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Divergent integrals of post-Newtonian gravity: Nonanalytic terms in the near-zone expansion of a gravitationally radiating system found by matching

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We study the divergent integrals that occur in all post-Newtonian (PN) treatments of radiation reaction in slow-motion, gravitationally bound systems in general relativity. The PN methods implicitly assume that the near-zone metric has a valid asymptotic expansion in *powers* of the small velocity parameter ϵ. We first show explicitly (for the gauge to be used here) that a PN-approximation method leads to a divergent integral at 4-PN order. This divergence arises from the second iteration. Matching arguments are then used to calculate a near-zone term of $O(\ln\epsilon)$ larger than 4-PN order. On the basis of this calculation and several previous model problems, we argue that the PN divergences signify the breakdown of the PN power-series assumptions, rather than a breakdown of the near and wave zones. Our results suggest that the PN calculations in fact give correct answers at least up to the orders at which divergences appear. The nonanalytic term of $O(\ln\epsilon)$ beyond 4-PN order arises in the near zone via matching to the wave-zone expansion when we include terms of $O(\epsilon^3)$ beyond linearized order. We also solve the wave-zone equations at $O(\epsilon^6)$ beyond linearized order and analyze the inner expansion of the solutions. Matching gives rise to a nonanalytic term in the wave zone at $O(\epsilon^{11}\ln\epsilon)$, i.e., at $O(\epsilon^6\ln\epsilon)$ beyond linearized order. A straining technique is used in the wave-zone expansion to give a sufficiently accurate approximation to the null surfaces near past and future null infinity. The lowest-order strained solution at first appears to contribute a large, anomalous, time-odd piece to the reaction potential. However, after analyzing the contribution of higher-order wave-zone terms, we obtain agreement with the Burke reaction potential. Our results thus strongly support the usual quadrupole formula.

I. INTRODUCTION

At present, no general-relativistic perturbation calculations with rigorous error estimates[1] exist for the gravitational radiation damping forces acting on a binary system such as PSR 1913 + 16. Such estimates may well be beyond the reach of current perturbation methods. One does not yet even have a proof that the approximate solutions obtained are uniformly valid asymptotic expansions[2]; such proofs are rarely attainable.[3] The best available *guess* for the error of a p-term asymptotic expansion is usually the size of the $(p+1)$st term. If the $(p+1)$st term is divergent, one can have very little confidence in the accuracy of the p-term expansion.

No divergent integrals have occurred up to the orders so far studied in the matching approaches to slow-motion radiation damping. Burke[4] first applied matched asymptotic expansions to calculate the mechanical energy lost from gravitationally bound system containing bodies with weak internal gravity. Burke's calculation utilized a weak-field, slow-motion expansion in the near zone together with a weak-field (but *not* slow-motion) expansion in the wave zone. Kates[5] later refined this calculation by adding a third type of matching zone to describe a body with strong internal gravi-

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ty, such as in PSR 1913 + 16.

On the other hand, the post-Newtonian (PN) expansion methods[6–8] for deriving radiation reaction all contain apparently divergent integrals, as first pointed out by Ehlers *et al.*[9] Ehlers[10] and Kerlick[11] modified the Anderson-Decanio[6] method to postpone its apparent divergences until 3-PN order. Recently, it has been found by Schutz[12] and by Breuer and Rudolph[13] that at least some of these divergences disappear if one performs angular integrations first. However, examination of Kerlick's second iteration[14] shows that a nonvanishing divergent integral occurs at 4-PN order.

Because of these divergences, one cannot yet regard[9] the PN calculations and their descendents[15] as self-consistent derivations of the radiation reaction on a slow-motion system. Nevertheless, since none of the known approximation methods (including matching) comes with rigorous error estimates, it is desirable to have several independent derivations[16] of the quadrupole formula. To this end, we present evidence that, up to the order in which the divergent integrals appear, the results of the PN methods are in fact valid. Furthermore, these divergences appear to arise from attempts to expand terms which are nonanalytic in the slow-motion expansion parameter as a power series in this parameter.

In order to test the hypothesis[17–20] that nonanalytic terms might be responsible for the PN infinities, Kates and Kegeles[21] (paper I) studied the simpler problem of a slowly varying, radiating scalar field on a Schwarzschild background. (In the present problem, the static monopole part of the gravitational field will also be represented by a Schwarzschild background.) In the monopole radiation problem of paper I, an expansion method analogous to the PN approach was shown to produce divergent integrals at $O(\epsilon^4)$, where ϵ is the slow-motion parameter. Matching was then used to show that finite, nonanalytic terms at $O(\epsilon^4 \ln\epsilon)$ in the near zone precisely replace these divergent integrals. Lower-order terms were found to agree with their counterparts from the PN-expansion method.

Our treatment of the gravitational problem proceeds similarly. In Sec. II, we make use of Thorne's[22] analysis and reduce our problem to the familiar Zerilli[23] equation for perturbations of a Schwarzschild background. In Sec. III, we show that a PN treatment of the Zerilli equation leads to a divergent integral in the second iteration at 4-PN order. This divergence is "real" in the sense that it remains even after angular integration. In Sec. IV we compute a term of $O(\ln\epsilon)$ larger than 4-PN order in the near-zone expansion. This term arises from matching to the $O(\epsilon^8)$ part of the wave-zone expansion whose inner expansion contains a term proportional to $\ln\epsilon$. It replaces the divergent term of 4-PN order found in Sec. III.

The wave-zone equations to be treated here require a refinement of the usual expansion about flat space: if one expresses the lowest-order radiation in terms of the Minkowskian retarded variable $u_{\text{flat}} = t - r$, one encounters in the next order terms proportional to $\ln r/r$. Since for sufficiently large r these terms become larger than the lowest-order terms, the resulting expansion is not uniform in r. To avoid such nonuniformities[24], we introduce a strained null retarded variable $u = t - r^*$, where r^* is the Schwarzschild "tortoise coordinate." As in paper I, the use of the strained variable r^* must be accompanied by the remaining corrections of $O(M)$ smaller than the linear approximation; otherwise, spurious time-odd terms larger than the Burke[4] reaction term arise. The cancellation of such spurious terms in higher order is shown in Sec. V.

Although the infinity found here arises from only the *second* iteration, the *third* iteration is also of interest because it too contributes to the lowest-order radiation reaction. Moreover, it is always possible that still higher-order terms in the wave-zone expansion might contribute logarithmic terms of 4-PN or even lower near-zone order, due to the interplay between powers of r and ϵ in matching. For these two reasons, we investigate the $O(\epsilon^{11})$ part of the wave-zone expansion in the Appendix. (This order corresponds to the third iteration.) By means of a matching argument, we find a nonanalytic term of $O(\epsilon^{11}\ln\epsilon)$, or $O(\epsilon^6 \ln\epsilon)$ beyond linearized order in the *wave-zone expansion*. However, any possible nonanalytic terms arising from matching in this term are at most $O(\ln\epsilon)$*11/2-PN.

II. REDUCTION OF PROBLEM TO ANALYSIS OF THE ZERILLI EQUATION

We consider a system satisfying the following slow-motion assumptions: Let l be a typical length scale for the sources and T_0 a typical time scale for the motion (i.e., orbital period). At distances of order T_0, time and spatial derivatives scale with T_0. At distances of order l, time derivatives still scale with T_0, while spatial derivatives scale with l. The dimensionless parameter $\epsilon << 1$ is defined as l/T_0 and represents a typical velocity. All coordinates and dimensional parameters such as the mass M are assumed to be expressed in the appropriate units of T_0. We assume that the system is gravitationally bound. Therefore, $M=O(\epsilon^3)$, and a typical near-zone Newtonian potential U_0 is of $O(\epsilon^2)$.

Thorne[22] derived slow-motion radiation reaction for gravitationally bound systems using the Regge-Wheeler metric decomposition and gauge. His approximate metric takes the form

$$\begin{aligned}g\sim&\left[1+\nu(t,r)+\sum_{l=1}^{\infty}\sum_{m=-1}^{l}\mathscr{H}^{l}_{0m}(t,r)Y^l_m(\theta,\phi)\right]dt^2-\left[1+\lambda(t,r)-\sum_{l=1}^{\infty}\sum_{m=-1}^{l}\mathscr{H}^{l}_{2m}(t,r)Y^l_m(\theta,\phi)\right]dr^2\\&-r^2\left[1-\sum_{l=2}^{\infty}\sum_{m=-1}^{l}\mathscr{K}^{l}_{m}(t,r)Y^l_m(\theta,\phi)\right](d\theta^2+\sin^2\theta\,d\phi^2)+2\left[\sum_{l=1}^{\infty}\sum_{m=-1}^{l}\mathscr{H}^{l}_{1m}(t,r)Y^l_m(\theta,\phi)\right]dt\,dr\\&+2\sum_{A=\theta,\phi}\left[\sum_{l=1}^{\infty}\sum_{m=-1}^{l}h^{l}_{0m}(t,r)\Phi^l_{mA}(\theta,\phi)\right]dt\,dx^A+2\sum_{A=\theta,\phi}\left[\sum_{l=2}^{\infty}\sum_{m=-1}^{l}h^{l}_{1m}(t,r)\Phi^l_{mA}(\theta,\phi)\right]dr\,dx^A,\quad A=\theta,\phi\ ,\end{aligned}\tag{2.1}$$

where the Y^l_m are scalar and the Φ^l_{mA} are vector spherical harmonics. The Newtonian metric functions $\nu(t,r)$ and $\lambda(t,r)$ represent the static, monopole part of the metric; they are $O(M)=O(\epsilon^3)$ in the wave zone and $O(\epsilon^2)$ in the near zone. (The coupling between wave-zone and near-zone orders is given in Fig. 1 of Thorne's paper.[22]) The metric functions $\mathscr{H}_0$, $\mathscr{H}_1$, etc. contain the linearized radiation. Let us suppose that quadrupole radiation of even parity is dominant in our system— the usual case. Our slow-motion assumptions then imply that the quadrupole radiation is of $O(\epsilon^5)$ in the wave zone.

We will depart from Thorne[22] by keeping all terms in $\nu(t,r)$ and $\lambda(t,r)$ up to $O(M^2O)=O(\epsilon^6)$. We will also keep terms in the wave-zone expansion of $\mathscr{H}_0$, $\mathscr{H}_1$, etc. up to $O(M^2)$ beyond the linearized orders kept by Thorne. [For the quadrupole case, our treatment thus covers not only the usual $O(\epsilon^5)$, but also $O(\epsilon^8)$ and $O(\epsilon^{11})$ in the wave zone.] *These higher-order, post-linear terms are responsible for the PN divergences and for the nonanalytic near-zone orders that we will later encounter via matching.* The most convenient way of keeping these post-linear terms is to write our metric expansion in the form of (even-parity) perturbations[25] on a Schwarzschild background,

$$\begin{aligned}g\sim&(1-2M/r)dt^2-(1-2M/r)^{-1}dr^2-r^2(d\theta^2+\sin\theta\,d\phi^2)\\&+\sum_{l,m}[(1-2M/r)dt^2+(1-2M/r)^{-1}dr^2]\mathscr{H}^{l}_{0m}(t,r)Y^l_m(\theta,\phi)+r^2\left[\sum_{l,m}\mathscr{K}^{l}_{m}(t,r)Y^l_m(\theta,\phi)\right](d\theta^2+\sin^2\theta\,d\phi^2)\\&+2\left[\sum_{l,m}\mathscr{H}^{l}_{1m}(t,r)Y^l_m(\theta,\phi)\right]dt\,dr,\quad l\neq1\ ,\end{aligned}\tag{2.2}$$

where we treat $M=O(\epsilon^3)$ as a small parameter and keep terms up to $O(M^2)$ in all quantities. For the $L=2$, even-parity perturbations of interest, the metric (2.2) becomes

$$\begin{aligned}g\sim&(1-2M/r)dt^2-(1-2M/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2)\\&+[(1-2M/r)dt^2+(1-2M/r)^{-1}dr^2]\mathscr{H}(t,r;M)Y^2_m\\&+(2dt\,dr)\mathscr{H}_1(t,r;M)Y^2_m+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\mathscr{K}(t,r;M)Y^2_m\ ,\end{aligned}\tag{2.3}$$

where we have dropped the l,m subscripts on $\mathscr{H}$, $\mathscr{H}_1$, and $\mathscr{K}$.

Let us suppose further that the Regge-Wheeler functions $\mathscr{H}$, $\mathscr{H}_1$, and $\mathscr{K}$ can be expressed as sums over components varying nearly harmonically with time,

$$\mathscr{H}=\sum_{\omega}e^{-i\omega t}H(r;\omega,M)\ ,\tag{2.4a}$$

$$\mathscr{H}_1=\sum_{\omega}e^{-i\omega t}H_1(r;\omega,M)\ ,\tag{2.4b}$$

$$\mathscr{K}=\sum_{\omega}e^{-i\omega t}K(r;\omega,M)\ .\tag{2.4c}$$

[If the frequencies vary only over a damping time—$O(T_0/\epsilon^5)$—the frequency derivatives will not enter at the orders considered here.]

To the approximation considered, the sourceless Einstein field equations for the functions H, K, and H_1 are equivalent to the single $l=2$ Zerilli[23] equation

$$\frac{d^2\hat{K}}{dr^{*2}}+[\omega^2-V(r;\epsilon)]\hat{K}=0\ , \tag{2.5}$$

$$r^*\equiv r+2M\ \ln(r-2M)\ , \tag{2.6}$$

$$V(r;\epsilon)\equiv(1-2M/r)\left[\frac{24r^3+24Mr^2+36M^2r+18M^3}{r^3(2r+3M)^2}\right] \tag{2.7}$$

for a function $\hat{K}(r;\omega,M)$, together with the following equations relating the metrics to $\hat{K}$:

$$K(r)=g\hat{K}+\hat{R}\ , \tag{2.8}$$

$$H_1(r)=\omega(h\hat{K}+k\hat{R})\ , \tag{2.9}$$

$$H(r)=(2r+3M)^{-1}(aK+bH_1)\ , \tag{2.10}$$

$$\hat{R}(r)\equiv d\hat{K}/dr^*=(1-2M/r)d\hat{K}/dr\ , \tag{2.11}$$

$$g(r)\equiv\frac{6r^2+6Mr+6M^2}{r^2(2r+3M)}\ , \tag{2.12}$$

$$H(r)\equiv i\frac{-2r^2+6Mr+3M^2}{(r-2M)(2r+3M)}\ , \tag{2.13}$$

$$k(r)\equiv\frac{-ir^2}{r-2M}\ , \tag{2.14}$$

$$a(r)\equiv 2r-\frac{\omega^2r^4+M(r-3M)}{r-2M}\ , \tag{2.15}$$

$$b(r)\equiv i\omega r^2+\frac{3M}{i\omega r}\ , \tag{2.16}$$

provided one keeps the terms of $O(1)$, $O(M)$, and $O(M^2)$ in Eq. (2.5).

To $O(1)$ in M, Eqs. (2.8)–(2.16) simplify to

$$k=\frac{3\hat{K}}{r}+\frac{d\hat{K}}{dr}\ , \tag{2.17}$$

$$H=K-\omega^2r\hat{K}\ , \tag{2.18}$$

$$H_1=-i\omega\left[\hat{K}+r\frac{d\hat{K}}{dr}\right]\ . \tag{2.19}$$

III. DIVERGENT INTEGRALS IN PN EXPANSION AT 4-PN ORDER

In this section, we show for the Regge-Wheeler gauge that a PN expansion leads to divergent integrals at 4-PN order. This 4-PN divergence also occurs in the deDonder gauge, as noted above.

We assume that the material sources are contained with some radius $r_0=O(\epsilon)$. We therefore consider the $l=2$ Zerilli equation (2.5) with a compact source term to represent the matter

$$\frac{d^2\hat{K}}{dr^{*2}}+[\omega^2-V(r;\epsilon)]\hat{K} = \text{compact source terms}\ . \tag{3.1}$$

For convenience, we first define

$$\hat{\phi}\equiv\frac{\hat{K}}{r}\ . \tag{3.2}$$

Collecting flat-space terms on the left, one can express Eq. (3.1) as

$$(\nabla^2+\omega^2)(\hat{\phi}Y_m^2)=S(r)Y_m^2\ , \tag{3.3}$$

where

$$S(r)=\text{compact source terms} +M\left[4r^{-1}\frac{d^2\hat{\phi}}{dr^2}+6r^{-2}\frac{d\hat{\phi}}{dr}-26r^{-3}\hat{\phi}\right] +O(M^2)\ . \tag{3.4}$$

Making use of the Helmholz Green's function, one can rewrite Eq. (3.3) as

$$\hat{\phi}(r)Y_m^2=-\frac{1}{4\pi}\int\frac{S(r)e^{i\omega|x-x'|}Y_m^2(\theta',\phi')}{|x-x'|}d^3x'\ . \tag{3.5}$$

Expressing the Green's function in the integrand of Eq. (3.5) in terms of spherical harmonics and integrating over θ',ϕ', one has

$$\hat{\phi}=-i\omega\left[\left[\int_{r\le r'}S(r')h_2^{(1)}(\omega r')r'^2dr'\right]j_2(\omega r)+\left[\int_{r>r'}S(r')j_2(\omega r')r'^2dr'\right]h_2^{(1)}(\omega r)\right]\ , \tag{3.6}$$

where

$$h_2^{(1)}(x) \equiv \frac{ie^{ix}}{x}\left[1+\frac{3i}{x}-\frac{3}{x^2}\right], \tag{3.7}$$

$$j_2(x) \equiv \left[\frac{3}{x^3}-\frac{1}{x}\right]\sin x - \frac{3\cos x}{x^2} \tag{3.8}$$

$$\left[j_2(x) \sim \frac{x^2}{15} + \cdots,\ x << 1\right] \tag{3.9}$$

are spherical Bessel functions.

Since $M = O(\epsilon^3)$ and $S = O(M)$, one seeks $\hat{\phi}$ as an expansion,

$$\hat{\phi} \sim {}_0\hat{\phi} + {}_1\hat{\phi} + \cdots , \tag{3.10}$$

where each ${}_p\hat{\phi}$ is obtained from ${}_0\hat{\phi}$ through ${}_{p-1}\hat{\phi}$ by formal substitution of Eq. (3.10) into Eq. (3.6). Each iteration therefore introduces a correction of $O(M)$ smaller than the preceding one.

Assume that the source generates a first iterate

$${}_0\hat{\phi} = ch_2^{(1)}(\omega r) . \tag{3.11}$$

The second iterate ${}_1\hat{\phi}$ is then given by

$${}_1\hat{\phi} = I_< + I_> , \tag{3.12}$$

$$I_> \equiv -i\omega\left[\left[\int_{r\le r'} S_1(r')j_2(\omega r')r'^2 dr'\right]h_2^{(1)}(\omega r)\right], \tag{3.13}$$

$$I_< \equiv -i\omega\left[\left[\int_{r>r'} S_1(r')j_2(\omega r')r'^2 dr\right]h_2^{(1)}(\omega r)\right], \tag{3.14}$$

where $S(r) = S_1(r)$ is defined by

$$S_1(r) \equiv Mce^{i\omega r}\left[\frac{4\omega^4}{r^2} + 14\frac{i\omega^3}{r^3} - \frac{18\omega^2}{r^4} - \frac{12i\omega}{r^5} + \frac{12}{r^6}\right] + M(\text{compact source corrections}) . \tag{3.15}$$

The PN slow-motion assumption now consists of treating $\omega r'$ as if it were a small quantity and expanding the integrands of Eqs (3.13) and (3.14) in powers of $\omega r'$. It is then assumed that one may integrate the resulting expansion term by term. (Note that the assumption $\omega r' << 1$ is actually violated at the upper limit.) The $r' < r$ integrals are finite and lead to no complications. However, for $r' > r$, one obtains

$$I_> \equiv i\omega c j_2(\omega r)\left[\frac{-36}{r'^6} + \frac{15\omega^2}{2r'^4} + \frac{6\omega^4}{r'^2} - \frac{2i\omega^5}{5r'} - 4\omega^6 \ln r'\right]\Bigg|_r^\infty + \cdots . \tag{3.16}$$

The fifth term in Eq. (3.16) diverges logarithmically.

The PN-order of this divergence can be computed as follows: The second iterate ${}_1\hat{\phi}$ is of $O(M) = O(\epsilon^3)$ smaller than ${}_0\hat{\phi}$. From Eqs. (3.2) and (2.8)–(2.16) or (2.17)–(2.19), one sees that in the near zone, the r^{-3} part of ${}_0\hat{\phi}$ corresponds to terms of Newtonian order in the metric functions H and K of Eq. (2.3). The divergent integral of Eq. (3.16) multiplies $j_2(\omega r)$, which goes as r^2 for small r; this factor of r^5 implies a factor of ϵ^5 in the near zone, making ϵ^8 in all. Therefore, the divergence (3.16) of the second iterate ${}_1\hat{\phi}$ occurs at $O(\epsilon^8)$ beyond Newtonian order—that is, at 4-PN order in the metric functions H and K.

IV. LOWEST-ORDER "STRAINED" WAVE-ZONE SOLUTION OF THE ZERILLI EQUATION; MATCHING TO NEAR ZONE

In the wave zone, it is convenient to work with a quantity $\psi(r;\omega,M)$ defined from $\hat{K}$ by

$$\hat{K}=Q(\omega)\psi(r;\omega,M)e^{i\omega r^*}, \tag{4.1}$$

where r^* is the "tortoise coordinate" of Eq. (2.6), and where Q is an amplitude to be discussed below; ψ obeys the equation

$$\left[1-\frac{2M}{r}\right]\frac{d^2\psi}{dr^2}+\left[\frac{2M}{r^2}+2i\omega\right]\frac{d\psi}{dr} -3\left[2+\frac{12}{r}+\frac{3M^2}{r^2}\right]\left[r^{-2}-3Mr^{-3} + \tfrac{27}{4}M^2r^{-4}\right]\psi +O(M^3)=0 . \tag{4.2}$$

This decomposition facilitates the application of outgoing-wave boundary conditions based on approximately Schwarzschild null surfaces. We require that

$$\psi\sim\sum_{n=0}^{\infty} b_n r^{-n} . \tag{4.3}$$

The outgoing-wave condition (4.3) and the condition that the wave-zone and near-zone expansions match order by order determine the two constants appearing at each order in $\hat{K}$.

We expand ψ in the form

$$\psi\sim\psi_0+M\psi_1+M^2\psi_2+\cdots . \tag{4.4}$$

The wave-zone expansion for $\hat{K}$ is strained in the sense that even the lowest-order approximation to $\hat{K}$ contains the implicit M dependence of $r^*=r+2M\ln(r-2M)$ in the exponential $e^{i\omega r^*}$.

The equation for ψ_0 is obtained by keeping the terms of $O(1)$ in Eq. (4.2) with (4.4) substituted for ψ:

$$\frac{d^2\psi_0}{dr^2}+2i\omega\frac{d\psi_0}{dr}-\frac{6}{r^2}\psi_0=0 . \tag{4.5}$$

Equation (4.5) has outgoing solutions of the form

$$\phi_0=A(3r^{-2}-3i\omega r^{-1}-\omega^2) . \tag{4.6}$$

We absorb the constant A into the amplitude $Q(\omega)$ without loss of generality.

The corresponding solutions for H, H_1, and K were obtained directly by Thorne.[22] These solutions can also be obtained from Eqs. (4.1), (4.6), and (2.17)–(2.19). The Regge-Wheeler gauge has an apparent large-r divergence which can be removed by a gauge transformation given in Ref. 25. The resulting gauge-transformed fields have the expected r^{-1} behavior at large r.

The small-r expansion of $\hat{K}$ (to order unity in M) is

$$\hat{K}\sim Q\left[3r^{-2}+\frac{\omega^2}{2}+\frac{\omega^4r^2}{8}+\frac{i\omega^5r^3}{15}+\cdots\right] +O(M) . \tag{4.7}$$

From Eqs. (2.8)–(2.16), the metric functions H and K become, to this order,

$$H\sim Q\epsilon^2\left[3R^{-3}-\tfrac{3}{2}\epsilon^2\omega^2R^{-1}+\tfrac{1}{8}\epsilon^4\omega^4R + \tfrac{2}{5}i\epsilon^5\omega^5R^2+\cdots\right]+O(M) , \tag{4.8}$$

$$K\sim Q\epsilon^2\left[3R^{-3}+\tfrac{3}{2}\epsilon^2\omega^2R^{-1}+\tfrac{5}{8}\epsilon^4\omega^4R + \tfrac{2}{5}i\epsilon^5\omega^5R^2+\cdots\right]+O(M) , \tag{4.9}$$

where we have rewritten H and K in terms of a near-zone variable $R\equiv r/\epsilon$. $Q(\omega)$ is determined as in Thorne[22] by matching out the quadrupole part of the Newtonian potential, which goes as $\epsilon^2R^{-3}=\epsilon^5r^{-3}$ for large R. Thus,

$$Q(\omega)=O(\epsilon^5) . \tag{4.10}$$

The first time-odd term in H goes as R^2 and matches[22] into the Regge-Wheeler version of the familiar Burke[4] reaction potential. Note that the Burke resistive potential is of order ϵ^7 in H and K, or $\frac{5}{2}$-PN order.

IV. WAVE-ZONE SOLUTION TO $O(M)$ BEYOND LINEARIZED ORDER; NONANALYTIC TERM IN NEAR-ZONE EXPANSION FOUND BY MATCHING

The equation for ψ_1 is obtained from the term of order M in Eqs. (4.2) and (4.4):

$$\frac{d^2\psi_1}{dr^2}+2i\omega\frac{d\psi_1}{dr}-6r^{-2}\psi_1 =6(2r^{-5}+3i\omega r^{-4}+2\omega^2r^{-3}) . \tag{5.1}$$

The most general solution to this equation is of the form

$$\psi_1=2r^{-3}+\tfrac{13}{2}i\omega r^{-2}-\omega^2r^{-1} +2i\omega e^{-2i\omega r}E_1(-2i\omega r)(3r^{-2}+3i\omega r^{-1}-\omega^2) +C_+(3r^{-2}-3i\omega r^{-1}-\omega^2) +C_-(3r^{-2}+3i\omega r^{-1}-\omega^2) , \tag{5.2}$$

where $E_1(z)$ is the exponential integral

$$E_1(z) \equiv \int_z^\infty t^{-1} e^{-t} dt \ . \tag{5.3}$$

The term involving the exponential integral in Eq. (5.2) represents backscatter off the Schwarzschild curvature because of the factor $e^{-2i\omega r}$.

To evaluate the constants appearing in Eq. (5.2), we will need the large- and small-z expansions[26] of $E_1(z)$:

$$E_1(z) \sim z^{-1} e^{-z} \left[\sum_{p=0}^{\infty} p!(-z)^{-p} \right], \quad z \to \infty \tag{5.4}$$

$$E_1(z) \sim -\gamma - \ln(z) - \sum_{p=1}^{\infty} \frac{(-z)^p}{pp!}, \quad z \to 0 \tag{5.5}$$

where γ=Euler constant. The constant C_- vanishes by virtue of Eqs. (4.1), (4.3), and (5.5).

The smaller-r expansion of $\hat{K}$ including terms of $O(M)$ beyond linearized orders is now given by

$$\hat{K} = Q e^{i\omega r^*} (\psi_0 + M\psi_1 + \cdots) , \tag{5.6}$$

$$\begin{aligned} e^{i\omega r^*}\psi_0 \sim & \left[\frac{3}{r^2} + \frac{\omega^2}{2} + \frac{\omega^4 r^2}{8} + \frac{i\omega^5 r^2}{15} + \cdots \right] \\ & + M \left[6i\omega \frac{\ln r}{r^2} + i\omega^3 \ln r + \frac{i\omega^5}{4} r^2 \ln r - \tfrac{2}{15}\omega^6 r^3 \ln r + \cdots \right] + O(M^2) , \end{aligned} \tag{5.7}$$

$$\begin{aligned} M e^{i\omega r^*}\psi_1 \sim M & \left[\frac{2}{r^3} + \frac{i\omega}{r^2} \left[\frac{17}{2} + 6\Delta - \frac{3i}{\omega} C_+ \right] + \frac{7}{2}\frac{\omega^2}{r} + \frac{i\omega^3}{6} \left[\frac{17}{2} + 6\Delta - \frac{3i}{\omega} C_+ \right] \right. \\ & \left. - 6i\omega \frac{\ln r}{r^2} - i\omega^3 \ln r - \frac{i\omega^5 r^2}{4} \ln r - \tfrac{2}{15}\omega^6 r^3 \ln r + \cdots \right] + O(M^2) , \end{aligned} \tag{5.8}$$

$$\begin{aligned} \hat{K} \sim Q \Bigg\{ & \left[\frac{3}{r^2} + \frac{\omega^2}{2} + \frac{\omega^4 r^2}{8} + \frac{i\omega^5 r^3}{15} + \cdots \right] \\ & + M \left[\frac{2}{r^3} + \frac{i\omega}{r^2} \left[\frac{17}{2} + 6\Delta - \frac{3i}{\omega} C_+ \right] + \frac{7}{2}\frac{\omega^2}{r} + \frac{i\omega^3}{6} \left[\frac{17}{2} + 6\Delta - \frac{3i}{\omega} C_+ \right] - \frac{4}{15}\omega^6 r^3 \ln r + \cdots \right] \\ & + O(M^2) \Bigg\} , \end{aligned} \tag{5.9}$$

$$\Delta \equiv -\gamma + \frac{i\pi}{2} - \ln(2\omega) \ . \tag{5.10}$$

Let us consider the terms proportional to M in Eq. (5.9) that grow most rapidly with $1/r$: the Mr^{-3} term can be shown to match to the larger-r expansion of the 1-PN part of the near-zone expansion of $\hat{K}$. The terms of $O(Mr^{-2})$ in Eq. (5.9) must match to the $\frac{3}{2}$-PN part of the near-zone expansion, which can be shown to vanish for the systems under consideration (see Fig. 1 of Thorne[22]). Setting the coefficient of Mr^{-2} equal to zero gives

$$C_+ = -i\alpha , \tag{5.11}$$

$$\alpha \equiv \omega \left[\tfrac{17}{6} + 2 \left[-\gamma + \frac{i\pi}{2} - \ln(2\omega) \right] \right] . \tag{5.12}$$

It is important to observe that the leading time-odd term in Eq. (5.9) corresponds to the Burke[4] reaction potential (as derived by Thorne[22]). This result rectifies an apparent paradox mentioned in Sec. I: Consider Eq. (5.7), which represents the linearized quadrupole radiation corrected for redshift. In the near zone, the time-odd term

$$e^{i\omega r^*}\psi_0 \sim \cdots + M \left[6i\omega \frac{\ln r}{r^2} + \cdots \right] \tag{5.13}$$

in (5.7) is of $O(\ln\epsilon)$ larger than $\frac{3}{2}$-PN order, whereas the Burke reaction potential

$$e^{i\omega r^*}\psi_0 \sim \left[\cdots + \frac{i\omega^5 r^3}{15} + \cdots \right] \tag{5.14}$$

is only of $\frac{5}{2}$-PN order. However, the inclusion of ψ_1 in $\hat{K}$ cancels the logarithmic contributions at $O(\epsilon^5 \ln\epsilon)$, $O(\epsilon^7 \ln\epsilon)$, $O(\epsilon^9 \ln\epsilon)$, $\cdots$. [An analogous result was found in Eq. (33) of paper I.]

Recall now the divergent integral of Sec. III at 4-PN order. The inevitability of this PN divergence can now be understood by the presence of the term

$$\hat{K} \sim Q[\,\cdots + M(\,\cdots - \tfrac{4}{15}\omega^6 r^3 \ln r + \cdots\,)] \tag{5.15}$$

in Eq. (5.9): Using Eqs. (2.17)–(2.19) for the metric functions and expressing the result in terms of a near-zone variable $R \equiv r/\epsilon$, one obtains

$$H \sim \cdots - \tfrac{8}{5} Q M \epsilon^2 \ln\epsilon (\omega^6 R^2) + \cdots , \tag{5.16}$$

$$K \sim \cdots - \tfrac{8}{5} Q M \epsilon^2 \ln\epsilon (\omega^6 R^2) + \cdots . \tag{5.17}$$

Since $Q = O(\epsilon^5)$ and $M = O(\epsilon^3)$, *one therefore needs to consider a term of* $O(\epsilon^{10}\ln\epsilon)$ in the near-zone expansions of H and K:

$$H \sim \cdots + \epsilon^{10} \ln\epsilon H_{NA} , \tag{5.18}$$

$$K \sim \cdots + \epsilon^{10} \ln\epsilon K_{NA} . \tag{5.19}$$

The terms in question are $O(\epsilon^8 \ln\epsilon)$ larger than Newtonian order, or $O(\ln\epsilon)$ larger than 4-PN order.

Substitution of expansions (5.18) and (5.19) with (2.3) and (2.4) and (2.6) into the Einstein field equations shows that H_{NA} and K_{NA} satisfy the $l=2$ radial part of Laplace's equation

$$\left[\frac{d^2}{dr^2} - \frac{6}{r^2}\right] H_{NA} = 0 , \tag{5.20}$$

$$\left[\frac{d^2}{dr^2} - \frac{6}{r^2}\right] K_{NA} = 0 . \tag{5.21}$$

The near-zone solutions of Eqs. (5.20) and (5.21) that match Eqs. (5.16) and (5.17) and are nonsingular at the origin are

$$K_{NA} = H_{NA} = -\frac{8}{5}\left[\frac{QM}{\epsilon^8}\right]\omega^6 R^2 , \tag{5.22}$$

where the quantity in parentheses is $O(1)$.

IV. CONCLUSIONS

Our calculation has achieved three results. First, we have found that the Burke[4] radiation reaction potential of Eqs. (4.8) and (4.9) is still the leading ($\frac{5}{2}$-PN) time-odd term in the near zone, despite the apparently larger ($\frac{3}{2}$-PN*$\ln\epsilon$) time-odd term of Eq. (5.13). This spurious time-odd term arose from the implict M dependence of r^* in the exponential $e^{i\omega r^*}$ in our lowest-order strained solution [given by Eqs. (4.1) (4.4) and (4.6)] for $\hat{K}$. [The straining had enabled us to write an accurate outgoing-wave condition in the convenient form (4.3) and to avoid nonuniformities at large r that would have occurred in an expansion about flat space.] We observe also that the cancellations of the sequence of time-odd logarithmic terms in Eqs. (5.7)–(5.9) were inevitable, because these time-odd terms could not have been matched to the near-zone expansion.

Our second goal has been to show that the post-Newtonian divergent integrals at 4-PN order represent a missing "nonanalytic" term of $O(\ln\epsilon)$ larger than 4-PN order. In Sec. III, we showed that a PN expansion gives a divergent integral at 4-PN order in the Regge-Wheeler gauge, just as one finds in Kerlick's[11] work in the deDonder gauge. In Sec. V. [Eqs. (5.15)–(5.22)], we found a near-zone term of $O(\epsilon^{10}\ln\epsilon)$ in the quadrupole metric functions H and K, i.e., $O(\ln\epsilon)$ larger than 4-PN order.

Since our nonanalytic near-zone metric terms are time-even, they would occur with any combination of (what we call) "incoming" and "outgoing" radiation. The presence of these terms is therefore independent of the question of whether the system satisfies a condition for the absence of incoming radiation at past null infinity. (We have not shown in this paper that such a condition is satisfied up to the orders considered.) For the same reason, these terms have no secular effects on the mechanical energy of the sources, and any periodic effects are far too small to generate measureable consequences.

In the Appendix, we solve the wave-zone equations at $O(M^2)$ beyond linearized order. Via matching arguments, we find a nonanalytic term in the *wave-zone* expansion at $O(\epsilon^{11}\ln\epsilon)$, or $O(\epsilon^6 \ln\epsilon)$ beyond linearized order. One can show that any possible nonanalytic contributions to the near zone from *this* part of the wave-zone expansion are $O(\epsilon^{13}\ln\epsilon)$, or $O(\ln\epsilon)$ larger than $\frac{11}{2}$-PN order, which is much smaller than our results from Sec. V. However, it is conceivable that still higher-order wave-zone terms might generate nonanalytic contributions larger than those found here.

The combined results of paper I and the present paper may shed some light on previous discussions[27] concerning the use of linearized equations in general relativity. Although relegating all nonflat terms to the right-hand side of the wave-zone field equations appears formally justified, the integrated effect of Schwarzschild curvature is important as one approaches past or future null infinity. For this reason, we strained the linearized approximation by correcting the null surfaces, as in Ref. 24. However, to avoid spurious time-odd terms in the near zone, one must include also post-linear corrections ($\psi_1, \cdots$) to the appropriate order in M/wavelength. Note that both the cancellation of spurious time-odd terms and the logarithmic time-even term depended on significant contributions from the "backscatter" terms of Eq. (5.2). Thus, although the radiation reaction one derives using a linearized approximation in the wave zone seems to give correct answers, one needs to look at the higher orders to make sure that these answers *remain* correct.

The results of this paper, paper I, and Refs. 17-20 suggest that nonanalytic behavior in the small parameter is a generic feature of radiation from slow-motion sources on curved backgrounds and may even be a generic feature of radiation from most nonlinear systems. The assumption of analyticity in the small parameter in such problems is likely to cause perplexing divergences.

In the present case, our finding of a nonanalytic term at $O(\ln\epsilon)$*4-PN order suggests strongly that the results of the post-Newtonian calculations[6–8,10,11] are in fact *correct*—except for their divergent integrals. However, we leave it to future work to verify that the lower-order terms not treated here do in fact agree with their counterparts from the post-Newtonian calculations.

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APPENDIX: NONANALYTIC TERMS ARISING FROM $O(M^2)$ BEYOND LINEARIZED ORDER IN WAVE ZONE

The equation for ψ_2,

$$\frac{d^2\psi_2}{dr^2}+2i\omega\frac{d\psi_2}{dr}-\frac{6}{r^2}\psi_2=S_2\ , \tag{A1}$$

where

$$\begin{aligned}S_2\equiv{}&\tfrac{261}{2}r^{-6}-i\omega r^{-5}(-\tfrac{7}{2}+\alpha-24\mathscr{E})+\omega^2r^{-4}(\tfrac{-195}{2}+18\alpha-84\mathscr{E})-i\omega^3r^{-3}(32+12\alpha+96\mathscr{E})\\&+\omega^4r^{-2}(8+56\mathscr{E})+16i\omega^5r^{-1}\mathscr{E}\end{aligned} \tag{A2}$$

and

$$\mathscr{E}\equiv e^{-2i\omega r}[E_1(-2i\omega r)] \tag{A3}$$

is obtained from the terms of order M^2 when Eq. (4.4) is substituted in Eq. (4.2). Variation of parameters yields the general solution

$$\begin{aligned}\psi_2={}&\tfrac{261}{28}r^{-4}+i\omega r^{-3}(\tfrac{701}{84}-2\alpha)+\omega^2r^{-2}(\tfrac{-279}{10}+\tfrac{13}{2}\alpha)+i\omega^3r^{-1}(\tfrac{-164}{35}+\alpha)\\&-\mathscr{E}\left[\frac{-4i\omega}{r^3}-\frac{\omega^2}{r^3}(6\alpha-\tfrac{949}{35})-\frac{i\omega^3}{r}(\tfrac{-1474}{35}+6\alpha)-\omega^4(-2\alpha+\tfrac{468}{35})\right]\\&-(-12\omega^2r^{-2}+12i\omega^3r^{-1}+4\omega^4)\int_r^\infty\frac{e^{-2i\omega s}}{s}[E_1(-2i\omega s)]ds\\&-(24\omega^2r^{-2}+24i\omega^3r^{-1}-8\omega^4)e^{-2i\omega r}\int_r^\infty\frac{E_1(-2i\omega s)}{s}ds\\&+D_+(3r^{-2}-3i\omega r^{-1}-\omega^2)+D_-(3r^{-2}+3i\omega r^{-1}-\omega^2)\ .\end{aligned} \tag{A4}$$

Integrals of the form

$$I_1 \equiv \int_r^\infty s^{-1} e^{-2i\omega s} E_1(-2i\omega s) ds \,, \quad \text{(A5)}$$

$$I_2 \equiv \int_r^\infty s^{-1} E_1(-2i\omega s) ds \,, \quad \text{(A6)}$$

in Eq. (A4) have large- and small-r expansions

$$I_1 \sim r^{-1} + \cdots \,, \quad r \to \infty \quad \text{(A7)}$$

$$I_2 \sim e^{2i\omega r} r^{-1} + E_1(-2i\omega r) + \cdots \,, \quad r \to \infty \quad \text{(A8)}$$

$$I_1 \sim \Delta \ln r - \tfrac{1}{2} \ln^2 r - \sum_{p=1}^{\infty} \frac{(2i\omega r)^p}{p p!} (\ln r + \Delta) + \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2i\omega r)^{p+m}}{p(p+m)p!m!} + \beta_1, \quad r \to 0 \quad \text{(A9)}$$

$$I_2 \sim \tfrac{1}{2} \ln^2 r - \sum_{p=1}^{\infty} \frac{(2i\omega r)^p}{p^2 p!} + \beta_2, \quad r \to 0 \,, \quad \text{(A10)}$$

where β_1 and β_2 are constants whose explicit values will not be needed here, and where Δ was defined in Eq. (5.10).

Applying the outgoing radiation condition (4.3) to the larger-r expansion of Eq. (A4), we find that D_- must vanish. We then obtain the following small-r expansion for the $O(M^2)$ part of $\hat{K}$:

$$K \sim \cdots + M^2 [\tfrac{261}{28} r^{-4} + \tfrac{214}{35} \omega^2 r^{-2} \ln r + (3D_+ + \text{constants}) r^{-2} + \cdots] + O(M^3) \,. \quad \text{(A11)}$$

The term going as r^{-4} should match the 2-PN near-zone metric. (However, we have not verified this matching explicitly.) The constant D_+ in the term going as r^{-2} is in principle determined by matching to the 3-PN metric, just as the constant C_+ was determined in Eq. (5.11) by matching to the $\frac{3}{2}$-PN metric. (Fortunately, the precise value of D_+ does not enter the remainder of our argument.) The $O(M^2)$ part of $\hat{K}$ is then in principle fully determined.

The term going as $r^{-2} \ln r$ in Eq. (A11) appears to demand near-zone metric corrections in H and K at $(\ln\epsilon)$* 3-PN order. Now, if there exist $O(\epsilon^8 \ln\epsilon)$ terms in H and K, they must be solutions of Eqs. (5.20) and (5.21) that are regular at the origin. However, from Eqs. (2.17)–(2.19), one can show that these $O(\epsilon^8 \ln\epsilon)$ near-zone corrections to H and K would be proportional to R^{-3}, and thus matching appears to be impossible.

However, by adding a homogeneous outgoing solution of the form (4.6) to the wave-zone expansion (4.4) at $O(\epsilon^{11} \ln\epsilon)$, we can resolve the apparent matching conflict. The wave-zone expansion (4.4) is thus modified to

$$\psi \sim \psi_0 + M\psi_1 + M^2\psi_2 - M^2(\tfrac{214}{35}\omega^2 \ln\epsilon) + O(M^3) \,. \quad \text{(A12)}$$

The total near-zone contribution from $\hat{K}$ at $O(\epsilon^8 \ln\epsilon)$ now vanishes.

The lowest order in which Eqs. (A12) and (A4) can now contribute to the near-zone expansion is at $O(\epsilon^{13} \ln\epsilon)$, because any such contibution must match to a homogeneous near-zone solution regular at the origin, and such a term in $\hat{K}$ would have to have the r dependence r^3.

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[1]J. Ehlers, in *Isolated Gravitating Systems in General Relativity,* edited by J. Ehlers (North-Holland, Amsterdam, 1979).

[2]R. E. Kates, Ann. Phys. (N.Y.) 132, 1 (1981).

[3]A. Erdelyi, *Asymptotic Expansions* (Dover, New York, 1956).

[4]W. L. Burke, J. Math. Phys. 12, 401 (1971).

[5]R. E. Kates, Phys. Rev. D 22, 1871 (1980).

[6]J. L. Anderson and T. C. Decanio, Gen. Relativ. Gravit. 6, 197 (1975).

[7]S. Chandrasekhar and F. P. Esposito, Astrophys. J. 160, 153 (1970).

[8]R. Epstein and R. V. Wagoner, Astrophys. J. 197, 717 (1975).

[9]J. Ehlers, A. Rosenblum, J. N. Goldberg, and P. Havas, Astrophys. J. 208, 177 (1975).

[10]J. Ehlers, Ann. N.Y. Acad. Sci. 336, 279 (1980).

[11]G. D. Kerlick, Gen. Relativ. Gravit. 12, 467 (1980); 12, 521 (1980).

[12]B. Schutz (private communication); T. Futamase, Ph.D. thesis, University of Wales, 1981 (unpublished), p. 27.

[13]R. Breuer and E. Rudolph Gen. Relativ. Gravit. (to be published).

[14]The expression "second iteration" is used loesely here to describe terms that contain two powers of G in the Anderson-Decanio scheme. One normally includes in each iteration only those terms that cannot be affected by subsequent iteration steps. The third and fourth iterations also contribute to 4-PN order.

[15]M. Walker and C. Will, Astrophys. J. 242, L129

(1980).

[16]M. Walker and C. Will, Phys. Rev. Lett. 45, 1741 (1980).

[17]J. L. Anderson, in *Isolated Gravitating Systems in General Relativity,* Ref. 1.

[18]J. L. Anderson and L. S. Kegeles, Gen. Relativ. Gravitit. 12, 633 (1980).

[19]W. G. Dixon, in *Proceedings of the Third Gregynog Relativity Workshop: Gravitational Radiation Theory,* edited by M. Walker (Max-Plank Institut, Müchen, 1979).

[20]J. L. Anderson and L. S. Kegeles (unpublished).

[21]R. E. Kates and L. S. Kegeles, preceding paper, Phys. Rev. D 25, 2030 (1982).

[22]K.S. Thorne, Astrophys. J. 158, 997 (1969).

[23]F. J. Zerilli, Phys. Rev. Lett. 24, 737 (1970).

[24]J. L. Anderson and R. E. Kates, J. Math. Phys. 22, 1081 (1981).

[25]R. Price and K. S. Thorne, Astrophys. J. 155, 163 (1969).

[26]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (U. S. Government Printing Office, Washington, 1968).

[27]F. I. Cooperstock, Phys. Rev. D 10, 3171 (1974); F. I. Cooperstock and P. H. Lim, *ibid.* 15, 2105 (1977); F. I. Cooperstock and D. W. Hobill, *ibid.* 20, 2995 (1979).