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High-energy electrons from bound-muon decay

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Expressions for the electron spectrum from bound-muon decay which exist in the literature are very complex. The aim of this paper is to gain some insight into these formulas. This becomes possible when the Dirac equation is used to derive Wronskian-type relations between electron and muon wave functions. We also show how a trivial modification of the formulas accounts for the nuclear-recoil effect to a very good approximation. We present numerical results to aid order-of-magnitude estimates of the bound-muon-decay contribution to the background in μe conversion experiments. We find that the electron spectrum seems insensitive to deviations of the weak-interaction parameters from standard values.

The aim of this paper is to study how the high-energy end of the electron spectrum from bound-muon decay depends on various weak-interaction parameters. This is of interest because bound-muon decay is an important source of background in experiments setting limits on neutrinoless μ-to-e conversion rates in nuclei. The other important contribution to the background for μe conversion experiments is from radiative pion and muon capture followed by internal or external conversion of the photon. The latter process has been discussed by Primakoff,[1] Hänni,[2] and others. The electron spectrum from bound-muon decay has been discussed by Hänggi *et al.*[3] and by Herzog and Alder.[4] They present formulas which are very useful to calculate the electron spectrum for any value of the electron energy. To gain some insight into the rather formidable formulas, I expand the spectrum about its end point in powers of δ, where

$$\delta=E_{\max}-E_e \ . \tag{1}$$

(Note that only the high-energy end of the electron spectrum is of interest for the μe conversion experiment.) I show how the results of Refs. 3 and 4 can be understood very simply for electron energies close to the end point. I also discuss the validity of their procedure for approximating the nuclear-recoil energy.

In the calculation of the high-energy end of the spectrum the effect of the nuclear-recoil energy on the neutrino phase space plays an important role. For a given electron energy the nuclear-recoil energy is not constant. It also depends on the neutrino momenta, and this complicates the integration over these (unobserved) momenta. This difficulty does not arise if the nuclear-recoil energy is approximated to be constant, i.e.,

$$E_{\rm rec}\approx E_e{}^2/2M_A \ , \tag{2}$$

where E_e is the electron energy and M_A is the mass of the recoiling nucleus. The uncertainty thus introduced into the recoil energy is of order δ^2/M_A. This should not affect the spectrum significantly since the recoil energy itself is very small. With this approximation for the nuclear-recoil energy the leading term for the electron spectrum takes the form

$$N(E_e)dE_e=CE_e{}^2(\delta_1/m_\mu)^5dE_e \ ,$$

where

$$\delta_1=E_\mu-E_e-E_e{}^2/2M_A \ . \tag{3}$$

The expression above comes from neglecting the variation of the weak-interaction matrix element with energy. The integration over the neutrino momenta then gives the factor of $\delta_1{}^5$. Because of the reduction in the phase space available to the neutrinos (due to nuclear recoil), δ_1 is a more appropiate expansion parameter than the δ of Eq. (1).

Hänggi *et al.* and Herzog and Alder have calculated the spectrum taking the nuclear-recoil energy exactly into account but assuming a point-charge Schrödinger wave function for the muon and neglecting the effect of the electrostatic potential on the electron. In what follows, this approximation will be called the Born approximation. Reference 3 gives the ratio of the spectrum with recoil to the spectrum without recoil in the above limit, which they call the recoil factor $A(E_e)$. The expression for $A(E_e)$ is very complicated.[4] However, the nuclear-recoil effect is important only for high electron energies, and Eqs. (2) and (3) should take the recoil effect into account quite well. To test this, we note that these equations predict $A(E_e)$ to be

$$A(E_e)\approx\frac{[m_\mu-B_s-E_e^2/(2M_A)-E_e]^5}{(m_\mu-B_s-E_e)^5}\ , \tag{4}$$

where $B_s=Z^2\alpha^2m_\mu/2$ and m_μ is the muon mass. Table I compares the calculations of Ref. 3 with the prediction of Eq. (4) for a few elements and a few electron energies. It is seen that the agreement is very good. To summarize, the formula of Ref. 3 for $A(E_e)$ does not make an expansion around the end point, and hence includes all terms in δ. Also, the dependence of the nuclear-recoil energy on the neutrino momenta is not neglected. Equation (4), on the other hand, is based on the approximations implicit in Eqs. (2) and (3). The good agreement in Table I seems to indicate that Eq. (2) is a good approximation for the nuclear recoil, and the other effects included in calculations of Hänggi *et al.* are not important.[5,6]

In the calculation of the electron spectrum using exact wave functions, Refs. 3 and 4 neglect the nuclear-recoil energy. However, the approximation of Eq. (2),[7] which reproduced Hänggi's recoil factor so well, can be included in their formulas by merely changing the limits of the neutrino integrations. With this modification, the constant C appearing in Eq. (3) can be evaluated with the help of the expressions in the paper by Herzog and Alder. Such a procedure is not exactly the same as expanding their expressions in a Taylor series about the end point. The difference is that Eq. (3) already contains the nuclear-recoil-energy effect and should not be multiplied by recoil correction factors. The weak-interaction Lagrangian that I use is

$$\mathscr{L}=\sum\frac{g_i}{\sqrt{2}}\bar{e}\Gamma_i\mu\bar{\nu}_\mu\Gamma^i\nu_e\ , \tag{5}$$

where $\Gamma_i=\gamma_\lambda(1-\gamma_5)$, 1, γ_5, and $\sigma_{\lambda\mu}$ for $i=(V-A)$, S, P, and T, respectively. In terms of the above coupling constants the expression for C is

$$C=\sum\frac{g_i^2m_\mu^5}{3\pi(2\pi)^3}B_5^i\ , \tag{6}$$

where B_5^i are expressions involving the lepton wave functions (interference terms between the different currents are absent as long as the neutrinos are not detected). Unlike in Ref. 3, I do not divide the spectrum by the free-muon decay rate. Table II gives the expressions for B_5^i in terms of the following:

$$\begin{aligned} p_k&=\int g_k^e g^\mu dr\ ,\\ q_k&=\int f_k^e g^\mu dr\ ,\\ r_k&=\int g_k^e f^\mu dr\ ,\\ s_k&=\int f_k^e f^\mu dr\ . \end{aligned} \tag{7}$$

g^μ and f^μ are the top and bottom components of the muon wave function and g_k^e and f_k^e are the electron wave functions with energy E_e, total angular momentum $(k-\frac{1}{2})$, and parity $(-1)^{k-1}$. The conventions regarding the Dirac equation and normalization can be found in the Appendix. As mentioned before, the electron spectrum has been evaluated in a closed form in the Born limit, i.e.,

TABLE I. Predictions of Eq. (4) for Hänggi's recoil factor $A(E_e)$.

Z	$\frac{E_e}{m_e}$	A from Hänggi's thesis	A from Eq. (4)
	180	0.904	0.900
8	190	0.828	0.826
	200	0.563	0.571
	170	0.964	0.961
14	185	0.924	0.922
	200	0.698	0.702
	180	0.970	0.969
29	190	0.939	0.938
	200	0.653	0.659

TABLE II. Expressions for B_5^i defined in Eq. (6). p, q, r, and s are defined in Eq. (7).

i	B_5^i	Column 2 in Born limit	Leading term of Born expression from Ref. 4
$(V-A)$	$\frac{16}{5}\left[p_1^2+\frac{s_1^2}{3}+\frac{2}{3}r_2^2\right]$	$\frac{2^8(Z\alpha m_\mu)^5}{5[E_e^2+(Z\alpha m_\mu)^2]^4}$	$\frac{2^8(Z\alpha m_\mu)^5}{5[E_e^2+(Z\alpha m_\mu)^2]^4}$
S,P	$\frac{1}{5}(p_1-s_1)^2$	$\frac{2^4(Z\alpha m_\mu)^5}{5[E_e^2+(Z\alpha m_\mu)^2]^4}$	$\frac{2^4(Z\alpha m_\mu)^5}{5[E_e^2+(Z\alpha m_\mu)^2]^4}$
T	$\frac{8}{15}[8r_2^2+(3p_1+s_1)^2]$	$\frac{3\times 2^7(Z\alpha m_\mu)^5}{5[E_e^2+(Z\alpha m_\mu)^2]^4}$	$\frac{3\times 2^7(Z\alpha m_\mu)^5}{5[E_e^2+(Z\alpha m_\mu)^2]^4}$

when $g^\mu=2r(Z\alpha m_\mu)^{3/2}\exp(-Z\alpha m_\mu r)$, $f_\mu=0$, $g_k^e=rj_{(k-1)}(E_e r)$, and $f_k^e=rj_k(E_e r)$. One can check the correctness of Eq. (6) by seeing if the entries of Table II reduce in the appropriate limit to the leading term in a Taylor series expansion of the Born expression. This comparison is made in Table II, and it can be seen that the two expressions do agree in the appropriate limit.

One might wonder how close the Born expression is to the actual spectrum for high-energy electrons. The Born approximation predicts the right order of magnitude for p_1, as seems reasonable. But it cannot be used for quantitative estimates, as I now describe. The following expression for the muon lifetime will prove useful in the discussion:

$$\Gamma_\mu=(g_S^2+g_p^2+16g_{VA}^2+24g_T^2)\frac{m_\mu^5}{3\times 2^7(2\pi)^3}\ . \quad (8)$$

Herzog and Alder fix the muon lifetime and consider the electron spectrum for four cases, namely, pure S, P, $V-A$, and T currents. (When one neglects the electron mass, as we do, there is no difference between the S and P cases.) Now, from Table II and Eqs. (6) and (8) we see that the Born approximation predicts that the high-energy end of the electron spectrum is independent of the case considered. However, the calculations of Herzog and Alder with correct wave functions show that at high energies the tensor spectrum is $\frac{4}{3}$ times the $V-A$ spectrum and the scalar and pseudoscalar spectra are very small, about one-thousandth the $V-A$ spectrum. Why does the Born approximation fail so badly in predicting these ratios, and why do the calculated spectra have the simple relations mentioned above? The Born approximation fails because it neglects the bottom component of the muon wave function, i.e., it assumes that $s_1=r_2=0$. However, because the electron and the muon are in the same potential and have the same energy, the following interesting relation can be proved using the Dirac equation (the proof is given in the Appendix):

$$p_1=s_1\ . \quad (9)$$

This simple relation predicts that for high energies the tensor spectrum must be $\frac{4}{3}$ the $V-A$ spectrum and the scalar spectrum must be small (the leading term for the latter case vanishes). This can be seen from an inspection of Table II and Eqs. (6) and (8). Thus, we see that Eqs. (3) and (6) give us insight into the high-energy end of the electron spectrum. Equation (9) is valid when the muon and electron have the same energy. Since the maximum electron energy $E_{\max}$ differs from E_μ by an amount $E_\mu^2/(2M_A)$, there are small corrections of the order of the nuclear-recoil energy to the above ratios. Neglecting these corrections the leading term for the spectrum becomes

$$N(E_e)dE_e=\frac{32E_e^2 m_\mu^5}{45\pi(2\pi)^3}(r_2^2+2p_1^2)(g_{VA}^2+2g_T^2)\left[\frac{\delta_1}{m_\mu}\right]^5 dE_e\ , \quad (10)$$

where r_2 and p_1 are defined in Eq. (7) and δ_1 is defined in Eq. (3). Since g_2^e and f_1^e both reduce to $rj_1(E_e r)$ in the limit that the electrostatic potential is turned off, and since the electron is energetic, the difference between g_2^e and f_1^e must be small. Thus, we see that $r_2\approx s_1$. Hence, from Eq. (9), $r_2\approx p_1$. For the light elements one can estimate the order of magnitude and the Z dependence of

$(r_2{}^2+2p_1{}^2)\approx 3p_1{}^2$ using the approximate wave functions given below Eq. (7). One finds that

$$p_1{}^2\approx\left[\int_0^\infty j_0(E_e r)2(Z\alpha m_\mu)^{3/2}e^{-Z\alpha m_\mu r}r^2dr\right]^2 \approx\frac{16(Z\alpha m_\mu)^5}{[E_e{}^2+(Z\alpha m_\mu)^2]^4}\ . \tag{11}$$

The coefficient varies as Z^5 for the light elements. The nuclear finite-size effect reduces the coefficient from the above estimate. This effect is substantial for the heavy elements. The numerical values of the coefficient for different elements have been calculated using the correct wave functions and will be presented later.

I will now discuss how the high-energy electron spectrum changes with deviations of the weak-interaction parameters from their standard values. One has very good measurements of the free-muon decay rate and the Michel ρ parameter,

$$\rho=\frac{12(g_{VA}{}^2+2g_T{}^2)}{(g_S{}^2+g_p{}^2+16g_{VA}{}^2+24g_T{}^2)}\ . \tag{12}$$

From Eq. (10) it is clear that the high-energy electron spectrum depends only on these parameters[8] and hence does not change very much for deviations of the weak current from the $(V-A)$ form. Since the neutrinos carry away little momentum the effect of nonzero neutrino mass should be considered. Since the weak current is $(V-A)$ (to a good approximation at least), a nonzero neutrino mass will not change the weak-interaction matrix element. The phase space will get modified. If one of the neutrinos emitted has mass m the effect can by obtained by replacing $\delta_1{}^5$ in Eq. (3) or (10) by $[(\delta_1-m)^5+5m(\delta_1-m)^4]$. In the presence of neutrino mixings the experimental spectrum would be a sum of such terms with the coefficients depending on the mixing angles. Only the limit on the τ neutrino mass is greater than the experimental resolution. Given present limits on the mixing angles the effect of a nonzero τ-neutrino mass is likely to be undetectable.

If we include the next higher term in δ_1 the expression for the spectrum takes the form

$$N(E_e)dE_e=(E_e/m_\mu)^2(\delta_1/m_\mu)^5 \times\left[D+E\left[\frac{\delta_1}{m_\mu}\right]+F\left[\frac{\delta}{m_\mu}\right]\right]dE_e\ , \tag{13}$$

where the term containing F arises because C in Eq. (3) involves integrations over the electron wave function and hence is a function of E_e. D is equal to $m_\mu{}^2C(E_{\max})$. For a $(V-A)$ current

$$E=\frac{32g_{VA}{}^2m_\mu{}^7}{135\pi(2\pi)^3}[(q_{1,1}-r_{1,1})2p_1+r_2(s_{2,1}+p_{2,1})]\ ,$$

$$F=-\frac{64g_{VA}{}^2m_\mu{}^7}{45\pi(2\pi)^3}\left[\frac{3p_1}{2}\frac{dp_1}{dE_e}+\frac{p_1}{2}\frac{ds_1}{dE_e}+r_2\frac{dr_2}{dE_e}\right]\ , \tag{14}$$

where $p_{k,m}$, $q_{k,m}$, etc., are similar to p, q, etc., in Eq. (7), except that the integrand involves an extra factor r^m. The Born expression can be used to check Eq. (14), as we did for (6) (see Table II). In the appropriate limit Eq. (14) also reduces to the form predicted by the Born expression. Table III lists the values of D and E and estimates of F for a few elements. These were calculated using a Fermi charge distribution[9] with $c=1.07A^{1/3}$ fm (where A is atomic weight) and $a=0.55$ fm. (For $Z=29$, however, the same c and a were used as in Hänggi's thesis, namely, $c=4.26$ fm and $a=0.578$ fm.) We note that E and F are not small compared to D and hence higher terms are important for precise numerical calculations. Figure 1 shows a comparison of the predictions of Eq. (13) with Hänggi's exact calculations for $Z=29$. For $E_e=200\ m_e$ [i.e., $(\delta/m_\mu)=0.013$] Eq. (13) reproduces Hänggi's result to within 10%. The contributions of the E and F terms are 4% and 8% of the total value, respectively. For $E_e=190m_e$ $[(\delta/m_\mu)=0.06]$ the value of Eq. (13) is about 60% of the value in Ref. 3. Our conclusion, that the spectrum depends mainly on the Michel ρ parameter, should not change in spite of the large contribution of higher terms. Certainly, our success in understanding the qualitative features of the results of Refs. 3 and 4 has been encouraging.

In conclusion, we have seen how a trivial modification of the formulas of Ref. 4 can account for the nuclear-recoil effect to a very good approximation. We have explained the qualitative features of the results of Refs. 3 and 4, using Wronskian-type relations which follow from the Dirac equation. We have presented numerical results which should aid in making a quick rough estimate, for different elements, of the bound-muon decay contribution to the background in μe conversion experiments. Our study indicates that the high-energy electron spec-

TABLE III. Numerical values of D, E, and F [defined in Eq. (13)] for a range of elements.

Z	$10^{21}D$	$10^{21}E$	$10^{21}F$	Spectrum end point E_{max} (MeV)
12	0.28	0.7	2	105.0
16	0.71	2.0	4	104.8
18	1.02	3.0	6	104.7
23	2.35	7.2	16	104.2
25	3.10	9.7	21	104.0
29	4.74	15.4	35	103.5
34	7.01	23.8	54	102.8
45	13.8	51.0	119	101.2
55	18.3	73.5	176	99.6
65	22.8	98.4	241	97.8
82	27.4	132	332	94.9

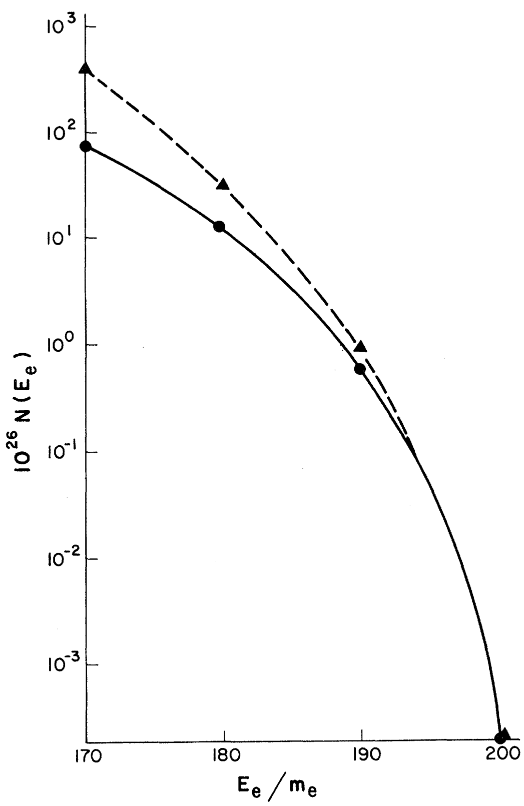

FIG. 1. Comparison of the prediction of Eq. (13) with Hänggi's results for copper. The dotted line is Hänggi's result and the solid line is from Eq. (13). At $E_e = 200m_e$ the two data points come together.

trum is insensitive to deviations of the weak-interaction parameters from their standard values.

The author has gained from many discussions with Lincoln Wolfenstein and from his suggestions. He also appreciates useful discussions with Harold Fearing.

APPENDIX

In this appendix the Dirac equation is written down to specify the wave-function conventions and normalizations. Certain relations are derived which help us in understanding the ratios of the electron spectra for different types of weak-interaction currents. The Dirac equation for a particle of mass m, energy E, total angular momentum $j = |\lambda| - \frac{1}{2}$, and parity $(-1)(\lambda - 1/2)\text{sign}(\lambda) - 1/2$ is given by

$$g' = \frac{\lambda}{r} g - [E - V(r) + m] f \, , \qquad f' = [E - V(r) - m] g - \frac{\lambda}{r} f \, . \tag{A1}$$

For the electron the mass m is taken to be zero, and the wave functions are normalized such that g is positive and both are regular at the origin, and g varies as sin $[Er + Z\alpha \ln(r) + \Phi]$ for $r \to \infty$. With this normalization g and f would reduce as $Z\alpha \to 0$ to $rj_{(\lambda-1)}(Er)$ and $rj_{(\lambda)}(Er)$, respectively, for positive λ. For negative λ they would reduce to $rj_{(-\lambda)}(Er)$ and $-rj_{(-\lambda-1)}(Er)$, respectively. The muon spends most of its time in the ground state, and hence λ takes the value 1 and E becomes the muon mass minus the ground-state binding energy. In our convention both g and f are positive for the muon wave function.

The Dirac equation can be used to find relations between the different terms occurring in the electron spectrum. For example, consider the derivative of the following determinant:

$$\frac{d}{dr}\begin{vmatrix} g^{\mu} & f^{\mu} \\ g^{e}_{\lambda} & f^{e}_{\lambda} \end{vmatrix} . \tag{A2}$$

Using Eq. (A1) the above determinant can be written as

$$\frac{\lambda-1}{r}\begin{vmatrix} g^{\mu} & f^{\mu} \\ g^{e}_{\lambda} & -f^{e}_{\lambda} \end{vmatrix} + m\begin{vmatrix} g^{\mu} & f^{\mu} \\ f^{e}_{\lambda} & g^{e}_{\lambda} \end{vmatrix} - (E_e - E_\mu)\begin{vmatrix} g^{\mu} & f^{\mu} \\ f^{e}_{\lambda} & -g^{e}_{\lambda} \end{vmatrix} . \tag{A3}$$

For $\lambda = 1$ and $E_\mu = E_e$ one can integrate both sides and find that

$$p_1 - s_1 = 0 , \tag{A4}$$

which explains the ratios of the electron spectra for different weak currents as calculated by Herzog and Alder. Another relation useful for the δ^6 term of the electron spectrum can be derived by putting $\lambda = 2$. Multiplying by r and integrating gives

$$m(p_{2,1} - s_{2,1}) = 2r_2 . \tag{A5}$$

Another relation useful for the electron spectrum expression can be derived by considering the derivative of the determinant:

$$\frac{d}{dr}\begin{vmatrix} g^{\mu} & f^{\mu} \\ -f^{e}_{\lambda} & g^{e}_{\lambda} \end{vmatrix} = \frac{\lambda+1}{r}\begin{vmatrix} g^{\mu} & f^{\mu} \\ f^{e}_{\lambda} & g^{e}_{\lambda} \end{vmatrix} - m\begin{vmatrix} g^{\mu} & f^{\mu} \\ -g^{e}_{\lambda} & f^{e}_{\lambda} \end{vmatrix} + (E_\mu - E_e)\begin{vmatrix} g^{\mu} & f^{\mu} \\ g^{e}_{\lambda} & f^{e}_{\lambda} \end{vmatrix} . \tag{A6}$$

For $\lambda = 1$ and $E_\mu = E_e$, multiplying by r, integrating, and using Eq. (A4) gives

$$m(q_{1,1} + r_{1,1}) = p_1 + s_1 = 2p_1 . \tag{A7}$$

I conclude this appendix with a remark on the numerical integration formula used in the calculations, since others may find it useful in their numerical work. If the interval of integration is divided into n equal parts of length δ each, then

$$\int_{x_0}^{x_n} f(x)\,dx \approx \delta\left[\frac{f_0}{2} + f_1 + f_2 + \cdots f_{n-1} + \frac{f_n}{2}\right] + \frac{f_0' - f_n'}{12}\delta^2 , \tag{A8}$$

where

$$f' = \frac{df}{dx} . \tag{A9}$$

The error is of order $n\delta^5$, just as in the commonly used Simpson's method. The advantage of Eq. (A8) over Simpson's method is that all the intermediate points are weighted equally, making it more efficient for computer use. (In Simpson's method the program has to treat odd and even points differently.) Of course, one has to know the values of the derivative of the function at the end points, but in practice one usually has that information. Equation (A8) was derived by fitting a cubic polynomial to the values of the function and its first derivative at the end points of the basic interval, and integrating the polynomial. The resulting formula was then applied to each interval in turn (just as in Simpson's method), leading to Eq. (A8).

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[1]H. Primakoff, Rev. Mod. Phys. 31, 802 (1959).
[2]H. Hänni, doctoral thesis, Bern, 1979 (unpublished).
[3]P. Hänggi, R. D. Viollier, U. Raff, and K. Alder, Phys. Lett. 51B, 119 (1974); P. Hänggi, master's thesis, Basel, 1973 (unpublished).
[4]F. Herzog, doctoral thesis, Basel (unpublished); F. Herzog and K. Alder, Helv. Phys. Acta 53, 53 (1980).
[5]The recoil-correction factor for different weak currents

is discussed in Ref. 4. Since the nuclear-recoil correction is important only near the spectrum end point, the formulas of that reference can be approximated quite accurately by Eq. (4). For the scalar and pseudoscalar cases of Ref. 4 the use of a recoil-correction factor calculated in the Born approximation is not justified. This is because a cancellation occurs in the weak matrix element evaluated with correct wave functions. This cancellation does not occur in the Born approximation, and hence the latter is a bad approximation. For the scalar and pseudoscalar cases the leading term in the actual spectrum is δ_1^7 while the Born approximation predicts a δ_1^5 variation. The next paragraph describes how the nuclear-recoil effect can be directly incorporated into the spectrum calculation, without the need for using the multiplicative recoil-correction factor.

[6]H. Fearing and G. Brookfield (private communication) have considered other recoil effects not included in Hänggi's formula.

[7]We recollect that Eq. (2) neglects the dependence of the nuclear-recoil energy on the neutrino momenta.

[8]This fact is not obvious from Eq. (6). One has to use the Wronskian-type relations derived in the Appendix to see this fact.

[9]W. F. Hornyak, *Nuclear Structure* (Academic, New York, 1975).