Beyond the wave function at the origin: Some momentum-dependent effects in the nonrelativistic quark model

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Although the u, d, and s quarks of nonrelativistic potential models are moving at high speeds, the approximation $p/m \simeq 0$ is used not only in spectroscopy (where its effects may be hidden), but also in calculating transition amplitudes. We have considered a simple scheme for going beyond the static approximation and found that many discrepancies disappear, including those for g_A/g_V and $\langle 0|A^\mu|A_1\rangle$.

I. INTRODUCTION

The success of the quark model in its various guises (the nonrelativistic quark model, the bag model, the quark-parton model, current algebra,...) is now beyond doubt, but in many cases the relation between these various manifestations remains obscure. In this paper we discuss some momentum-dependent effects in the nonrelativistic quark model with the aim of shedding some light on the relation of nonrelativistic quarks to the relativistic quarks of the bag model and to the quarks of current algebra.

The nonrelativistic quark model, especially when supplemented with ideas from QCD, has proved remarkably successful in describing the spectroscopy, static properties, and decay amplitudes of hadrons. Yet it is almost certainly *not* true that $v/c << 1$ in hadrons made of u, d, and s quarks, so this success requires some rationalization. There are, as well, a few results of the nonrelativistic quark model which stand in curious contradiction to other quark models and experiment. The standard result[1] that annihilation decay amplitudes are proportional to the "wave function at the origin" implies (since the wave function for orbitally excited states vanishes at the origin) that such decays are possible only for $l=0$ mesons. In particular, the prediction $\langle 0|A^\mu|A_1\rangle=0$ (where A^μ is the axial-vector current and A_1 is the lowest-lying axial-vector isovector meson) is in contradiction to current algebra. As another example, the prediction of $\frac{5}{3}$ for the ratio g_A/g_V of the nucleon axial-vector and vector coupling constants is a long-standing problem in the nonrelativistic quark model, the deviation from the experimental value of 1.25 usually being ascribed to relativistic effects.[2] Indeed, we show below that both these discrepancies may be resolved by taking into account the quark momenta.

In the next section we extend the "nonrelativistic" quark model to include some momentum-dependent effects.[2,3] In Sec. III we show that predictions for magnetic moments, magnetic dipole transitions, and S-wave $q\bar{q}$ annihilation processes are not qualitatively changed by such an extension, but that g_A/g_V comes into line with its measured value, and P-wave annihilations become strongly allowed. This last observation has applications to the recently measured decay $\tau\to A_1\nu_\tau$. In the final section we briefly discuss our results and their implications.

II. AN EXTENSION OF THE NONRELATIVISTIC QUARK MODEL

The successful predictions of the nonrelativistic quark model are, roughly speaking, based on an expansion in the p/m of the quarks keeping the first nontrivial term. The most basic such expansion is of course $(m^2+p^2)^{1/2}\simeq m[1+\frac{1}{2}(p/m)^2]$, but similar approximations occur in the calculation of hadronic matrix elements: thus in $\pi\to\mu\bar{\nu}$ one uses

$$\bar{v}(\bar{p}\,\bar{s})\gamma^\mu\gamma_5 u(ps)\simeq\delta^{\mu 0}\chi^\dagger_{-\bar{s}}\chi_s+0(p/m)\ , \qquad (1)$$

where the χ's are Pauli spinors. It is our intention to extend such calculations in a rather trivial way and to observe the effects: *we shall calculate such matrix elements to all orders* in p/m.

To carry out such calculations we require a simple model for the quark wave functions. We have chosen for simplicity to assume that the quark momentum wave functions are Gaussians of the form

$$\phi(\vec{p})=(\beta^2\pi)^{-3/4}e^{-p^2/2\beta^2}, \qquad (2)$$

$$\phi(\vec{p})_{\pm}=\mp(\beta^2\pi)^{-3/4}\left[\frac{p\pm}{\beta}\right]e^{-p^2/2\beta^2}, \qquad (3)$$

for $l=0$ and $l=1$ mesons, respectively, and

$$\phi(\vec{p}_\rho,\vec{p}_\lambda)=(\alpha^2\pi)^{-3/2}\exp[-(p_\rho{}^2+p_\lambda{}^2)/2\alpha^2], \qquad (4)$$

for $l=0$ baryons, where $\vec{p}$, $\vec{p}_\rho$, and $\vec{p}_\lambda$ are the momenta conjugate (nonrelativistically) to $\vec{r}\equiv\vec{r}_q-\vec{r}_{\bar{q}}$, $\vec{\rho}\equiv(1/\sqrt{2})(\vec{r}_1-\vec{r}_2)$, and $\vec{\lambda}\equiv(1/\sqrt{6})(\vec{r}_1+\vec{r}_2-2\vec{r}_3)$; any wave function with a similar mean value of p/m will, however, give similar results. Since our primary aim here is to show the *relation* between our calculations and those of the usual nonrelativistic quark model, our sensitivity to these choices is further reduced.

To carry out our calculations we also require a prescription for dealing with ambiguities that arise in any nonrelativistic quark model calculation: such models are ambiguous when the hadron mass M_H is very different from the sum of the constituent quark masses since $M_H\simeq\sum_i m_i$ in the nonrelativistic limit. For the purposes of this paper we shall adopt the following prescription[4] to deal with this ambiguity (as with our choice of wave functions, *comparisons* between our calculations and the usual lowest-order calculations will be rather insensitive to this prescription):

The mock-hadron prescription. To calculate a hadronic matrix element, express the physical matrix element $\mathscr{M}$ in terms of Lorentz covariants with Lorentz-scalar coefficients A. Then define a mock hadron $\widetilde{H}$ to be a collection of free quarks with the wave function of the bound quarks in a physical hadron H, and a mock mass $\widetilde{M}_H$ equal to the mean total energy of the quarks in $\widetilde{H}$. In many simple cases the mock-hadronic matrix element $\widetilde{\mathscr{M}}$ will be of the same form as $\mathscr{M}$; in these cases, take $A=\widetilde{A}$.

This prescription lets us calculate hadronic amplitudes as integrals over free-quark amplitudes. To make this basic idea concrete, we consider two examples: $\omega\to\pi\gamma$ and $\pi\to\mu\bar{\nu}$.

The hadronic matrix element operative in $\omega\to\pi\gamma$ may be written in general as

$$\langle\pi(k')\,|\,j^\mu_{\rm em}(0)\,|\,\omega(ek)\rangle\equiv(2\pi)^{-3}\mu_{\pi\omega}\epsilon^{\mu\nu\rho\sigma}e_\nu(k'-k)_\rho(k'+k)_\sigma, \qquad (5)$$

so that only the Lorentz-scalar coefficient $\mu_{\pi\omega}$ need be calculated (this corresponds to the quantity A defined above). We do this via mock mesons by first defining mock ω and π mesons in the limit $\widetilde{k}_i$, $\widetilde{k}'_i\to 0$ $(i=1,2,3)$ by

$$|\,\widetilde{\omega}(e\widetilde{k})\rangle=(2\widetilde{M}_\omega)^{1/2}\int\frac{d^3p}{(E\bar{E}/m^2)^{1/2}}\phi_\omega(\vec{p})\chi^e_{s\bar{s}}\Phi^\omega_{q\bar{q}}\,\Big|\,q\left[\frac{\widetilde{k}_i}{2}+p_i,s\right]\bar{q}\left[\frac{\widetilde{k}_i}{2}-p_i,\bar{s}\right]\Big\rangle, \qquad (6a)$$

$$|\,\widetilde{\pi}(\widetilde{k}')\rangle=(2\widetilde{M}_\pi)^{1/2}\int\frac{d^3p'}{(E'\bar{E}'/m^2)^{1/2}}\phi_\pi(\vec{p}\,')\chi^0_{s'\bar{s}'}\Phi^\pi_{q'\bar{q}'}\,\Big|\,q'\left[\frac{\widetilde{k}'_i}{2}+p'_i,s'\right]\bar{q}\left[\frac{\widetilde{k}'_i}{2}-p'_i,\bar{s}'\right]\Big\rangle, \qquad (6b)$$

where ϕ, χ, and Φ are momentum, spin, and flavor wave functions. We then calculate $\langle\widetilde{\pi}(\widetilde{k}')\,|\,j^\mu_{\rm em}(0)\,|\,\widetilde{\omega}(e\widetilde{k})\rangle$ with these superpositions of free quarks; this involves calculating the matrix elements $\langle q'(\widetilde{k}'_i/2+p'_i,s')\,|\,j^\mu_{\rm em}\,|\,q(\widetilde{k}_i/2+p_i,s)\rangle$ and $\langle\bar{q}'(\widetilde{k}'_i/2-p'_i,\bar{s}')\,|\,j^\mu_{\rm em}\,|\,\bar{q}(\widetilde{k}_i/2-p_i,\bar{s})\rangle$, which we can do exactly using full Dirac spinors. Since $\widetilde{\omega}$ and $\widetilde{\pi}$ have the same quantum numbers as ω and π, we then write

$$\langle\widetilde{\pi}(\widetilde{k}')\,|\,j^\mu_{\rm em}(0)\,|\,\widetilde{\omega}(e\widetilde{k})\rangle=(2\pi)^{-3}\widetilde{\mu}_{\pi\omega}\epsilon^{\mu\nu\rho\sigma}e_\nu(\widetilde{k}'-\widetilde{k})_\rho(\widetilde{k}'+\widetilde{k})_\sigma \qquad (7)$$

and find $\widetilde{\mu}_{\pi\omega}$ as the explicit integral over mock meson wave functions given in (12) below. The mock hadron procedure is consummated by taking $\mu_{\pi\omega}=\widetilde{\mu}_{\pi\omega}$. To emphasize that, within the prescription, these results are independent of any arbitrary mass factors in our definitions of the amplitudes, consider next the decay $\pi\to\mu\bar{\nu}$. Here we must examine the matrix element

$$\langle 0\,|\,A^\mu_{1+i2}(0)\,|\,\pi(k)\rangle\equiv(2\pi)^{-3/2}if_\pi M_\pi k^\mu, \qquad (8)$$

TABLE I. The parameters of the standard and extended models (all in GeV).

Parameter	Significance	Standard value	Extended value
$m_d = m_u = m$	mass of u and d	0.34	0.22
m_s	mass of s	0.51	0.43
β	meson Gaussian parameter	0.22	0.30
α	baryon Gaussian parameter	0.25	0.25

where we have chosen to make f_π dimensionless. By a similar procedure to that outlined above, the prescription will equate the Lorentz-scalar coefficient $f_\pi\, M_\pi$ with its mock hadron analog $\tilde{f}_\pi \tilde{M}_\pi$ [the result is given in (14) below]. Clearly the physics is independent of our definition of f_π.

Of course the standard nonrelativistic quark model results can also be calculated in this way by keeping only the lowest-order terms in p/m in the calculation of the free-quark matrix elements.

III. BEYOND THE WAVE FUNCTION AT THE ORIGIN

We shall take the standard nonrelativistic quark model and our extension of it to be defined by the preceding description and the parameters of Table I. We could make our model much more elaborate [by considering SU(6) breaking in α and β (from hyperfine mixing, quark mass differences, etc.), recoil effects, and so on], but since our main goal is the qualitative one of comparing these two models, the resulting complexity would probably be counterproductive.

With the "extended" parameters we find that

$$\left\langle \frac{p^2/2m}{m} \right\rangle_p \simeq 0.6 \quad \text{with } (r_{E,p}{}^2)^{1/2} \simeq 0.9 \text{ fm} \tag{9}$$

and

$$\left\langle \frac{p^2/2m}{m} \right\rangle_\pi \simeq 1.3 \quad \text{with } (r_{E,\pi}{}^2)^{1/2} \simeq 0.6 \text{ fm} \tag{10}$$

(the charge radii quoted here are from the Appendix) so that these systems are certainly relativistic, but perhaps not hopelessly so: the discrepancy $\langle (m^2+p^2)^{1/2} - m - p^2/2m \rangle$ even in mesons is only about 25%. The extended model therefore seems to be a good testing ground for the effects of large p/m in a setting where relativistic effects have not—as in the bag—become completely dominant.

The results of our calculations are displayed in Table II. These results follow from the techniques described in Sec. II; for purposes of illustration we list here a representative formula for each type of calculation:

$$\mu_p = \frac{e}{2m} \int\int d^3p_\rho d^3p_\lambda \, |\phi(p_\rho, p_\lambda)|^2 \left[\frac{m^2 + 2mE_3}{3E_3{}^2} \right], \tag{11}$$

where $E_3 \equiv (\frac{2}{3} p_\lambda{}^2 + m^2)^{1/2}$;

$$\mu_{\pi\omega} = \frac{e}{2m} \cos(\theta_V - \theta_{\text{ideal}}) \int d^3p \, \phi_\pi^*(p) \phi_\omega(p) \left[\frac{m^2 + 2mE}{3E^2} \right], \tag{12}$$

where $E \equiv (p^2 + m^2)^{1/2}$;

$$\Gamma(\pi \to \gamma\gamma) = \frac{2\pi\alpha^2}{3m^2} \left[\frac{M_\pi}{\tilde{M}_\pi} \right]^3 \left| (2\pi)^{-3/2} \int d^3p \, \phi_\pi(p) \frac{m^2}{Ep} \ln \left[\frac{E+p}{m} \right] \right|^2, \tag{13}$$

$$f_\pi = \frac{2\sqrt{3}(2\pi)^{-3/2}}{M_\pi \tilde{M}_\pi{}^{1/2}} \int d^3p \, \phi_\pi(p) \left[\frac{m}{E} \right], \tag{14}$$

TABLE II. Comparison of the standard and extended models.

Amplitude/process	Standard	Extended	Experiment (Ref. 12 unless otherwise noted)
baryon magnetic moments	(units of μ_n)		
μ_p	2.77	2.77	2.79
μ_n	−1.85	−1.85	−1.91
μ_Λ	−0.62	−0.62	−0.61±0.01
μ_{Σ^+}	2.67	2.67	2.33±0.13
μ_{Σ^-}	−1.02	−1.02	−1.41±0.25
μ_{Ξ^0}	−1.44	−1.44	−1.25±0.02[c,d]
μ_{Ξ^-}	−0.51	−0.51	−0.75±0.06[c,d]
$\mu_{\Delta^{++}}$	5.53	5.53	5.7±1.0
meson transition moments[a]	(units of μ_N)		
$\mu_{\pi\rho}$	0.92	0.72	0.67±0.04[e]
$\mu_{\pi\omega}$	2.75	2.14	2.29±0.13[f]
$\mu_{\pi\phi}$	0.23	0.18	0.13±0.02
$\mu_{\eta\rho}$	1.95	1.52	1.68±0.21
$\mu_{\eta\omega}$	0.58	0.44	0.37±0.15
$\mu_{\eta\phi}$	0.92	0.80	0.69±0.07
$\mu_{\eta'\rho}$	1.95	1.52	1.50±0.31
$\mu_{\eta'\omega}$	0.72	0.57	0.44±0.12
$\mu_{\eta'\phi}$	0.81	0.71	
$\mu_{K^0K^{*0}}$	1.53	1.25	0.95±0.22
$\mu_{K^+K^{*+}}$	1.23	0.90	0.86±0.11[f]
baryon transition moments[b]	(units of μ_N)		
$\mu_{\Lambda\Sigma}$	1.60	1.60	1.82±0.20
$\mu_{N\Delta}$	3.20	3.20	3.76±0.19
annihilation decays[a]			
$\Gamma_{\pi\to\gamma\gamma}$	14.4 eV	6.0 eV	7.9±0.5 eV
$\Gamma_{\eta\to\gamma\gamma}$	0.45 keV	0.19 keV	0.32±0.05 keV
$\Gamma_{\eta'\to\gamma\gamma}$	5.2 keV	2.8 keV	5.3±2.5 keV
f_π	1.32	0.88	0.95
f_K	0.33	0.29	0.32
f_ρ	0.15	0.21	0.20±0.02
f_ω	0.05	0.07	0.07±0.01
f_ϕ	0.05	0.07	0.07
f_{A_1}	zero	0.15	0.10±0.04[g]
$f_{K^*(892)}$	0.17	0.25	0.28±0.07[h]
g_A/g_V			
$n\to pe\bar{\nu}$	1.67	1.35	1.25±0.01
$\Lambda\to pe\bar{\nu}$	1.00	0.87	0.73±0.03[i]
$\Sigma^-\to ne\bar{\nu}$	0.33	0.29	0.39±0.07

[a]We have used "perfect" η-η' mixing $\eta(\eta')=(1/\sqrt{2})[(1/\sqrt{2})(u\bar{u}+d\bar{d})\mp s\bar{s}]$ in these results (Ref. 10) and the usual nearly "ideal" quadratic mixing angle $\theta_V=40°$ for the ω-ϕ system.
[b]For the $\Delta\to N\gamma$ transition we assumed the pure $M1$ Lorentz-covariant amplitude $\langle N(p's')|j^\mu|\Delta(ps)\rangle=(2\pi)^{-3}\mu_{N\Delta}\epsilon^{\mu\nu\rho\sigma}k_\rho\bar{u}(p's')\gamma_\sigma u_\nu(ps)$ where $u_\nu(ps)$ is the Rarita-Schwinger wave function (Ref. 11) for the Δ, and $k=p-p'$.
[c]Reference 13. [d]Reference 14. [e]Reference 15. [f]Reference 16.
[g]Reference 17. [h]Reference 18. [i] Reference 19.

$$f_\rho = \frac{\sqrt{6}\tilde{M}_\rho^{1/2}(2\pi)^{-3/2}}{M_\rho^2} \int d^3p\, \phi_\rho(p) \left[\frac{m+2E}{3E} \right] , \tag{15}$$

where we have defined $\langle 0 | j^\mu_{\rm em} | \rho(e,k) \rangle \equiv (2\pi)^{-3/2} e^\mu f_\rho M_\rho^2$;

$$f_{A_1} = \frac{\sqrt{3}\tilde{M}_{A_1}^{1/2}(2\pi)^{-3/2}}{M_{A_1}^2} \int d^3p \left[\frac{p_- \phi_{A_1}(p)_+ - p_+ \phi_{A_1}(p)_-}{E} \right] , \tag{16}$$

where we have similarly defined $\langle 0 | A^\mu_{1+i2} | A_1(e,k) \rangle \equiv (2\pi)^{-3/2} e^\mu f_{A_1} M_{A_1}^2$; and finally,

$$\left[\frac{g_A}{g_V} \right]_{n\to p} = \frac{5}{3} \int\int d^3p_\rho d^3p_\lambda \,|\phi(p_\rho, p_\lambda)| \left[\frac{2m+E_3}{3E_3} \right] . \tag{17}$$

One can easily recover the standard static quark model results from these formulas. For example, (12) and (14) become

$$\mu_{\pi\omega} = \frac{e}{2m} \cos(\theta_V - \theta_{\rm ideal}) \int d^3p\, \phi_\pi^*(p) \phi_\omega(p) \tag{18}$$

and

$$f_\pi = \frac{2\sqrt{3}\,|\psi_\pi(0)|}{M_\pi \tilde{M}_\pi^{1/2}} , \tag{19}$$

respectively. In the last formula, the $\sqrt{3}$ comes from color and $\psi_\pi(0)$ is the spatial wave function of the pion at zero quark-antiquark separation.

IV. DISCUSSION AND CONCLUSIONS

There are many rather obvious qualitative conclusions to be drawn from these results. As already indicated by the similarity between the bag-model and nonrelativistic-quark-model results for the S-wave hadrons, "relativistic" effects do not seriously alter the calculations of S-wave static or transition magnetic moments: the nonrelativistic quark magneton $e/2m$ is, roughly speaking, replaced by $e/2m^*$ where m^* is some average effective mass. Since the S-wave spectroscopy is believed to be dominated by analogous color-magnetic effects, a similar explanation of the successes of the nonrelativistic quark model in this area emerges. Similar conclusions follow from studying processes that involve S-wave quark-antiquark annihilation, and we conclude that relativistic effects of the type we are considering here may for the most part be absorbed by a change of parameters.[5]

One outstanding exception to this observation occurs in the annihilation of non-S-wave quarks as in the $A_1^\pm \leftrightarrow W^\pm$ vertex operative in $\tau \to A_1 \nu_\tau$. There the standard result that this vertex is zero [since $\psi_{A_1}(0)=0$] is completely changed. This effect may be rationalized by recalling the necessity of interpreting the position operator $\vec{r}$ in relativistic wave mechanics as the *average* position operator; i.e., momentum dependence leads to a nonlocal interpretation of the wave function. So we may say that the quarks can "touch" even though $\psi(0)=0$ since they have amplitudes to be found at other than their average positions.[6] (The decay $^3D_1 \to e^+e^-$ of charmonium may result from this effect as well.[3]) The reduction[2] that the relativistic corrections presented here produce in the naive result for g_A/g_V, while not so striking an exception as f_{A_1}, is at least as welcome.[7] We also note that there *are* some small effects in other areas: e.g., the $V \to P\gamma$ amplitudes no longer require suppression by overlap integrals.[4,8]

We would like to close with some comments on the relation of the nonrelativistic quark model (with our extension) and current algebra. In the current-algebra picture with spontaneously broken chiral symmetry, the pion's low mass arises from its nature as a Goldstone boson, while in the quark model the ρ and π are a $^3S_1$ and $^1S_0$ ground-state pair and the low pion mass arises naturally from strong color-magnetic forces. As different as these two views seem, they may be reconciled if we picture spontaneous breakdown of chiral symmetry as occurring via the mechanism of creating (1) confined quarks on a distance scale $\Lambda^{-1}\sim 1$ fm (where $\alpha_s \simeq 1$), (2) a quark self-mass of order $\Lambda \sim 200$ MeV, and (3) various residual interactions of strength $\alpha_s/\Lambda^{-1}\sim 200$ MeV. Chiral symmetry tells us that these effects must conspire to make $m_\pi^2 \simeq 0$ and while this conspiracy looks "miraculous" when viewed in the constituent-quark model, it suggests that there may exist a massive-constituent model with confinement and residual interactions which is physically equivalent to the chiral-symmetry picture. There are already a number of such equivalences demonstrated, including $m_\pi^2 \simeq 0$, $\Gamma(\pi^0 \to 2\gamma)$, and the K_{l3} form factors.[4] We can now add the ratio g_A/g_V (Ref. 9) and the matrix element $\langle 0 | A^\mu | A_1 \rangle$ to this list. On the other hand the chiral-symmetry predictions for the $\pi\pi$ and πN scattering lengths have not yet been discussed in the constituent-quark model; clearly, it would be interesting to do so.

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APPENDIX: CHARGE RADII IN THE QUARK MODEL

In calculating the charge radii of Eqs. (10) and (11), we do not simply associate the r_E^2's with the expectation values of $\sum e_i r_i^2$ in the Fourier transforms of the wave functions (2) and (4). Rather, as demanded by the approach of this paper, we calculate

$$G_{E,p}(\vec{q}^{\,2})=(2\pi)^3\left\langle p\left[\frac{\vec{q}}{2},+\right]\,|\rho(0)|\,p\left[-\frac{\vec{q}}{2},+\right]\right\rangle, \tag{A1}$$

$$F_\pi(\vec{q}^{\,2})=(2\pi)^3\left\langle \pi\left[\frac{\vec{q}}{2}\right]\,|\rho(0)|\,\pi\left[-\frac{\vec{q}}{2}\right]\right\rangle, \tag{A2}$$

for small $\vec{q}$ and then take

$$r_{E,p}{}^2=-6\left.\frac{dG_{E,p}}{d\vec{q}^{\,2}}\right|_{\vec{q}^{\,2}=0}, \tag{A3}$$

$$r_{E,\pi}{}^2=-6\left.\frac{dF_\pi}{d\vec{q}^{\,2}}\right|_{\vec{q}^{\,2}=0}. \tag{A4}$$

We find in this way that

$$r_{E,p}{}^2=\langle \sum_i e_i r_i^2\rangle_p+\tfrac{3}{4}\int d^3p_\rho\, d^3p_\lambda\,|\phi(p_\rho p_\lambda)|^2\left[\frac{4E_3{}^3+E_3M^2+M^3}{3E_3{}^4(E_3+M)}\right]-\frac{3}{4\tilde{M}_p{}^2}, \tag{A5}$$

$$r_{E,\pi}{}^2=\langle \sum_i e_i r_i^2\rangle_\pi+\tfrac{3}{4}\int d^3p\,|\phi(p)|^2\left[\frac{4E^3+EM^2+M^3}{3E^4(E+M)}\right]. \tag{A6}$$

In each case the first term is the ordinary nonrelativistic result and the second term is an additional smearing over distances of order m^{-1} due to the relativistic position averaging discussed in Sec. IV. Note that such a term is present even in the nonrelativistic limit $p \to 0$, but in that limit m^{-2} is much less than $\langle \sum e_i r_i^2\rangle$. In the case of the proton there is a third term which we identify as an artifact of our extending the mock-hadron prescription to order q^2 for the first time. In this order a variety of new effects appear, including a dependence of the momentum-space wave functions on the total hadron momentum. Fortunately, this probably spurious term is very small (~ 0.03 fm^2) and we believe that these results are at least qualitatively correct. In fact, we think it probable that the smearing terms in (A5) and (A6) account for the long-standing discrepancy between quark model wave functions for the pion and the proton and their measured charge radii.

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