,\dots)=\lambda^+\lambda^-\Big\langle 0\Big|T\Big\{&a_\mu(x_1)\phi(x_2)\bar{\phi}(x_3)\cdots\\&\times\exp\left[-i\int_{t_{\text{in}}}^{t_{\text{out}}}dt\,d^3x\left[j\cdot a^T+\tfrac{1}{2}j_0\int d^3x'\,Gj_0'\right]\right]\Big\}\Big|0\Big\rangle\ ,\end{aligned} \tag{5.15}$$

where the time derivative in (5.14) has been discarded since $\partial_\mu a^\mu$ has no contractions with any other operator appearing in the S matrix. The reader immediately recognizes (5.15) as the standard Coulomb-gauge representation of the time-ordered products. Of course, expression (5.15) exhibits the infrared-divergence problem previously discussed because (5.14) does not adiabatically switch off at asymptotic times. Note that the c number $\epsilon_0(t)$ defined in (2.14) has been suppressed throughout this derivation. This is because it is canceled by the factor $\lambda^+\lambda^-$, which is given by[23]

$$\lambda^+\lambda^- = \left\langle 0 \left| T\left\{ \exp\left[i\int_{t_{\rm in}}^{t_{\rm out}} dt\, d^3x \left[j\cdot a^T + \tfrac{1}{2} j_0 \int d^3x'\, G j_0' \right]\right]\right\}\right| 0 \right\rangle . \tag{5.16}$$

As a final note, the series (5.15) is the unrenormalized Green's function. As mentioned in the Introduction, the renormalization of the theory lies outside the scope of this paper.

In order to find the form of the asymptotic fields (5.3) it is necessary to assume a nonconstant $\tilde{U}$. The IAS assumption demands that the argument of the S matrix, i.e., the effective interaction Hamiltonian (5.8), must vanish at asymptotic times. This means that

$$\underset{t\to t_{\rm as}}{\text{w-lim}}\, [\dot{\tilde{U}}\tilde{U}^{-1} + \tilde{U}(\dot{V}V^{-1} + V\dot{Z}Z^{-1}V^{-1})\tilde{U}^{-1}] = 0 . \tag{5.17}$$

Inverting (5.17) gives

$$\underset{t\to t_{\rm as}}{\text{w-lim}}\, \tilde{U}^{-1}\dot{\tilde{U}} = -\underset{t\to t_{\rm as}}{\text{w-lim}}\, (\dot{V}V^{-1} + V\dot{Z}Z^{-1}V^{-1}) . \tag{5.18}$$

Thus, the expression for $\tilde{U}^{-1}\dot{\tilde{U}}$ may be found by applying the same analysis to the effective interaction as was applied to (2.14). A minor complication occurs because of the Coulomb interaction appearing in expression (5.14). Inserting the plane-wave expansion (2.4a) into the Coulomb interaction shows that there occur operator combinations of the form

$$\frac{e^2}{(2\pi)^3} \sum_{s_1,s_2,s_3,s_4} \int d^3p\, d^3k\, d^3q\, m^2(\epsilon_{\vec{p}}\epsilon_{-\vec{p}-\vec{k}}\epsilon_{-\vec{q}}\epsilon_{\vec{q}-\vec{k}})^{-1/2} b_{\vec{p}}^{s_1\dagger} d_{-\vec{p}-\vec{k}}^{s_2\dagger} d_{-\vec{q}}^{s_3} b_{\vec{q}-\vec{k}}^{s_4} u_{\vec{p},s_1}^\dagger v_{-\vec{p}-\vec{k},s_2}$$
$$\times |\vec{k}|^{-2} v_{-\vec{q},s_3}^\dagger u_{\vec{q}-\vec{k},s_4} \exp[-i(\epsilon_{\vec{p}} + \epsilon_{-\vec{p}-\vec{k}} - \epsilon_{\vec{q}} - \epsilon_{\vec{q}-\vec{k}})t] \tag{5.19a}$$

and

$$\frac{e^2}{(2\pi)^3} \sum_{s_1,s_2,s_3,s_4} \int d^3p\, d^3k\, d^3q\, m^2(\epsilon_{\vec{p}}\epsilon_{\vec{p}+\vec{k}}\epsilon_{\vec{q}}\epsilon_{\vec{q}-\vec{k}})^{-1/2} b_{\vec{p}}^{s_1\dagger} b_{\vec{p}+\vec{k}}^{s_2} b_{\vec{q}}^{s_3\dagger} b_{\vec{q}-\vec{k}}^{s_4} u_{\vec{p},s_1}^\dagger u_{\vec{p}+\vec{k},s_2}$$
$$\times |\vec{k}|^{-2} u_{\vec{q},s_3}^\dagger u_{\vec{q}-\vec{k},s_4} \exp[-i(\epsilon_{\vec{p}} - \epsilon_{\vec{p}+\vec{k}} + \epsilon_{\vec{q}} - \epsilon_{\vec{q}-\vec{k}})t] \tag{5.19b}$$

which, because of their time dependence, may not vanish in the asymptotic limit. All forms which do not contain equal numbers of annihilation and creation will vanish. In the limit $|t|$ becomes very large the exponentials become

$$\lim_{t\to t_{\rm as}} \exp[-i(\epsilon_{\vec{p}} - \epsilon_{\vec{p}+\vec{k}} + \epsilon_{\vec{q}} - \epsilon_{\vec{q}-\vec{k}})t] = \exp\left[i\left(\frac{\vec{k}\cdot\vec{p}}{\epsilon_{\vec{p}}} - \frac{\vec{k}\cdot\vec{q}}{\epsilon_{\vec{q}}} \right) t \right] , \quad \vec{k}\approx 0 , \tag{5.20a}$$

and

$$\lim_{t\to t_{\rm as}} \exp[-i(\epsilon_{\vec{p}} + \epsilon_{-\vec{p}-\vec{k}} - \epsilon_{-\vec{q}} - \epsilon_{\vec{q}-\vec{k}})t] = \exp\left[-i\left(\frac{\vec{k}\cdot\vec{p}}{\epsilon_{\vec{p}}} + \frac{\vec{k}\cdot\vec{q}}{\epsilon_{\vec{q}}} \right) t \right] , \quad \vec{p}=\vec{q} , \quad \vec{k}\approx 0 . \tag{5.20b}$$

The terms of the form (5.19a) then vanish because the spinor products are zero for $\vec{k}\approx 0$. The Coulomb interaction then becomes

$$\underset{t\to t_{\rm as}}{\text{w-lim}} \int d^3x\, d^3x'\, j_0(\vec{x},t) G(\vec{x}-\vec{x}') j_0(\vec{x}',t) = \int d^3x\, d^3x'\, j_0^{\rm as}(\vec{x},t) G(\vec{x}-\vec{x}') j_0^{\rm as}(\vec{x}',t) . \tag{5.21}$$

The asymptotic form of the effective interaction is

$$\tilde{U}^{-1}\dot{\tilde{U}} = i\int d^3x\, \vec{j}^{\rm as}(x)\cdot\vec{a}^T(x) + \tfrac{1}{2} i \int d^3x\, d^3x'\, j_0^{\rm as}(\vec{x},t) G(\vec{x}-\vec{x}') j_0^{\rm as}(\vec{x}',t) , \tag{5.22}$$

where the terms proportional to $\partial_\mu a^\mu$ have been dropped since they will have no contractions with any other fields.

Repeating the steps of Sec. II B gives

$$\tilde{U}(t_2)\tilde{U}^{-1}(t_1)=\exp\left[i\int_{t_1}^{t_2}dt\,d^3x\,\vec{j}^{\,\rm as}\cdot\vec{a}^{\,T}\right]\exp\left[\tfrac{1}{2}i\int d^4x\,d^4x'\,j_i^{\rm as}(x)\Delta_{R,T}^{ik}(x-x')j_k^{\rm as}(x')\right]$$
$$\times\exp\left[\tfrac{1}{2}i\int_{t_1}^{t_2}d^3x\,d^3x'\,dt\,j_0^{\rm as}(\vec{x},t)G(\vec{x}-\vec{x}\,')j_0^{\rm as}(\vec{x}\,',t)\right], \tag{5.23}$$

where $\Delta_{R,T}^{ik}$ is the transverse retarded Green's function given by

$$\Delta_{R,T}^{ij}(x-x')=-i\theta(t-t')[a_i^T(x),a_j^T(x')]$$
$$=-i\theta(t-t')\int\frac{d^3k}{(2\pi)^3}(2\omega_{\vec{k}})^{-1}\left[\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\right][e^{ik(x-x')}-e^{-ik(x-x')}], \tag{5.24}$$

and the sum in (5.23) runs over spatial indices only. It follows that $\tilde{U}$ has the representation

$$\tilde{U}(t)=\exp[-\tilde{R}(t)]\exp[-i\tilde{\beta}(t)], \tag{5.25}$$

where

$$\tilde{R}(t)=\frac{e}{(2\pi)^{3/2}}\int d^3p\,d^3k(2\omega_{\vec{k}})^{-1/2}\frac{p_i}{k_\mu p^\mu}[a_i^T(\vec{k})e^{ikpt/\epsilon_{\vec{p}}}-a_i^{T\dagger}(\vec{k})e^{-ikpt/\epsilon_{\vec{p}}}]\rho(\vec{p}), \tag{5.26a}$$

and

$$\tilde{\beta}(t)=\frac{e^2}{(2\pi)^3}\int d^3p\,d^3k\,d^3q\,\rho(\vec{p})\rho(\vec{q})\left\{\left[2\omega_{\vec{k}}^{\;2}\left[\frac{\vec{k}\cdot\vec{p}}{\epsilon_{\vec{p}}}-\frac{\vec{k}\cdot\vec{q}}{\epsilon_{\vec{q}}}\right]\right]^{-1}\sin\left[\left[\frac{\vec{k}\cdot\vec{p}}{\epsilon_{\vec{p}}}-\frac{\vec{k}\cdot\vec{q}}{\epsilon_{\vec{q}}}\right]t\right]\right.$$
$$\left.+[\vec{p}\cdot\vec{q}-\omega_{\vec{k}}^{\;2}(\vec{p}\cdot\vec{k})(\vec{q}\cdot\vec{k})]\left[2\omega_{\vec{k}}\epsilon_{\vec{p}}\epsilon_{\vec{q}}(kq)\left[\frac{kp}{\epsilon_{\vec{p}}}-\frac{kq}{\epsilon_{\vec{q}}}\right]\right]^{-1}\sin\left[\left[\frac{kp}{\epsilon_{\vec{p}}}-\frac{kq}{\epsilon_{\vec{q}}}\right]t\right]\right\}. \tag{5.26b}$$

In (5.26a) a_i^T is the transverse operator satisfying

$$\vec{k}\cdot\vec{a}^{\,T}(\vec{k})=0,\quad [a_i^T(\vec{k}),a_j^T(\vec{p})]=\left[\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\right]\delta^3(\vec{k}-\vec{p}). \tag{5.27}$$

The IAS form of the amplitudes (5.1) is given by

$$F_\alpha(x_1,x_2,x_3,\ldots)$$
$$=\lambda^{+\prime}\lambda^{-\prime}\left\langle 0\left|T\left\{\tilde{a}_\mu^{\rm as}(x_1)\tilde{\phi}^{\rm as}(x_2)\tilde{\bar{\phi}}^{\rm as}(x_3)\cdots\right.\right.\right.$$
$$\times\exp\left[-i\int_{t_{\rm in}}^{t_{\rm out}}dt\,d^3x\left[\vec{j}(\tilde{\phi}^{\rm as}(x))\cdot\tilde{\vec{a}}^{\,T\,\rm as}(x)+\tfrac{1}{2}j_0(\tilde{\phi}^{\rm as}(x,t))\int d^3x'\,G(\vec{x}-\vec{x}\,')j_0(\tilde{\phi}^{\rm as}(x',t))\right.\right.$$
$$\left.\left.\left.\left.\left.-\vec{j}^{\,\rm as}(x)\cdot\tilde{\vec{a}}^{\,T\,\rm as}(x)-\tfrac{1}{2}j_0^{\rm as}(x)\int d^3x'\,G(\vec{x}-\vec{x}\,')j_0^{\rm as}(x',t)\right]\right]\right\}\right|0\right\rangle. \tag{5.28}$$

Again, the c-number time dependence in the exponential is canceled by the phase factor $\lambda^{+\prime}\lambda^{-\prime}$, given by

$$\lambda^{+\prime}\lambda^{-\prime}=\left\langle 0\left|T\left\{\exp\left[i\int_{t_{\rm in}}^{t_{\rm out}}dt\,\tilde{H}(t)\right]\right\}\right|0\right\rangle. \tag{5.29}$$

It is now apparent that the perturbative representation of the amplitudes associated with physical scattering processes derived through the LSZ reduction technique is equivalent to a set of Coulomb-gauge Feynman rules, as opposed to a set of Feynman-gauge Feynman rules. This point will be discussed further in the Conclusions. It is also obvious that the time dependences of β and $\tilde{\beta}$ are irrelevant to the amplitude, since it has no effect upon contractions of either field appearing in the perturbation series. In addition the asymptotic form of $\tilde{a}_i^{T\,\mathrm{as}}$ may be deleted and replaced by the free field a_i^T since

$$\langle 0 \,|\, T\{a_i^{T\,\mathrm{as}}(x)\tilde{a}_j^{T\,\mathrm{as}}(x')\} \,|\, 0\rangle = \langle 0 \,|\, T\{a_i^T(x)a_j^T(x')\} \,|\, 0\rangle \ . \tag{5.30}$$

This shows that the infrared divergences are removed by changing the propagator for the charged particles, and thus that $\tilde{R}$ is the important part of $\tilde{U}$.

Using (5.28) the Yang-Feldman equation for the interpolating fields may be derived. It is straightforward to show that, in the weak limit,

$$A_i(x) = \tilde{a}_i^{\mathrm{as}}(x) + \int d^4y\,\Delta_{ij}^{R,T}(x-y)[J_j(y) - j_j^{\mathrm{as}}(y)] \ , \tag{5.31}$$

where $\Delta_{ij}^{R,T}$ is the retarded transverse propagator defined by (5.24), and $\vec{J}$ is the current written in terms of the interpolating spinor fields.

At this point Källén's argument can be invalidated. It is based on the fact that the current commutator must have the representation

$$\langle 0 \,|\, [J_i(x),J_j(y)] \,|\, 0\rangle = \int d^4p\, e^{ip(x-y)}\epsilon(p)(\delta_{ij}p^2 - p_ip_j)\pi(p^2) \ , \tag{5.32}$$

where $\pi(p^2)$ can be related to the spectral function, and the assumption that A_μ has the Yang-Feldman representation

$$A_i(x) = a_i^{\mathrm{as}}(x) + \int d^4y\,\Delta_R^{ij}(x-y)J_j(y), \quad \Delta_R = -i\theta(t-t')\langle 0 \,|\, [\tilde{a}_i^{\mathrm{as}},\tilde{a}_j^{\mathrm{as}}] \,|\, 0\rangle \ . \tag{5.33}$$

When the commutator $[A_i(x),A_j(y)]$ is calculated using (5.32) and (5.33), the terms proportional to p_ip_j in (5.32) leave a nonvanishing contribution at equal time. If instead of (5.33) expression (5.31) is used in the commutator, it is straightforward to show that these terms are canceled due to the transverse nature of the propagator. This allows both (2.15) and causality to hold in the interpolating region.

VI. CONCLUSIONS

The goal of this paper is to reexamine the problem of deriving the LSZ reduction formulas for Feynman-gauge electrodynamics and to resolve the role of the Gupta-Bleuler condition in this procedure. It is relevant to review the steps of the paper. By examining the asymptotic limit of the Feynman-gauge interaction Hamiltonian the infrared limit of the interpolating Feynman gauge fields was found. However, the particle states which evolve into the Feynman gauge fields were found not to satisfy the Gupta-Bleuler condition. Those states which do satisfy the Gupta-Bleuler condition evolve into the Feynman gauge fields with a phase on the spinor fields. When perturbatively analyzed these amplitudes are represented by a Coulomb-gauge series rather than a Feynman-gauge series because of the phase on the spinor fields. In effect, applying the Gupta-Bleuler condition leads directly to a theory which is dynamically equivalent to Coulomb-gauge electrodynamics, while still allowing all the degrees of freedom associated with manifestly covariant quantization.

There are several points to be made regarding the results derived in this paper. The first is that, as has already been shown, the Gupta-Bleuler condition and the demand for manifest gauge invariance are very similar. Satisfying this restriction led to a nonlocal instantaneous result. That this should happen can be seen from the equation of motion (2.1) for ψ. If the new field $\tilde{\psi}$ is defined by

$$\psi(x) = \tilde{\psi}(x)e^{-iC_H(x)} \ , \tag{6.1}$$

and is inserted into (2.1), it follows that

$$(i\gamma^\mu\partial_\mu - m)\tilde{\psi} = e\vec{A}^T\cdot\vec{\gamma}\tilde{\psi} + \tfrac{1}{2}e\vec{\gamma}\tilde{\psi}\cdot\vec{\nabla}\int d^3x'\,G(\vec{x}-\vec{x}\,')\partial_\mu A^\mu(\vec{x}\,',t)$$
$$+\tfrac{1}{2}e\gamma_0\tilde{\psi}\int d^3x'\,G(\vec{x}-\vec{x}\,')\partial_\mu \dot{A}_\mu(\vec{x}\,',t) - e^2\gamma_0\tilde{\psi}\int d^3x'\,G(\vec{x}-\vec{x}\,')\bar{\tilde{\psi}}(x')\gamma_0\tilde{\psi}(x') \ . \tag{6.2}$$

The equation of motion (6.2) is invariant under a gauge transformation solely upon A_μ if the gauge function satisfies (4.24). Thus, (6.1) is the decomposition of the bispinor field which decouples it from its gauge phase. That the phase must be instantaneous is now apparent, and this also illustrates the fact that the gauge transformation on the bispinor field ψ is automatically induced by a gauge transformation on the vector field through relation (6.1).

The second point is that, for the case of quantum electrodynamics, the static potential between two charged particles appears in the effective interaction (5.14) or (5.28) if the subsidiary condition is satisfied, and is not caused to appear by the infrared analysis. This is an extremely interesting point because of the current interest in finding the static potential associated with non-Abelian gauge theories. However, such topics are beyond the scope of this paper.

The final point to be made is that strict satisfaction of the Gupta-Bleuler condition is not necessary in quantum electrodynamics. As the reader is undoubtedly aware, most scattering amplitudes for the NAS assumption are calculated using a set of manifestly covariant Feyman rules, i.e., the S-matrix operator is defined as

$$S=T\left\{\exp\left[-i\int_{t_{\rm in}}^{t_{\rm out}} d^4x\, j_\mu(x)a^\mu(x)\right]\right\}. \qquad (6.3)$$

It has been shown that the Green's functions calculated with (6.3) and those associated with (5.15) differ only in terms which vanish when the amplitude is placed on its energy shell.[17] Thus, using (6.3) for an on-shell physical process will not give an incorrect answer. Another way to see that the Gupta-Bleuler condition is not critical to the reduction scheme in QED is to examine the contribution of zero norm states to scattering derived from states which have not been transformed by V. In terms of Heisenberg fields this would be given by

$$\langle 0|T\{\psi(x_1)\bar\psi(x_2)\cdots\}|\text{ghost}\rangle=i\int d^4x\, f^{(-)}(x)\Box_x\langle 0|T\{\partial_\mu A^\mu(x)\bar\psi(x_2)\psi(x_1)\cdots\}|0\rangle . \qquad (6.4)$$

Because the V operator is absent the spinor fields do not have the phase attached in (6.4). Applying the d'Alembertian to the Green's function gives

$$\begin{aligned}\langle 0|T\{\psi(x_1)\bar\psi(x_2)\cdots\}|\text{ghost}\rangle=&\,i\int d^4x\, f^{(-)}(x)\langle 0|T\{\Box\partial_\mu A^\mu(x)\psi(x_1)\bar\psi(x_2)\cdots\}|0\rangle\\ &+i\int d^4x\, f^{(-)}(x)\langle 0|T\{[\partial_\mu \dot A^\mu(x),\psi(x_1)]\delta(t-t_1)\bar\psi(x_2)\cdots\\ &+\psi(x_1)[\partial_\mu \dot A^\mu(x),\bar\psi(x_2)]\delta(t-t_2)\cdots+\cdots\}|0\rangle ,\end{aligned} \qquad (6.5)$$

where the second term on the right-hand side is the sum of equal-time commutators between $\partial_\mu \dot A_\mu$ and all other field operators in the time-ordered product. Clearly, the first term vanishes from (2.2). The second term requires care because the commutators do not necessarily vanish. In fact,

$$[\partial_\mu \dot A^\mu(\vec x,t),\psi(\vec x\,',t)]=e\psi(\vec x\,',t)\delta^3(\vec x-\vec x\,') \qquad (6.6a)$$

and

$$[\partial_\mu \dot A^\mu(\vec x,t),\bar\psi(\vec x\,',t)]=-e\bar\psi(\vec x\,',t)\delta^3(\vec x-\vec x\,') . \qquad (6.6b)$$

However, the set of commutators in (6.5) sum to zero for any amplitude for which electric charge is conserved, since for such a process there are equal numbers of the two types of commutators (6.6). This particular problem does not occur for amplitudes where the spinor fields have the phase attached because

$$[\partial_\mu \dot A^\mu(\vec x,t),\psi(\vec x\,',t)e^{iC_H(\vec x\,',t)}]=0 . \qquad (6.7)$$

Thus, physical processes calculated by either set of Feynman rules (5.16) or (6.3) will be the same in quantum electrodynamics because of the conservation of electric charge.

There are several extensions of the work presented here. An immediate problem is in understanding how the infrared problem is resolved in a path-integral formulation of QED. In recent work[24] it was shown that the Coulomb-gauge path integral can be derived using intermediate Feynman-gauge interaction-picture coherent states which satisfy the Gupta-Bleuler condition. It is not clear how the infrared structure of the asymptotic particle states is related to this result, and since path integrals are in such wide usage this needs to be understood. A second project, already undertaken by many authors, is to extend these results to non-Abelian gauge theories. Several authors[25] have argued that the extension of the

Gupta-Bleuler condition to the non-Abelian case is given by demanding that the generator of the Becchi-Rouet-Stora transformation annihilate physical states. It is possible that an analysis similar to the one provided for QED would generate the non-Abelian static potential, but this must remain speculation for the time being.

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