Model-independent dispersion approach to proton Compton scattering

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The proton Compton scattering at low and intermediate energies is studied by means of a dispersion framework which exploits in an optimal way the (fixed-momentum-transfer) analyticity properties of the amplitudes in conjunction with the consequences of the (s-channel) unitarity. The mathematical background of the work consists in methods specific to boundary-value problems for analytic vector-valued functions and interpolation theory. In comparison with previous related work, the extremal problems to be solved now are much more difficult because of the inclusion of the photoproduction input and also lead to additional computational complications. The lower bounds on the differential cross section, obtained without any reference to subtractions and annihilation-channel contributions, appear sufficiently restrictive to provide rigorous evidence of some inconsistencies between results of single-pion-photoproduction multipole extractions and proton Compton-scattering data.

I. INTRODUCTION

The main physical information used in the dispersion treatments of the proton Compton effect is represented by the supposedly known photoproduction data which establish, according to the unitarity condition, the s,u-channel absorptive parts of the Compton-scattering amplitudes. However, the knowledge of the photoproduction matrix elements alone is not sufficient to determine fully the Compton scattering since two (out of the six independent) amplitudes describing the process require subtractions in the fixed momentum-transfer dispersion relations, due to their presumably bad (as predicted by the normal Regge-pole model) high-energy asymptotic behavior. It is at this point that the real difficulties begin to appear and they manifest themselves especially at low energies where the theoretical results depend to a great extent on the way in which the highly model-dependent annihilation-channel contributions to the needed t-channel absorptive parts are treated. Efforts have been made to evaluate the t-channel exchanges[1-3] (particularly in connection with dispersion estimates of the proton electromagnetic polarizabilities) in the approximation of disregarding higher than two-pion states in the (t channel) unitarity sum but the situation will continue to remain rather confused until more reliable information on the $\gamma\gamma\to\pi\pi$ reaction will become available. First experimental results from the study of the reactions $\gamma p\to\gamma p\pi, ee\to ee\pi\pi$ and radiative scattering of pions on nuclear Coulomb fields are soon expected and they may bring important clarifications on the $\gamma\gamma\to\pi\pi$ (or $\gamma\pi\to\gamma\pi$) matrix elements. For the time being, it is perhaps better to look more carefully at what has really been achieved in the present theoretical understanding of the proton Compton effect at low and intermediate energies, by trying to work as much as possible in a model-independent framework in order to isolate the space actually left for model building from that part of the conclusions which remain in fact untouched by assumption regarding high-energy asymptotic behaviors and/or annihilation-channel contributions. We have undertaken such an attempt in the work reported here.

Some recent new results in the dispersion theory of hadron Compton scattering[4-7] which originated in themes previously put forward in Refs. 8 to 10 have made possible the present investigation. The progress realized so far refers mainly to the construction[4] of six new (analytic unitary) amplitudes for proton Compton scattering which share with the invariant amplitudes (free of kinematical singularities and zeros) their simple s-u crossing properties and analyticity structure in ν at fixed t ($\nu=\frac{1}{4}(s-u)$, $s,t,u=$ Mandelstam invariants) while, on the unitarity cut in the complex ν^2 plane, they are related to the usual helicity amplitudes by a unitary matrix. The unpolarized differential cross section on the boundary of the analyticity region can thereby be written as a sum of moduli squared of analytic objects. Since this analytic diagonalization of the cross section was accomplished, the question of finding (optimal) analyticity restrictions on the values of the amplitudes or their partial derivatives with respect to ν^2 at interior points of the analyticity domain is brought[5,6] to the application of more or less known methods and results from the current mathematical literature. Especially, one has to deal with the interpolation theory and with boundary-value problems for vector-valued analytic functions.

In Refs. 6 and 7 optimal bounds on the amplitudes and cross section in the (physical) low-energy region below the threshold ν_0^2 of single-pion photoproduction have been concretely obtained (taking the unpolarized differential cross section as known above ν_0^2) and subsequently numerically tested using experimental data. Those of the bounds which referred only to directly measurable quantities appeared satisfied but rather weak in spite of the complete exploitation of the gauge invariance, s-u crossing, and fixed-t analyticity properties of the amplitudes. This weakness is understandable because of the reduced dynamical input and the generality of the underlying assumptions. The question of including rigorously in the formalism the main dynamical ingredient, i.e., the pion photoproduction data, still remained open although in the last section of Ref. 7 an attempt had been made to illustrate in a simple manner how productive such an enlargement of the starting information might be in strengthening the bounds. In the present paper within a certain particular but precise formulation of this problem we are able to solve it to the very end. The absorptive parts of all the six invariant (or helicity) amplitudes of the process are taken as known (in terms of single-pion-photoproduction multipoles) over the interval $\nu_0^2 \leqslant \nu^2 \leqslant \nu_{in}^2$, where ν_{in}^2 is the threshold of double-pion photoproduction (in practice ν_{in}^2 may be chosen well above this second inelastic threshold since the double-pion-photoproduction contributions remain comparatively very small up to photon laboratory energies $\omega \sim 800$ MeV); above ν_{in}^2 the unpolarized differential cross section for proton Compton scattering is taken as known from experiment. This information (supplemented with the knowledge of the s,u-channel Born-poles residua which are entirely specified by the proton's charge and anomalous magnetic moment) is optimally exploited to get, by means of special techniques, the domain of the allowed values of the amplitudes at any wanted ν^2 point. Particularly, for ν^2 in the physical region from ν_{min}^2 up to ν_{in}^2 (i.e., in the low- and intermediate-energy region including the zone of the N^* resonance) one therefrom obtains, through a subsequent extremization, the optimal domain of allowed values for the polarized or unpolarized differential cross sections. The interval of (physical) t values in which the present considerations hold is sufficiently large to cover all the physical angles and is substantially larger than the domain of the t allowed in Refs. 10, 4, 6, and 7.

The complete treatment of the above physical input leads to a problem of functional extremization, related to a combined finite Pick-Nevanlinna and infinite Schur-Carathéodory interpolation problem for vector-valued analytic functions. This kind of problem is treated by techniques of functional analysis in the specialized literature.[11-13] Such techniques have already been applied in the scalar case (i.e., the case of only one analytic function) in other studies.[14-16] In order to get numbers to be compared with experimental values, complex computational methods and programming devices were employed, some of them being adapted from material already existing.[17]

This paper is organized as follows. In Sec. II the problem to be solved is precisely formulated starting from a certain physical input accepted as known. The solution of the problem is obtained in Sec. III. In Sec. IV an optimal-sum-rule inequality relating the proton anomalous magnetic moment to the single-pion-photoproduction multipoles and the Compton-scattering unpolarized differential cross section is obtained and tested numerically. In Sec. V (analyticity *and* unitarity) bounds on the unpolarized or polarized differential cross sections are found and compared with experiment. The lower bounds improve previous results and give rise to some violations which evidentiate inconsistencies between the results of the pion-photoproduction multipole extractions and proton Compton-scattering data. Section VI is devoted to some final comments.

II. FORMULATION OF THE PROBLEM

We shall work here with the Bardeen-Tung amplitudes $A_i(\nu,t)$ used in Ref. 1 or rather with the (nucleon) pole-free (s-u crossing symmetric) amplitudes

$$\overline{A}_i(\nu^2,t)=K_i 4(\nu^2-\nu_B{}^2)A_i(\nu,t) \quad (i=1,\ldots,6),$$

$$K_1=\frac{1}{m}, \quad K_2=K_4=1,$$

$$K_3=\frac{m}{\nu}, \quad K_5=m, \quad K_6=\frac{m^2}{\nu};$$

$\nu_B{}^2=t^2/16=$ the position of the s-u-channel Born poles. $\overline{A}_i(\nu_B{}^2,t)$ are known functions of t, specified completely in terms of the proton charge and anomalous magnetic moment [see, for instance, Eq. (2.9) of Ref. 4]. $\overline{A}_i(\nu^2,t)$ are real analytic functions $[\overline{A}_i^*(\nu^{2*},t)=\overline{A}_i(\nu^2,t)]$ in the complex ν^2 plane cut along the real axis from ν_0^2 to ∞. To the lowest order in the fine-structure constant the first inelastic threshold ν_0 is given by

$$\nu_0=\tfrac{1}{2}\left(s_0-m^2+\frac{t}{2}\right)\;[s_0=(m+\mu)^2],\quad \nu_0=\tfrac{1}{2}\mu(\mu+2m)+\frac{t}{4}$$

(μ = pion mass, m = proton mass). The physical region of the process in the ν, t variables is

$$t\leq 0,\quad \nu\geq\nu_{\rm min}=\tfrac{1}{2}(-t)^{1/2}\left(m^2-\frac{t}{4}\right)^{1/2}.$$

We recall the kinematical relations

$$\nu=m\omega+\frac{t}{4},\quad \cos\theta_{\rm c.m.}=1+t\frac{1+2\omega/m}{2\omega^2},$$

where ω denotes the photon energy in the laboratory system and $\theta_{\rm c.m.}$ the center-of-mass scattering angle.

The essential dynamical information about the Compton process is provided by unitarity, which gives an exact connection between the absorptive parts $\text{Im}\bar{A}_i$ above the pion-photoproduction threshold $s_0=(m+\mu)^2$ and the amplitudes describing various photoproduction processes. Since in the present work we use this information only at low and intermediate energies we retain only two-particle (i.e., πN) intermediate states in the s-channel unitarity sum and compute the imaginary parts $\text{Im}\bar{A}_i$ in terms of the multipoles for single-pion photoproduction

$$\text{Im}\bar{A}_i(\nu^2,t)=\rho_i(\nu^2,t)\quad(i=1,\ldots 6),\quad \nu_0^{\,2}\leq\nu^2\leq\nu_{\rm in}^{\,2}. \tag{2.1}$$

The concrete form of the functions $\rho_i(\nu^2,t)$ we have worked with is that employed in Ref. 1 [we recall that the pion-photoproduction multipoles were taken from Ref. 18 in the ω (photon laboratory energy) region from 180 to 250 MeV and from Ref. 19 for 250 MeV $\leq\omega\leq$ 1210 MeV]. Rigorously, Eq. (2.1) is exact only up to the threshold of double-pion photoproduction $\nu_{\rm in}^{\,2}=(s_{\rm in}-m^2+t/2)^2/4$, $s_{\rm in}=(m+2\mu)^2$. However, the contribution of the multipion photoproduction is believed to be rather small well beyond this threshold, as suggested for instance by the estimations performed in Ref. 3. Therefore in the actual calculations we shall take for the threshold $\nu_{\rm in}^{\,2}$ various values below that corresponding to ω = 1210 MeV, the limit up to which the multipoles are tabulated in Ref. 19. Above the threshold $\nu_{\rm in}^{\,2}$ the unpolarized differential cross section (UDCS) of the elastic γ-proton scattering will be considered as known. This yields a boundary condition on the amplitudes $\bar{A}_i$ of the general form[4]

$$\sum_{i,j=1}^{6}M_{ij}(\nu^2,t)\bar{A}_i^*(\nu^2,t)\bar{A}_j(\nu^2,t)=\sigma(\nu^2,t),\quad \nu^2>\nu_{\rm in}^{\,2}, \tag{2.2}$$

where the matrix M is Hermitian and positive definite on the cut and σ is related to the UDCS in the center-of-mass system by

$$\sigma(\nu^2,t)=128\pi^2 s\left(\frac{d\sigma}{d\Omega}\right)_{\rm c.m.}. \tag{2.3}$$

The concrete expression of $M_{ij}(\nu^2,t)$ is to be read off, for instance, from Eqs. (2.10)–(2.12) of Ref. 4.

In the present paper we establish and exploit a dispersion formalism which takes into account in an optimal way the physical information contained in the relations (2.1) and (2.2). As a consequence, constraints upon the values of the amplitudes $\bar{A}_i$ or their derivatives with respect to ν^2 at fixed t in an arbitrary number of points inside the analyticity domain ($\nu^2<\nu_0^{\,2}$) or on the cut, in the region $\nu_0^{\,2}<\nu^2<\nu_{\rm in}^{\,2}$, can be derived and expressed in terms of this physical input. Our derivation relies on functional-analysis methods used in the theory of extremal problems for vector-valued analytic functions.[11-13] Below we shall formulate rigorously the extremum problem to be solved in connection with the physical boundary conditions (2.1) and (2.2) and the solution of this problem will be worked out in detail in the next section.

We start by noticing that the present considerations generalize the approach developed in Refs. 4 to 7 to the more difficult case in which consequences of the unitarity condition are being incorporated. The main point of the procedure applied in these references was an analytic factorization of the bilinear form from the left-hand side of Eq. (2.2).

A new set of six "analytic unitary amplitudes" having the same good analyticity structure in ν^2 as the invariant amplitudes $\bar{A}_i$ but connected with the usual helicity amplitudes f_i by a matrix which is unitary on the cut was found. For treating the condition (2.2) as it appears now, i.e., only along a part of the unitarity cut, it is natural to resort to the same technique as in Ref. 4 properly adapted to the new situation. Accordingly, let us perform first the conformal mapping

$$z=\frac{(\nu_{\rm in}^{\,2}-\nu_B^{\,2})^{1/2}-(\nu_{\rm in}^{\,2}-\nu^2)^{1/2}}{(\nu_{\rm in}^{\,2}-\nu_B^{\,2})^{1/2}+(\nu_{\rm in}^{\,2}-\nu^2)^{1/2}}, \tag{2.4}$$

which brings the cut ν^2 plane onto the interior of the unit disk $|z|\leq 1$ cut along the real segment $[x_0, 1]$, where

$$x_0=\frac{(\nu_{\rm in}^{\,2}-\nu_B^{\,2})^{1/2}-(\nu_{\rm in}^{\,2}-\nu_0^{\,2})^{1/2}}{(\nu_{\rm in}^{\,2}-\nu_B^{\,2})^{1/2}+(\nu_{\rm in}^{\,2}-\nu_0^{\,2})^{1/2}}.$$

From (2.4) this segment is seen to be the image of the part $[\nu_0^{\,2},\nu_{\rm in}^{\,2}]$ of the unitarity cut, the upper and lower borders of the remaining part $\nu^2\geq\nu_{\rm in}^{\,2}$ being applied onto the upper and lower semicircles, respectively. For convenience the point ν_B^2 was applied through (2.4) in the origin $z=0$.

We introduce further the outer real analytic function[12]

$$\mathcal{S}(z)=\exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\sigma(\theta)d\theta\right] \quad (2.5)$$

which, by definition, has the modulus equal to $[\sigma(\theta)]^{1/2}$ on the boundary $z=e^{i\theta}$ and has no zeros inside the unit disk $|z|<1$. Particularly $\mathcal{S}(x)$ is seen to be positive for real x.

By the conformal mapping (2.4) the condition (2.2) is imposed along the circle $|z|=1$, exactly as it was in Ref. 4. One can therefore apply step by step the procedure developed there for diagonalizing the bilinear form expressing σ. Specially, we shall consider the 6×6 matrix

$$N(z)=N_{\rm in}(\nu^2,t)=\begin{pmatrix} N_{\rm I} & 0\\ 0 & N_{\rm II}\end{pmatrix},$$

$$N_{\rm I}=\frac{m(-t)^{1/2}}{2\sqrt{2}(L_1)^2}\begin{bmatrix} 0 & 0 & \dfrac{-4L_2[\nu_{\rm in}+(\nu_{\rm in}^2-\nu^2)^{1/2}]}{m(4m^2-t)^{1/2}}\\ (-t)^{1/2}(4m^2-t)^{1/2} & 0 & \dfrac{-4\nu^2(-t)^{1/2}}{m^2(4m^2-t)^{1/2}}\\ 0 & t & -\dfrac{4\nu^2}{m^2}\end{bmatrix}, \quad (2.6)$$

$$N_{\rm II}=\frac{L_2}{4\sqrt{2}(L_1)^2(2\nu_{\rm in})^{1/2}[\nu_{\rm in}+(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}]^{1/2}}$$

$$\times\begin{bmatrix} tL_2 & 0 & \dfrac{2}{m^2}L_2[\nu_{\rm in}+(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}]\times[\nu_{\rm in}+(\nu_{\rm in}^2-\nu^2)^{1/2}]\\ -8mL_2(2\nu_{\rm in})^{1/2}\times[\nu_{\rm in}+(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}]^{1/2} & -\dfrac{L_2}{m}(4m^2-t)^{1/2}(2\nu_{\rm in})^{1/2}\times[\nu_{\rm in}+(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}]^{1/2} & 0\\ \dfrac{-2t}{\nu_{\rm min}}\{\nu_{\rm min}+[\nu_{\rm in}+(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}]\times[\nu_{\rm in}+(\nu_{\rm in}^2-\nu^2)^{1/2}]\} & 0 & \dfrac{-4\nu_{\rm min}}{m^2}\{\nu^2+[\nu_{\rm in}+(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}]\times[\nu_{\rm in}+(\nu_{\rm in}^2-\nu^2)^{1/2}]\}\end{bmatrix}$$

$$L_1=2[(\nu_{\rm in}^2-\nu_B^2)^{1/2}+(\nu_{\rm in}^2-\nu^2)^{1/2}],\quad L_2=2[(\nu_{\rm in}^2-\nu_{\rm min}^2)^{1/2}+(\nu_{\rm in}^2-\nu)^{1/2}],$$

which is obtained from the matrix $N_0(\nu^2,t)$ defined in Eqs. (2.21)–(2.24) of Ref. 6 by simply replacing the threshold ν_0^2 with the new threshold $\nu_{\rm in}^2$ relevant in the present case. By construction the matrix $N(z)$ is real analytic and nonsingular $[\det N(z)\neq 0]$ in $|z|<1$. Therefore the new set of amplitudes $\varphi_i(z)$ defined by the relations

$$\varphi_i(z)=\frac{1}{\mathcal{S}(z)}\sum_{j=1}^{6}N_{ij}(z)\bar{A}_j(z)\quad (i=1,\ldots,6) \quad (2.7)$$

has the same analyticity properties as the amplitudes $\bar{A}_i$, i.e., they are real analytic in the disk $|z|<1$ with a cut along the segment $[x_0,1]$. Since both $\mathcal{S}(z)$ and $N(z)$ are real along this segment, being by construction analytic and of real type in $|z|<1$, the discontinuity of the functions $\varphi_i(z)$ across this cut, obtained from Eqs. (2.1) and (2.7) is

$$\operatorname{Im}\varphi_i(x)=\frac{1}{\mathcal{S}(x)}\sum_{j=1}^{6}N_{ij}(x)\rho_j(x),\quad x\in[x_0,1]\quad (i=1,\ldots,6) \quad (2.8)$$

and the condition (2.2) expressed in terms of these amplitudes takes on the simple form

$$\sum_{i=1}^{6}|\varphi_i(\theta)|^2=1,\quad \theta\in[-\pi,\pi]. \quad (2.9)$$

We shall now take into account explicitly the analyticity properties of the functions $\varphi_i(z)$ by splitting each of them into two terms, one being analytic in $|z|<1$ and the other having a cut along the segment $[x_0,1]$ with a given discontinuity (2.8) across it. Of course this separation is not unique but as we shall see below the result will not depend upon this arbitrariness. For our purposes it is convenient to write $\varphi_i(z)$

$$\varphi_i(z) = g_i(z) + \frac{1}{\pi}\int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-z} dz \quad (i=1,\ldots,6)\,, \tag{2.10}$$

where $\eta > 0$ is arbitrary and $\mathrm{Im}\varphi_i(x)$ for $x>1$ are arbitrary continuous extensions of the functions (2.8) known for $x \leq 1$. Since the last term in (2.10) has inside $|z|<1$ a cut for $x_0 \leq x \leq 1$ with the same discontinuity as φ_i, the functions $g_i(z)$ are real analytic in $|z|<1$, having in particular real values for $x_0 \leq x \leq 1$.

By introducing the relations (2.10) into (2.9), the boundary conditions of the problem, expressed by Eqs. (2.1) and (2.2), can be formulated in terms of the functions $g_i(z)$ in the following compact form:

$$\sum_{i=1}^{6} \left| g_i(\theta) + \frac{1}{\pi}\int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-e^{i\theta}} dx \right|^2 \leq 1\,, \quad \theta \in [-\pi,\pi]\,. \tag{2.11}$$

The sign $\leq$ was introduced above instead of the equality sign in order to preserve, in further considerations, the freedom of using, instead of the cross section given for $\nu^2 > \nu_{\mathrm{in}}^2$, a quantity majorizing it.

Let now ν_k^2, $1 \leq k \leq n$ be n points, which we assume for simplicity to be distinct and situated on the real axis, below the threshold ν_{in}^2 and $\overline{A}_i(\nu_k^2, t)$ the values taken in these points by the amplitudes $\overline{A}_i$, which are real if $\nu_k^2 < \nu_0^2$ and complex on the cut, i.e., for $\nu_k^2 > \nu_0^2$. Denoting by z_k the images of the points ν_k^2 through the conformal mapping (2.4) one has from Eqs. (2.10) and (2.7)

$$\begin{aligned} g_i(z_k) &= \frac{1}{\mathcal{S}(z_k)} \sum_{j=1}^{6} N_{ij}(z_k)\overline{A}_j(z_k) \\ &\quad - \frac{1}{\pi}\int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-z_k} dx\,, \quad z_k < x_0\,, \\ &\quad (i=1,\ldots,6)\,, \\ g_i(z_k) &= \frac{1}{\mathcal{S}(z_k)} \sum_{j=1}^{6} N_{ij}(z_k)\,\mathrm{Re}\overline{A}_j(z_k) \\ &\quad - \frac{\mathrm{P}}{\pi}\int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-z_k} dx\,, \quad x_0 < z_k < 1 \end{aligned} \tag{2.12}$$

the last integral being evaluated as a principal part.

The problem we have to solve is to find the optimal domain of the allowed values $\overline{A}_i(\nu_k^2, t) \equiv \overline{A}_i(z_k)$, consistent with Eqs. (2.1) and (2.2). Since for $\nu_k^2 > \nu_0^2$ the imaginary parts are known from (2.1), we shall derive in this case constraints upon the unknown real parts $\mathrm{Re}\overline{A}_i(\nu_k^2, t)$. But, as the matrix N is nonsingular, the values of interest, $\overline{A}_i(z_k)$ for $z_k < x_0$ and $\mathrm{Re}\overline{A}_j(z_k)$ for $z_k > x_0$, are seen from (2.12) to be fully specified by $g_i(z_k)$. We must therefore find the admissible domain of the values $\{g_i(z_k)\}_{i=1,\ldots,6}^{k=1,\ldots,n}$, when the functions $g_i(z)$ are subject to the unique boundary condition (2.11), and then, using (2.12), express it in terms of the values $\overline{A}_j(z_k)$ of interest. This problem will be completely solved in the next section.

III. SOLUTION OF THE PROBLEM

We first notice that the values $\{g_i(z_k)\}_{i=1,\ldots,6}^{k=1,\ldots,n}$, compatible with the condition (2.11), form a convex and closed domain $\mathfrak{D}$ in the Euclidean space R^{6n}. Let us take a point of coordinates $\{g_i(z_k)\}_{i=1,\ldots,6}^{k=1,\ldots,n}$ inside this domain. This means that one can find at least one set of analytic functions $g_i(z)$ which take these values in the points z_k and satisfy (2.11) on the whole boundary. Therefore if one calculates the L^∞ norm, i.e., the essential supremum with respect to $\theta \in [-\pi,\pi]$ of the left-hand side of (2.11) for a fixed set of analytic functions g_i, having at z_k the fixed values $g_i(z_k)$, and then takes the infimum with respect to all these functions, the result surely will be less than or equal to one. On the other hand, if a point $\{g_i(z_k)\}_{i=1,\ldots,6}^{k=1,\ldots,n}$ is outside $\mathfrak{D}$, the L^∞ norm of the expression appearing in (2.11) will be strictly greater than one for all the analytic functions g_i taking in z_k the prescribed values $g_i(z_k)$. If one considers again the infimum of all such L^∞ norms and takes into account the fact that it is effectively attained by some analytic functions g_i (Refs. 12 and 13) one will also obtain a number strictly greater than one. From the above arguments it follows that the domain $\mathfrak{D}$ of admissible values $\{g_i(z_k)\}_{i=1,\ldots,6}^{k=1,\ldots,n}$ is exactly described by the inequality

$$\mu_\infty = \min_{\substack{g_i \in H^\infty \\ g_i(z_k)=\text{given} \\ (i=1,\ldots,6;\,k=1,\ldots,n)}} \left\| \left[\sum_{i=1}^{6} \left| g_i(\theta) + \frac{1}{\pi}\int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-e^{i\theta}} dx \right|^2 \right]^{1/2} \right\|_\infty \leq 1\,, \tag{3.1}$$

which is actually saturated if the points $\{g_i(z_k)\}_{i=1,\ldots,6}^{k=1,\ldots,n}$ are situated on the frontier of $\mathfrak{D}$. In this relation we have restricted the minimization to analytic functions g_i of class H^∞ (Ref. 12), i.e., bounded in $|z| \leq 1$. Actually this restriction did not appear in our previous discussion. However, since the nonanalytic terms added to g_i's in Eq. (3.1) are by construction bounded on the frontier of the unit disk, it is enough to consider in the minimization process only bounded analytic functions g_i, the L^∞ norm being otherwise infinite and hence of no interest for us.

In order to solve the constrained minimization problem (3.1) it is convenient to express first the functions $g_i(z)$ as

$$g_i(z)=\sum_{k=1}^{n}A_k^{(i)}B_k(z)+B_{n+1}(z)h_i(z)\quad (i=1,\ldots,6), \tag{3.2}$$

where $B_k(z)$ are products of Blaschke factors[12] defined recurrently by

$$B_1(z)=1,\quad B_k(z)=B_{k-1}(z)\frac{z-z_{k-1}}{1-zz_{k-1}},\quad k=2,\ldots,(n+1), \tag{3.3}$$

and $A_k^{(i)}$ are real coefficients determined from the triangular system of equations

$$g_i(z_k)=\sum_{j=1}^{k}A_j^{(i)}B_j(z_k),\quad k=1,\ldots,n\quad (i=1,\ldots,6), \tag{3.4}$$

In the expressions (3.2) the additional constraints upon the values of the functions g_i in the points z_k, appearing in Eq. (3.1), are automatically fulfilled, the functions $h_i(z)$ being arbitrary bounded analytic functions in $|z|\le 1$. By introducing Eq. (3.2) into Eq. (3.1) and taking into account the fact that the Blaschke factors have modulus equal to one on the boundary, the minimization problem (3.1) written in terms of h_i becomes

$$\mu_\infty=\min_{h_i\in H^\infty}\left\|\left[\sum_{i=1}^{6}|h_i(\theta)-\chi_i(\theta)|^2\right]^{1/2}\right\|_\infty, \tag{3.5}$$

where

$$\chi_i(\theta)=-\sum_{j=1}^{n}A_j^{(i)}\frac{B_j(\theta)}{B_{n+1}(\theta)}-\frac{1}{\pi B_{n+1}(\theta)}\int_{x_0}^{1+\eta}\frac{\mathrm{Im}\varphi_i(x)}{x-e^{i\theta}}dx\quad (i=1,\ldots,6) \tag{3.6}$$

are the boundary values of a set of six functions, nonanalytic inside the unit disk, and the analytic functions h_i, unlike g_i in Eq. (3.1), are subject to no additional constraints.

The solution of the minimization problem (3.5) can be obtained by applying a duality theorem.[11,12] Usually, by duality a minimum norm problem in an abstract space is related to a problem of maximization in the dual space, which is sometimes simpler than the initial one. For completeness we first state below the general duality theorem for minimum norm problems,[11,12] specifying afterwards its particular form as needed in our case.

Let X be a Banach space and X^* its normed dual (i.e., the space of all linear continuous functionals acting on X). We denote by $\langle x^*,x\rangle$ the value taken by the functional $x^*\in X^*$ acting upon the element $x\in X$. Then if x and x^* are fixed vectors in X and X^*, respectively, M a closed subspace of X, and $M^\perp\subset X^*$ the orthogonal complement (the annihilator) of M (i.e., $\langle m^*,m\rangle=0$ for every $m\in M$ and $m^*\in M^\perp$), the following equalities hold:

$$\inf_{m\in M}\|x-m\|=\max_{\substack{m^*\in M^\perp\\ \|m^*\|\le 1}}|\langle m^*,x\rangle|, \tag{3.7}$$

$$\min_{m^*\in M^\perp}\|x^*-m^*\|=\sup_{\substack{m\in M\\ \|m\|\le 1}}|\langle x^*,m\rangle|. \tag{3.7'}$$

The maximum in the right-hand side of Eq. (3.7) and the minimum in the left-hand side of Eq. (3.7′) are attained.

In the present problem we are dealing with the spaces $\vec{L}^p$ $(1\le p\le\infty)$ of vector-valued functions $\vec{f}=\{f_i(\theta)\}_{i=1}^{6}$ (measurable on $[-\pi,\pi]$ with respect to the measure $d\theta/2\pi$) with the norms

$$\|\vec{f}\|_p=\left\|\left[\sum_{i=1}^{6}|f_i(\theta)|^2\right]^{1/2}\right\|_p\equiv\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\left[\sum_{i=1}^{6}|f_i(\theta)|^2\right]^{p/2}\right\}^{1/p}<\infty,\quad 1\le p<\infty$$

$$\|\vec{f}\|_\infty=\left\|\left[\sum_{i=1}^{6}|f_i(\theta)|^2\right]^{1/2}\right\|_\infty=\operatorname*{ess\,sup}_{\theta\in[-\pi,\pi]}\left[\sum_{i=1}^{6}|f_i(\theta)|^2\right]^{1/2}<\infty \tag{3.8}$$

[where ess sup = essential (i.e., up to a null-measure set) supremum] and with the subspaces $\vec{H}^p$ of $\vec{L}^p$ consisting of (boundary values of) analytic vector-valued functions in $|z|<1$. According to the Riesz theorem[11,12] the dual of $\vec{L}^p$ is the space $\vec{L}^q$ (with $1/p+1/q=1$, $1\le p<\infty$) and every linear continuous functional on $\vec{L}^p$ has the form

$$\langle\vec{F},\vec{f}\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{i=1}^{6}F_i(\theta)f_i(\theta)d\theta,\quad \vec{f}\in\vec{L}^p,\quad \vec{F}\in\vec{L}^q. \tag{3.9}$$

Moreover, the annihilator of $\vec{H}^p$, required in the duality theorem (3.7), consists[11,12] of all the functions having the form $\vec{F}(z)=z\vec{G}(z)$, with $\vec{G}\in\vec{H}^q$. With these preliminaries we apply the general relation (3.7′) to the minimum-norm problem (3.5), where we identify X^* and $M^\perp$ to be $\vec{L}^\infty$ and $\vec{H}^\infty$, respectively, and the element x^* the boundary value of the nonanalytic function $\vec{\chi}=\{\chi_i(\theta)\}_{i=1}^{6}\in\vec{L}^\infty$. Using (3.7′) and (3.9) we obtain

$$\mu_\infty=\min_{\vec{h}\in\vec{H}^\infty}\|\vec{h}-\vec{\chi}\|_\infty=\sup_{\substack{\vec{F}\in\vec{H}^1\\ \|\vec{F}\|_1\le 1}}\left|\frac{1}{2\pi i}\int_{|\zeta|=1}\sum_{i=1}^{6}F_i(\zeta)\chi_i(\zeta)d\zeta\right|, \tag{3.10}$$

where $\zeta=e^{i\theta}$ and $d\zeta=i\zeta\,d\theta$.

In order to evaluate the supremum over the functions F_i we first evaluate the integral appearing in (3.10), using the expressions (3.6) of the functions χ_i $(\zeta=e^{i\theta})$ and applying the residua theorem. The first term of χ_i contains only poles (due to the Blaschke factor in the denominator) and brings in the contribution

$$I_1=\sum_{i=1}^{6}\sum_{j=1}^{n}A_j^{(i)}\sum_{k=j}^{n}F_i(z_k)\left[\frac{(z-z_k)B_j(z)}{B_{n+1}(z)}\right]_{z=z_k}$$

to the integral in Eq. (3.10). Permuting the sums upon j and k, i.e., $\sum_{j=1}^n \sum_{k=j}^n = \sum_{k=1}^n \sum_{j=1}^k$ and taking into account Eqs. (3.4) one finds

$$I_1 = \sum_{i=1}^{6} \sum_{k=1}^{n} F_i(z_k) g_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k}.$$

The contribution of the second term entering χ_i can be evaluated by applying again the residua theorem but noticing that now the integrand has a cut along the segment $[x_0, 1]$. We first write

$$I_2 = \sum_{i=1}^{6} \sum_{z_k<x_0} F_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} \frac{1}{\pi} \int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-z_k}\, dx + I_2', \tag{3.12}$$

where we have picked up the contribution of the poles z_k situated below the threshold x_0, the remaining term I_2' containing an additional integral around the segment $[x_0, 1]$

$$I_2' = -\frac{1}{2\pi i} \int_{x_0}^{1} dy \frac{1}{\pi} \times \int_{x_0}^{1} dx \left\{ \sum_{i=1}^{6} \left[\frac{F_i(y+i\epsilon)\,\mathrm{Im}\varphi_i(x)}{B_{n+1}(y+i\epsilon)(x-y-i\epsilon)} - \frac{F_i(y-i\epsilon)\,\mathrm{Im}\varphi_i(x)}{B_{n+1}(y-i\epsilon)(x-y+i\epsilon)} \right] \right\}.$$

By using the known identity

$$\frac{1}{\pi} \int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-y\pm i\epsilon}\, dx = \frac{\mathrm{P}}{\pi} \int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-y}\, dx \mp i\, \mathrm{Im}\varphi_i(y)$$

and by applying once again the residua theorem for the poles of the Blaschke factors situated inside the contour of integration, i.e., for $z_k > x_0$, one can write I_2' as

$$I_2' = \sum_{i=1}^{6} \left\{ \sum_{z_k>x_0} F_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} \frac{\mathrm{P}}{\pi} \int_{x_0}^{1+\eta} \frac{\mathrm{Im}\varphi_i(x)}{x-z_k}\, dx - \frac{\mathrm{P}}{\pi} \int_{x_0}^{1} \frac{F_i(x)\,\mathrm{Im}\varphi_i(x)}{B_{n+1}(x)} \right\}. \tag{3.13}$$

We now introduce the relations (3.11), (3.12), and (3.13) in the right-hand side of Eq. (3.10). Using the relations (2.12) one notices that the intermediate values $g_i(z_k)$ as well as the arbitrary functions $\mathrm{Im}\varphi_i(x)$ for $x>1$ eventually disappear from the result which is expressed only in terms of the input values $\varphi_i(z_k)$ and $\mathrm{Im}\varphi_i(x)$ for $x \le 1$ as

$$\mu_\infty = \sup_{\substack{\vec{F} \in \vec{H}^1 \\ \|\vec{F}\|_1 \le 1}} \left| \sum_{i=1}^{6} \left\{ \sum_{z_k<x_0} F_i(z_k) \varphi_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} + \sum_{z_k>x_0} F_i(z_k)\, \mathrm{Re}\varphi_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} - \frac{\mathrm{P}}{\pi} \int_{x_0}^{1} \frac{F_i(x)\,\mathrm{Im}\varphi_i(x)}{B_{n+1}(x)}\, dx \right\} \right|. \tag{3.14}$$

For the further evaluation of the supremum (3.14) it is convenient to use a factorization of the function $\vec{F} \in \vec{H}^1$.[11,5] We first define the function

$$w(z) = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln\left[\sum_{i=1}^{6} |F_i(\theta)|^2\right]^{1/4} d\theta \right\}, \tag{3.15}$$

which is, by construction, an outer analytic function belonging to the Hilbert space H^2. From the constraint upon the functions $F_i(z)$ it follows that w obeys the condition

$$\|w\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |w(\theta)|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{i=1}^{6} |F_i(\theta)|^2\right]^{1/2} d\theta \le 1. \tag{3.16}$$

We now define the functions G_i through the relations

$$F_i(z) = w(z) G_i(z) \quad (i=1,\ldots,6) \tag{3.17}$$

and notice that they belong to H^2 and moreover satisfy [in view of Eqs. (3.14) and (3.16)] the inequality

$$\|\vec{G}\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{i=1}^{6} |G_i(\theta)|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{i=1}^{6} |F_i(\theta)|^2\right]^{1/2} d\theta \le 1. \tag{3.18}$$

The supremum (3.14), written in terms of $w(z)$ and $G_i(z)$ becomes

$$\mu_\infty = \sup_{\substack{w \in H^2,\ \vec{G} \in \vec{H}^2 \\ \|w\|_2 \le 1,\ \|\vec{G}\|_2 \le 1}} \left| \sum_{i=1}^{6} \left\{ \sum_{z_k<x_0} w(z_k) G_i(z_k) \varphi_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} + \sum_{z_k>x_0} w(z_k) G_i(z_k)\, \mathrm{Re}\varphi_i(z_k) \left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} - \frac{\mathrm{P}}{\pi} \int_{x_0}^{1} \frac{w(x) G_i(x)\,\mathrm{Im}\varphi_i(x)}{B_{n+1}(x)}\, dx \right\} \right| \tag{3.19}$$

and it can actually be expressed as a norm of an operator acting from $\vec{H}^2$ into H^2. Indeed, let us first develop $w(z)$ and $G_i(z)$ as a Taylor series

$$w(z) = \sum_{m=0}^{\infty} w_m z^m, \qquad G_i(z) = \sum_{m=0}^{\infty} G_m^{(i)} z^m \quad (i=1,\ldots,6). \tag{3.20}$$

Then Eq. (3.19) can be written as

$$\mu_\infty = \sup_{\substack{\sum_{m=0}^{\infty} w_m^2 \le 1 \\ \sum_{i=1}^{6} \sum_{k=0}^{\infty} G_k^{(i)2} \le 1}} \left| \sum_{m,k=1}^{\infty} w_{m-1} \sum_{i=1}^{6} H_{mk}^{(i)} G_{k-1}^{(i)} \right|, \tag{3.21}$$

where $H_{mk}^{(i)}$ are the elements of the Hankel matrices $H^{(i)}$,

$$H^{(i)}_{mk}=c^{(i)}_{-(m+k-1)};\quad m,k=1,2,\ldots\quad (i=1,\ldots,6), \tag{3.22}$$

defined in terms of the real coefficients $c^{(i)}_{-(j+1)}$

$$\begin{aligned} c^{(i)}_{-(j+1)} &= \sum_{z_k<x_0} z_k^j \varphi_i(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} \\ &+\sum_{z_k>x_0} z_k^j \operatorname{Re}\varphi_i(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} \\ &-\frac{\mathrm{P}}{\pi}\int_{x_0}^{1}\frac{x^j \operatorname{Im}\varphi_i(x)}{B_{n+1}(x)}dx, \\ j&=0,1,\ldots\quad (i=1,\ldots 6). \end{aligned} \tag{3.23}$$

In Eq. (3.21) there appears the norm of an infinite matrix acting from l^2 into l^2.[11] For its computation it is useful to truncate the matrices $H^{(i)}$ at a finite dimension N by setting all the coefficients $c^{(i)}_{-(j+1)}=0$ for $j\geq N$ and to take into account the fact that the norms obtained in this way tend, for $N\to\infty$, towards the exact result (3.21).[13,15]

Let us evaluate explicitly the supremum (3.21) by means of the Kuhn-Tucker theorem for convex optimization,[20] assuming that the matrices are already truncated at the dimension N. The Lagrangian of the problem is

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{6}\sum_{k,m=1}^{N} w_{m-1}G^{(i)}_{k-1}(H^{(i)}_N)_{km} \\ &+\lambda_N\left(\sum_{m=1}^{\infty} w_{m-1}{}^2-1\right) \\ &+\kappa_N\left(\sum_{i=1}^{6}\sum_{k=1}^{\infty} G^{(i)2}_{k-1}-1\right) \end{aligned}$$

with λ_N and κ_N being non-negative Lagrange multipliers. By setting the derivatives of $\mathcal{L}$ with respect to w_m and $G^{(i)}_k$ equal to zero one has

$$\begin{aligned} &\sum_{i=1}^{6}\sum_{k=1}^{N}(H^{(i)}_N)_{mk}G^{(i)}_{k-1}=\lambda_N w_{m-1},\quad m=1,2,\ldots,N; \\ &\sum_{m=1}^{N}(H^{(i)}_N)_{km}w_{m-1}=\kappa_N G^{(i)}_{k-1},\quad k=1,2,\ldots,N\quad (i=1,\ldots,6); \\ &w_m=0,\quad m\geq N;\quad G^{(i)}_k=0,\quad k\geq N\quad (i=1,\ldots 6). \end{aligned} \tag{3.24}$$

If one now multiplies the first set of equations by w_{m-1} and the second one by $G^{(i)}_{k-1}$, sums upon m and i,k, respectively, and takes into account the fact that for nonzero λ_N and κ_N the constraints upon w_m and $G^{(i)}_k$ must be saturated,[20] one obtains

$$\lambda_N=\kappa_N=\sum_{k,m=1}^{N} w_{m-1}\sum_{i=1}^{6}(H^{(i)}_N)_{mk}G^{(i)}_{k-1}. \tag{3.25}$$

Using this equality in Eqs. (3.24), one also finds

$$\sum_{j,k=1}^{N}\sum_{i=1}^{6}(H^{(i)}_N)_{mj}(H^{(i)}_N)_{jk}w_{k-1}=\lambda_N{}^2 w_{m-1} \tag{3.26}$$

which shows that $\lambda_N{}^2$ is an eigenvalue of the real matrix

$$H_N=\sum_{i=1}^{6}H^{(i)2}_N. \tag{3.27}$$

In Eq. (3.25) expressing λ_N one may recognize on the other hand the supremum to be evaluated. It is thus equal to the square root of the greatest eigenvalue $\lambda_{N,\max}{}^2$ of the matrix H_N defined in Eq. (3.27). In view of the starting formula (3.1) and of the above remarks one obtains the inequality

$$\mu_\infty{}^2=\lim_{N\to\infty}\mu_\infty^{(N)2}=\lim_{N\to\infty}\lambda_{N,\max}{}^2(\bar{A}_j(z_k))\leq 1, \tag{3.28}$$

which provides the required optimal description of the domain $\mathfrak{D}$ of the allowed values for $\bar{A}_j(\nu_k{}^2,t)$; $\lambda_{N,\max}{}^2$ depends implicitly on $\bar{A}_j(\nu_k{}^2,t)$, through the coefficients $c^{(i)}_{-(j+1)}$, as seen from Eqs. (3.22), (3.23), and (3.27).

The problem formulated in Sec. II is thereby solved. A first specific application of this result, together with details concerning the numerical calculations, will be presented in the next section, where we shall be interested in the case of a single interior point $\nu^2=\nu_B{}^2(z=0)$, actually corresponding to the location of the s, u-channel Born poles.

We point out also that the inequality (3.28) can be expressed equivalently as a non-negativity condition for the matrix $1-H_N$ when $N\to\infty$ which, in the particular case $\nu_{1n}{}^2=\nu_0{}^2$, can be reduced to the condition derived in Ref. 5 and tested in Ref. 7.

Before concluding this section we shall present, as an alternative to the above technique, another formulation useful in practical applications. The problem treated above was complicated since it actually was an infinite Schur-Carathéodory interpolation in the origin ($z=0$), combined with a finite Pick-Nevanlinna interpolation in distinct points z_k, formulated in the Banach space H^∞. Usually a minimum norm or an interpolation problem is considerably simplified if it can be considered in a Hilbert space such as H^2. In what follows we shall exploit this idea, embedding the problem from the initial H^∞ space into a large family of Hilbert spaces H^2. Let us first go back to Eq. (3.19) and assume that the function w is kept fixed, the maximization being performed only over the functions G_j. By this procedure the resulting supremum, depending on w and denoted further by $\mu_2(w)$ will be, for every w with the properties required in (3.19), less (in general) than the optimal value μ_∞. Therefore if this new supremum is majorized by one, a constraint upon the values $\bar{A}_j(\nu_k{}^2,t)$, weaker in general than the optimal one (3.28), is obtained. This means that the domain $\mathfrak{D}_w$ of admissible values for $\bar{A}_i(\nu_k{}^2,t)$ in this new context is (in general) larger than $\mathfrak{D}$ and includes it. On the other hand, as we shall show below, $\mathfrak{D}_w$, unlike $\mathfrak{D}$, can be explicitly found. Indeed, it is yielded by the inequality

$$\mu_2(w) = \sup_{\substack{\vec{G}\in\vec{H}^2 \\ \|\vec{G}\|_2\leq 1}} \left|\sum_{i=1}^{6}\left\{\sum_{z_k<x_0} w(z_k)G_i(z_k)\varphi_i(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} + \sum_{z_k>x_0} w(z_k)G_i(z_k)\operatorname{Re}\varphi_i(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} - \frac{P}{\pi}\int_{x_0}^{1}\frac{w(x)G_i(x)\operatorname{Im}\varphi_i(x)}{B_{n+1}(x)}dx\right\}\right| \leq 1 . \tag{3.29}$$

Developing as before the functions $G_i(z)$ in the Taylor series, this relation becomes

$$\mu_2(w) = \sup_{\Sigma_{i=1}^{6}\Sigma_{m=0}^{\infty}G_m^{(i)2}\leq 1} \left|\sum_{i=1}^{6}\sum_{m=0}^{\infty} G_m^{(i)}\gamma_{-(m+1)}^{(i)}\right| \leq 1 , \tag{3.30}$$

where the real coefficients $\gamma_{-(m+1)}^{(i)}$ are

$$\gamma_{-(m+1)}^{(i)} = \sum_{z_k<x_0} z_k^{\,m} w(z_k)\varphi_i(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} + \sum_{z_k>x_0} z_k^{\,m} w(z_k)\operatorname{Re}\varphi_i(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k} - \frac{P}{\pi}\int_{x_0}^{1}\frac{x^m w(x)\operatorname{Im}\varphi_i(x)}{B_{n+1}(x)}dx, \quad m=0,1,\ldots;(i=1,\ldots,6) . \tag{3.31}$$

The evaluation of the supremum (3.30) is trivial: using the Cauchy-Schwarz inequality we obtain

$$\mu_2^{\,2}(w) = \sum_{i=1}^{6}\sum_{m=0}^{\infty}\gamma_{-(m+1)}^{(i)2} \leq 1 . \tag{3.32}$$

By introducing in this relation the expressions (3.31) for $\gamma_{-(m+1)}^{(i)}$ and noticing that the summation upon m can be performed exactly, the following inequality describing explicitly the domain $\mathfrak{D}_w$ is obtained:

$$\mu_2^{\,2}(w) = \sum_{i=1}^{6}\left\{\sum_{j,k=1}^{n}\frac{\operatorname{Re}\varphi_i(z_k)\operatorname{Re}\varphi_i(z_j)}{1-z_kz_j}w(z_k)w(z_j)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k}\left[\frac{z-z_j}{B_{n+1}(z)}\right]_{z=z_j} - 2\sum_{k=1}^{n}\operatorname{Re}\varphi_i(z_k)w(z_k)\left[\frac{z-z_k}{B_{n+1}(z)}\right]_{z=z_k}\frac{P}{\pi}\int_{x_0}^{1}\frac{\operatorname{Im}\varphi_i(x)w(x)}{(1-xz_k)B_{n+1}(x)}dx + \frac{P}{\pi^2}\int_{x_0}^{1}\int_{x_0}^{1}\frac{\operatorname{Im}\varphi_i(x)\operatorname{Im}\varphi_i(y)w(x)w(y)}{(1-xy)B_{n+1}(x)B_{n+1}(y)}dx\,dy\right\} \leq 1 . \tag{3.33}$$

In this relation the points $z_k(|z_k|<1)$ lie below or above the threshold x_0. When $z_k<x_0$ the values $\varphi_i(z_k)$ are of course real and the notation $\operatorname{Re}\varphi_i(z_k)$ is for them redundant.

From the initial conditions (2.1) and (2.2) of the problem we have obtained in (3.33) a set of necessary constraints on the values $\bar{A}_j(\nu_k^{\,2},t)$, for every outer analytic function w in the unit sphere of H^2. Finding among these constraints the optimal one amounts to taking the supremum over w in $\mu_2(w)$. The practical usefulness of the procedure proposed here depends upon a successful guess of a simple and economical way of approaching closely the optimal result by a suitable choice of w. Of course, a complete maximization would lead again to the previous optimal result (3.28). The number $\mu_2(w)$ defined in (3.29) is also the solution of the minimum-norm problem:

$$\mu_2(w) = \min_{\substack{g_i\in H^\infty \\ g_i(z_k)=\text{given}}} \left\| w(\theta)\left[\sum_{i=1}^{6}\left| g_i(\theta) + \frac{1}{\pi}\int_{x_0}^{1+\eta}\frac{\operatorname{Im}\varphi_i(x)}{x-e^{i\theta}}dx\right|^2\right]^{1/2}\right\|_2 . \tag{3.34}$$

The equivalence between Eq. (3.29) and Eq. (3.34) may be established by applying again the duality theorem (3.7′), the spaces X^* and $M^\perp$ in the stated theorem being this time $\vec{L}^2$ and $\vec{H}^2$, respectively. In the relation (3.34) one may recognize an L^2 norm, weighted by a function $w\in H^2$ with $\|w\|_2\leq 1$ [actually this justifies our notation $\mu_2(w)$]. The mapping of the unit circle $|z|=1$ onto itself,

$$z' = \frac{z-\alpha}{1-z\alpha} , \tag{3.35}$$

where α is a real parameter $\alpha\in(-1,1)$, leaves invariant the L^∞ norm of a function f,

$$\|f\|_\infty = \operatorname{ess\,sup}|f(\theta)| , \quad \theta\in[-\pi,\pi] ,$$

while it changes the L^2 norm,

$$\|f\|_2 = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta\right]^{1/2} .$$

Actually, by taking the maximum of all such L^2 norms one obtains $\|f\|_\infty$. A suitable choice for w appears to be[16]

$$w(z) = \frac{(1-\alpha^2)^{1/2}}{1-z\alpha} \tag{3.36}$$

since $|w(\theta)|^2 = (1-\alpha^2)/(1-2\alpha\cos\theta+\alpha^2)$ coincides with the Jacobian $|d\theta'/d\theta|$ of the conformal mapping (3.35). By varying α in the interval $(-1,1)$ we expect to obtain a result very close to the optimal one (3.28).

IV. THE "ONE-POINT PROBLEM": OPTIMAL SUM RULES FOR THE BORN RESIDUA

In this section we perform the first numerical application of the technique presented above. We start by treating the case of a single interior point corresponding to the position of the s, u-channel Born poles, $\nu_B^2 = t^2/16$. The amplitudes $\bar{A}_i$ take at this point values completely specified in terms of t, the electric charge, and the anomalous magnetic moment of the target.[4] Optimal-sum-rule inequalities for these values are derived in Refs. 4 and 6 and were investigated numerically in Ref. 7. In this present context one makes use, in addition, of the absorptive parts $\mathrm{Im}\bar{A}_i$ as yielded by the unitarity condition.

We shall perform the numerical analysis using both the rather complicated L^∞ solution (3.28) and the sum rule (3.33). We particularize the general formulas derived in the previous section to the case of a single interior point ν_B^2 applied through the mapping (2.4) into $z=0$. The coefficients $c^{(i)}_{-(j+1)}$ defined in Eq. (3.23) are

$$c^{(i)}_{-(j+1)} = \varphi_i(0)\delta_{j0} - \frac{1}{\pi}\int_{x_0}^{1} x^{j-1}\,\mathrm{Im}\varphi_i(x)dx\,;$$

$$j=0,1,\ldots;\quad (i=1,\ldots,6). \tag{4.1}$$

We have first to construct the matrix H_N defined by Eqs. (3.22) and (3.27) and then to compute its greatest eigenvalue $\lambda_{N,\max}^2$, which depends implicitly on the values $\varphi_i(0)$ containing the Born-pole residua. In the limit $N\to\infty$, $\lambda_{N,\max}^2$ actually provides, according to the inequality (3.28), the optimal constraint looked for,

$$\mu_\infty^2 = \lim_{N\to\infty} \lambda_{N,\max}^2(\varphi_i(0)) \le 1\,. \tag{4.2}$$

On the other hand, according to the arguments expressed in the previous section, the inequality (3.33) which now takes on the particular form

$$\mu_2^2(w) = \sum_{i=1}^{6}\Big\{\varphi_i^2(0)w^2(0) - 2w(0)\varphi_i(0)\frac{1}{\pi}\int_{x_0}^{1}\frac{w(x)\,\mathrm{Im}\varphi_i(x)}{x}dx + \frac{1}{\pi^2}\int_{x_0}^{1}\int_{x_0}^{1}\frac{w(x)w(y)\,\mathrm{Im}\varphi_i(x)\,\mathrm{Im}\varphi_i(y)}{xy(1-xy)}dx\,dy\Big\} \le 1\,, \tag{4.3}$$

although it is in general nonoptimal, presents the great practical advantage of displaying an explicit dependence on $\varphi_i(0)$. In fact in Eq. (4.3) one has a family of sum rules (labeled by the arbitrary function w) which must approach the optimal result (4.2) when the maximum over w (with $\|w\|_2^2 \le 1$) is taken. In this section both the inequalities (4.2) and (4.3) will be examined in detail numerically.

For the purposes of the present analysis it is quite reasonable to test the above sum rules for several ν_{in}^2 values situated in the range corresponding to the photon (laboratory) energy region $\omega \le 1210$ MeV, where the pion-photoproduction multipoles are available.[18,19] Particularly, when $\nu_{\mathrm{in}}^2 = \nu_0^2$ we must obviously recover the corresponding results reported in Ref. 7. In the context of the present paper one has a much greater flexibility concerning the allowed range of values for the momentum transfer than in Refs. 5 and 7. Indeed, the limitation for t comes from the requirement that the region $\nu^2 \ge \nu_{\mathrm{in}}^2$ be physical (i.e., $\nu_{\mathrm{in}}^2 \ge \nu_{\mathrm{min}}^2$) since in this region one uses the cross section of the Compton scattering. This gives for t the allowed range

$$\frac{-4m\omega_{\mathrm{in}}^2}{2\omega_{\mathrm{in}}+m} \le t \le 0 \quad (\nu_{\mathrm{in}} = m\omega_{\mathrm{in}} + t/4)\,,$$

so that for $\omega_{\mathrm{in}} \ge 300$ MeV one may work practically in the whole interval $-12\mu^2 \le t \le 0$ in which the Legendre series for the imaginary parts $\mathrm{Im}\bar{A}_i$ converges.[21] This relaxation of the momentum-transfer domain of validity of the constraints is very important since it allows a model-independent treatment of the proton Compton effect in the whole N^* resonance region.

For carrying out the numerical investigation of the above inequalities a set of computational programs, dealing with the various steps of the calculations, were applied. In what follows we briefly describe the main points of these programs.

A first problem that occurred was the calculation of the outer function $\mathcal{S}(z)$ defined in Eq. (2.5). Using the equality $\sigma(-\theta)=\sigma(\theta)$ we wrote this relation in the equivalent form

$$\mathcal{S}(z) = \exp\left[\frac{1-z^2}{2\pi}\int_0^{\pi}\frac{\ln\sigma(\theta)}{1-2z\cos\theta+z^2}d\theta\right], \tag{4.4}$$

where the variable θ is related to ν^2 at fixed t by the relation

$$\theta = 2\arctan\frac{\sqrt{\nu^2-\nu_{\text{in}}^2}}{\sqrt{\nu_{\text{in}}^2-\nu_B^2}}\,,\quad \nu^2\geq \nu_{\text{in}}^2 \tag{4.5}$$

which follows from Eq. (2.4).

We have computed the integral (4.4) using the experimental data on the differential cross section $d\sigma/d\Omega$ considered as a function of ν^2 at fixed t. In the present work (apart from small differences in the evaluation of the high-energy contributions) we have adopted for $\sigma(\nu^2,t)$ the parametrizations discussed in detail in Ref. 7. $\mathcal{S}(z)$ depends essentially only upon the values of $\sigma(\nu^2,t)$ in the immediate vicinity of the threshold ν_{in}^2.

In our applications we shall often work with a threshold ν_{in}^2 corresponding to $\omega_{\text{in}}\simeq 700-1210$ MeV. As in Ref. 7, we have employed for σ a linear interpolation (in the ν^2 variable at fixed t) between the data of Jung *et al.*[22] at 700–850 MeV and the high-energy parametrizations quoted in Ref. 23. With such interpolations we have reproduced satisfactorily, or slightly overestimated, the experimental values at 700–1400 MeV reported recently in Ref. 24, for t in the allowed range. From the formulation of the problem given in the previous section it is clear that an overestimation of σ could weaken but not endanger the validity of the bounds. Therefore one has to be especially careful not to underestimate the input values of σ. On the other hand, a drastic overestimation of σ has to be avoided, too, since the inequalitities then risk becoming uninterestingly large. Of course, for completeness one must take into account also the uncertainties affecting σ and investigate their influence upon the results; a discussion of this point will be given below. The integral (4.4) and the integrals appearing in Eqs. (4.1) and (4.3) have been computed using a standard Gauss-quadrature method. For the L^∞ method leading to the condition (4.2) we had, in addition, to calculate the number μ_∞ [according to Eq. (3.28)] using for the construction of H_N the relations (3.27), (3.22), and (3.23). A similar evaluation was performed previously in another context in Ref. 14. In the present work we have applied, for the calculation of $\lambda_{N,\max}^2$ an algorithm described in Ref. 17, which consists in successively squaring the matrix H_N and taking the traces of the resulting matrices. It can be shown[17] that the quantity

$$\lambda_{N,\max}^{2(n)} = (\mathrm{Tr}\,H_N^{2^n})^{1/2^n}\,,$$

where the integer n specifies the step of iteration, tends to the greatest eigenvalue $\lambda_{N,\max}^2$ of H_N when $n\to\infty$. A theoretical estimation of the error of the procedure at the step n of the iteration and other programming details may be found in Ref. 17. In practice, in our case a good accuracy in the computation was achieved without difficulty.

The problem of the limit $N\to\infty$ in Eq. (3.28) was also carefully investigated, the required stability of the results against the increase of N being practically settled for $N\simeq 100$.

The techniques described above were applied in connection with the sum rules (4.2) and (4.3) for several values of the threshold ν_{in}^2 and of the momentum transfer t. We have computed both the number μ_∞^2 defined in (4.2) and $\mu_2^2(w)$ defined in (4.3), with w of the form (3.36). As expected, the maximum of the numbers $\mu_2^2(w)$ with respect to α was practically identical with μ_∞^2. This fact is illustrated in Table I where we list, for $\omega_{\text{in}}=1210$ MeV and $t=-0.5\,\mu^2$, the values $\mu_2^2(\alpha)$ for several values of α within the interval $(-1,1)$, together with the optimal result μ_∞^2. The maximum of $\mu_2^2(\alpha)$ (actually corresponding to $\alpha\approx 0.20$) is seen to coincide with the number μ_∞^2 obtained through a much more sophisticated procedure. This result is important in practice since it shows that the formulation leading to Eq. (4.3) [or more generally to Eq. (3.33)] does not imply a significant loss of optimality, being at the same time much more convenient from the computational point of view.

We present in Table II the results of the evaluation of the sum rules for the Born-pole residua. All the numbers μ_∞^2 [or max $\mu_2^2(w)$] written in Table II are less than one, which means that the inequalities (4.2) or (4.3) are not violated. The various pieces of physical information entering as input in these relations, i.e., the absorptive parts $\mathrm{Im}\bar{A}_i$ for $\nu^2\leq\nu_{\text{in}}^2$, the unpolarized differential cross section for $\nu^2\geq\nu_{\text{in}}^2$, and the target's static electromagnetic characteristics (charge and magnetic moment) are therefore consistent with each other. This consistency is actually very comfortable when the number μ_∞ is much less than one,

TABLE I. Evaluation of $\mu_2^2(w)$ defined in Eq. (4.3) for various weight functions $w(z)$ of the form (3.35); μ_∞^2 represents the optimal result Eq. (4.2) ($\omega_{\text{in}}=1210$ MeV, $t=-0.5\mu^2$).

α	$\mu_2^2(\alpha)$
−0.96	0.008
−0.84	0.029
−0.72	0.049
−0.60	0.066
−0.40	0.090
−0.20	0.108
0.0	0.118
0.20	0.120
0.40	0.112
0.60	0.094
0.96	0.021
$\mu_\infty^2=0.12$	

TABLE II. Test of the optimal-sum-rule inequality for the Born-pole residua Eq. (4.2), for several values of the second threshold ω_{in} and t.

ω_{in} (MeV) \ t/μ^2	−0.1	−0.6	−3.48	−6.0	−10.0
$\omega_0=\mu+\mu^2/2m$	0.43	0.32	0.10		
300	0.57	0.56	0.49	0.45	0.39
350	0.88	0.87	0.82	0.74	0.57
400	0.86	0.86	0.81	0.77	0.62
450	0.68	0.68	0.63	0.61	0.51
500	0.48	0.48	0.48	0.48	0.48
700	0.32	0.32	0.35	0.40	0.44
1210	0.12	0.12	0.14	0.19	0.18

as it happens if ω_{in} is taken either near the threshold ω_0 or at high energies ($\omega_{in}\simeq 1210$ MeV).

When the threshold ν_{in}^2 is taken in the N^* resonance region (i.e., $\omega_{in}=350$–400 MeV) the inequality (4.2) appears to be rather close to saturation, at least for some low values of $|t|$. Before discussing the significance of this result it is necessary to estimate the uncertainty upon the values given in Table II, produced by the error, experimental or computational, affecting the physical information used as input.

A reasonable way of investigating the possible effect of the neglected multipion photoproduction is to modify $\mathrm{Im}\bar{A}_i$ randomly by a factor not greater than 5%, this being the order of magnitude of the two-pion-photoproduction contribution suggested by the analysis performed in Ref. 3. We have found that the effect of this change in $\mathrm{Im}\bar{A}_i$ upon μ_∞^2 was not greater than 8–10%. This variation turned out to be monotonical (for instance, an overall increase of $\mathrm{Im}\bar{A}_i$ led to larger values for μ_∞^2).

The influence upon the results of the uncertainties affecting the cross section σ can, on the other hand, be established in a more precise way. Let us assume that the input cross section $\sigma(\nu^2,t)$ is replaced by $p\cdot\sigma(\nu^2,t)$, where p is constant with respect to ν^2. From the definition (2.5) and the maximum-modulus principle it follows that the outer function $\mathcal{S}(z)$ will become equal to $\sqrt{p}\ \mathcal{S}(z)$ at every point z. Since $\mathcal{S}(x)$ is positive for real x and it enters quadratically in the denominator of Eq. (4.3) it follows that $\mu_2^2(w)$ will become equal to $(1/p)\mu_2^2(w)$ and the same property will hold for μ_∞^2, too, since it is equal to the maximum of μ_2^2 with respect to w. If we now assume that the cross section taken as input in our calculations is lowered by 10–15%, i.e., if it is chosen at the lowest extremity of the experimental error bars usually given in Refs. 22 to 26, it may be seen from the above arguments that the numbers given in Table II will become generally greater by 10–15%. Particularly, if the threshold ω_{in} is taken in the critical region 350–400 MeV and t is sufficiently low, μ_∞^2 will reach 1 or become even slightly greater, which means that the sum rule (4.2) is saturated or even slightly violated. The significance of these possible violations will be discussed in the frame of the more general analysis performed in the next section.

V. THE "TWO-POINT PROBLEM": BOUNDS FOR THE COMPTON DIFFERENTIAL CROSS SECTION

In this section we start considering the case of two points, one of them being again ν_B^2 and the other an arbitrary point ν_2^2 situated either inside the analyticity domain or on the cut, below the second inelastic threshold ν_{in}^2.

Since the optimal L^∞ norm solution (3.29) is rather complicated, especially when many interior points z_k have to be considered and, on the other hand, the numerical analysis performed for the one-point problem in the previous section showed that the formulation (3.33) leads in practice to almost the same result, we shall use in this section only the latter, more simple formulation.

For the subsequent applications it is convenient to write the inequality (3.33) in the case of two interior points, $z=0$ and the image z_2 of ν_2^2, in the form

$$a\sum_{i=1}^{6}[\mathrm{Re}\varphi_i(z_2)]^2-2\sum_{i=1}^{6}b_i\,\mathrm{Re}\varphi_i(z_2)-c\leq 0\,, \qquad (5.1)$$

where

$$\begin{aligned}
a&=\frac{1-z_2^2}{z_2^2}w(z_2)\,,\\
b_i&=\frac{1-z_2^2}{z_2}w(z_2)\left[\frac{w(0)\varphi_i(0)}{z_2}+\frac{\mathrm{P}}{\pi}\int_{x_0}^{1}\frac{w(x)\mathrm{Im}\varphi_i(x)}{x(x-z_2)}dx\right]\quad (i=1,\ldots,6)\,,\\
c&=1-\sum_{i=1}^{6}\left[\frac{w^2(0)\varphi_i^2(0)}{z_2^2}+\frac{2w(0)\varphi_i(0)}{z_2}\frac{\mathrm{P}}{\pi}\int_{x_0}^{1}\frac{w(x)\mathrm{Im}\varphi_i(x)(1-xz_2)}{x(x-z_2)}dx\right.\\
&\qquad\left.+\frac{\mathrm{P}}{\pi^2}\int_{x_0}^{1}\int_{x_0}^{1}\frac{w(x)w(y)\mathrm{Im}\varphi_i(x)\mathrm{Im}\varphi_i(y)(1-xz_2)(1-yz_2)}{xy(1-xy)(x-z_2)(y-z_2)}dx\,dy\right].
\end{aligned} \qquad (5.2)$$

The above relations are valid for z_2 situated either above the threshold x_0 or below it [in the latter case the notation $\mathrm{Re}\varphi_i(z_2)$ is redundant since $\varphi_i(z_2)$ are real numbers; also the principal part in front of the integrals appearing in b_i and c becomes superfluous].

The function w entering the coefficients a, b_i, and c will be chosen again as in Eq. (3.36) with α left as a free parameter. For a fixed α, a, b_i, and c are completely determined in terms of the absorptive parts $\mathrm{Im}\bar{A}_i$ (for $\nu^2 \leqslant \nu_{\mathrm{in}}^2$), the cross section σ (for $\nu^2 > \nu_{\mathrm{in}}^2$) and the Born-pole residua entering $\varphi_i(0)$. The entire physical input being concentrated in these coefficients, the inequality (5.1) expresses a rigorous, model-independent constraint upon the values $\mathrm{Re}\varphi_i(z_2)$. In geometrical terms, the inequality (5.1) shows that the allowed values of $\mathrm{Re}\varphi_i(z_2)$ are restricted to the intersection of a set of convex domains d_w in the Euclidean space R^6. Since any physical observable of the Compton process at given $\nu_2{}^2$ and t is completely specified by the amplitudes $\bar{A}_i(\nu_2{}^2, t)$ or, equivalently, by $\varphi_i(z_2)$, and the imaginary parts $\mathrm{Im}\varphi_i(z_2)$ are already taken as known [particularly $\mathrm{Im}\varphi_i(z_2)=0$ for $z_2<x_0$], the condition (5.1) on $\mathrm{Re}\varphi_i(z_2)$ $(i=1,2,\ldots,6)$ induces rigorous upper and lower bounds upon every such physical quantity of interest. Next we shall investigate the unpolarized differential cross section $d\sigma/d\Omega$ as well as the differential cross sections with the incoming photon linearly polarized parallel or normal to the reaction plane ($d\sigma_{\parallel}/d\Omega$ and $d\sigma_{\perp}/d\Omega$) at some ν^2 points below and above the threshold $\nu_0{}^2$ for the single-pion photoproduction, especially in the N^* resonance region.

We display below the explicit expressions of the differential cross sections in terms of $\mathrm{Re}\varphi_i(z_2)$. Using the known kinematical relations for the proton Compton scattering and Eq. (2.7) one has for $\nu_2{}^2<\nu_{\mathrm{in}}{}^2$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{c.m.}}(\nu_2,t)=\left(\frac{d\sigma}{d\Omega}\right)^{\mathrm{UB}}_{\mathrm{c.m.}}(\nu_2,t)+\frac{[\mathcal{S}(z_2)]^2}{128\pi^2(2\nu_2+m^2-t/2)}\sum_{i,j=1}^{6}P_{ij}(\nu_2{}^2)\,\mathrm{Re}\varphi_i(z_2)\,\mathrm{Re}\varphi_j(z_2)\,, \tag{5.3}$$

$$\left(\frac{d\sigma_{\parallel,\perp}}{d\Omega}\right)_{\mathrm{c.m.}}(\nu_2,t)=\left(\frac{d\sigma_{\parallel,\perp}}{d\Omega}\right)_{\mathrm{c.m.}}(\nu_2,t)+\frac{[\mathcal{S}(z_2)]^2}{128\pi^2(2\nu_2+m^2-t/2)}\sum_{i,j=1}^{6}[Q_{\parallel,\perp}(\nu_2{}^2)]_{ij}\,\mathrm{Re}\varphi_i(z_2)\,\mathrm{Re}\varphi_j(z_2)\,. \tag{5.4}$$

Here P and $Q_{\parallel,\perp}$ are real, symmetric, and positive definite 6×6 matrices defined as

$$P_{11}(\nu^2)=\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[\nu_{\mathrm{in}}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}][(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}][(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}\,,$$

$$P_{22}(\nu^2)=P_{33}(\nu^2)=\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2}\,,$$

$$P_{44}(\nu^2)=\frac{[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^3[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[\nu_{\mathrm{in}}{}^2+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}]}{[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^3[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2\nu_{\mathrm{in}}[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}$$

$$P_{55}(\nu^2)=\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2}\,, \tag{5.5}$$

$$P_{46}(\nu^2)=P_{64}(\nu^2)=-\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}](\nu^2-\nu_{\mathrm{min}}{}^2)(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}(\nu_{\mathrm{min}}{}^2)^{1/2}}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^3\nu_{\mathrm{in}}[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}\,,$$

$$P_{66}(\nu^2)=\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}][\nu_{\mathrm{in}}{}^2-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}]}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]\nu_{\mathrm{in}}[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}\,,$$

all the other $P_{ij}=0$ and $Q_{\parallel,\perp}=P\mp R$ with

$$R_{14}(\nu^2)$$
$$=R_{41}(\nu^2)$$
$$=-\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2\nu_{\mathrm{min}}\nu^2}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2\{2\nu_{\mathrm{in}}[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}]\}^{1/2}[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2}\,,$$

$$R_{16}(\nu^2)=R_{61}(\nu^2)=-\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]\,\nu^2[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}]^{1/2}}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}(2\nu_{\mathrm{in}})^{1/2}[\nu_{\mathrm{in}}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2}$$

$$R_{25}(\nu^2)=R_{52}(\nu^2)=-\frac{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}{[(\nu_{\mathrm{in}}{}^2-\nu_B{}^2)^{1/2}-(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]^2[(\nu_{\mathrm{in}}{}^2-\nu_{\mathrm{min}}{}^2)^{1/2}+(\nu_{\mathrm{in}}{}^2-\nu^2)^{1/2}]}\,, \tag{5.6}$$

all the other $R_{ij}=0$. As far as the first terms appearing in Eqs. (5.3) and (5.4) are concerned, they are equal to

$$\left(\frac{d\sigma}{d\Omega}\right)^{\rm UB}_{\rm c.m.}=\frac{[\mathcal{S}(z_2)]^2}{128\pi^2(2\nu_2+m^2-t/2)}\sum_{i,j=1}^{6}P_{ij}(\nu_2{}^2)\,{\rm Im}\,\varphi_i(z_2)\,{\rm Im}\,\varphi_j(z_2) \tag{5.7}$$

and

$$\left(\frac{d\sigma_{\shortparallel,\perp}}{d\Omega}\right)^{\rm UB}_{\rm c.m.}=\frac{[\mathcal{S}(z_2)]^2}{128\pi^2(2\nu_2+m^2-t/2)}\sum_{i,j=1}^{6}[Q_{\shortparallel,\perp}(\nu_2{}^2)]_{ij}\,{\rm Im}\,\varphi_i(z_2)\,{\rm Im}\,\varphi_j(z_2)\,, \tag{5.8}$$

containing therefore the contribution of the absorptive parts. The superscript UB means "unitarity bound" and expresses the fact that these terms yield a natural lower bound on the cross sections[27] due only to unitarity requirements and obtained by setting the real parts ${\rm Re}\,\varphi_i(z_2)$ equal to zero (particularly this bound is trivial, i.e., zero, for points below the threshold $\nu_0{}^2$). One of our purposes was to improve this lower bound by taking into account the nonvanishing contribution of the real parts in a model-independent way, particularly avoiding any specific assumption about subtractions. Since the analyticity structure of the scattering amplitudes was ignored in purely unitarity bounds, a substantial improvement is now to be expected.

In order to derive bounds on the differential cross sections from Eqs. (5.3) and (5.4) one has to solve a standard finite-dimensional optimization problem. It consists in bounding a function

$$F(\xi_i)=\sum_{i,j=1}^{6}\mathcal{R}_{ij}\xi_i\xi_j \tag{5.9}$$

when the real parameters ξ_i satisfy the constraint

$$G(\xi_i)=a\sum_{i=1}^{6}\xi_i{}^2-2\sum_{i=1}^{6}b_i\xi_i-c\leq 0\,. \tag{5.10}$$

In these relations ξ_i denote the parameters ${\rm Re}\,\varphi_i(z_2)$, the matrix $\mathcal{R}$ being either P or $Q_{\shortparallel,\perp}$, depending on the case considered. Both the functions F, which must be extremized, and G, expressing the constraint, are convex functions of ξ_i. Indeed, F is a positive definite quadratic form while G results from the $\vec{L}^2$ norm of an analytic vector which is known to be a convex function of the values taken by the vector components in a specified point. For solving this problem we resort again to the Kuhn-Tucker convex optimization procedure with inequality constraints[20] already applied in Sec. III. For completeness we expose here in more detail the content of this method. Let $\xi_i^{(m)}$ and $\xi_i^{(M)}$ be the optimal values of the parameters ξ_i, realizing the constrained minimum and maximum of F, respectively. The Lagrangian of the problem writes as

$$\mathcal{L}(\xi_i,\lambda)=F(\xi_i)+\lambda G(\xi_i)\,. \tag{5.11}$$

According to the Kuhn-Tucker theorem[20] there exists a value $\lambda_m\geq 0$ and a value $\lambda_M\leq 0$ such that $\xi_i^{(m)}$ and $\xi_i^{(M)}$ are stationary points for the Lagrangian $\mathcal{L}$ in which λ is set equal to λ_m and λ_M, respectively. Moreover, the following alignment conditions hold:

$$\lambda_m G(\xi_i^{(m)})=0,\quad \lambda_m G(\xi_i^{(M)})=0\,. \tag{5.12}$$

Using the expressions (5.9) and (5.10) we obtain from the stationarity conditions $\partial\mathcal{L}/\partial\xi_i=0$ the equations

$$\sum_{j=1}^{6}(\mathcal{R}_{ij}-\lambda_{m,M}a\delta_{ij})\xi_j^{(m),(M)}=\lambda_{m,M}b_i\quad(i=1,\ldots,6)$$

which have the solutions

$$\xi_i^{(m),(M)}=\lambda_{m,M}\sum_{j=1}^{6}[(\mathcal{R}-1a\lambda_{m,M})^{-1}]_{ij}b_j \quad (i=1,\ldots,6) \tag{5.13}$$

(1 denotes the unit 6×6 matrix). From Eqs, (5.12) it follows that either the Lagrange multiplier is zero or the inequality (5.10) is saturated by the optimal values $\xi_i^{(m),(M)}$. The first alternative has to be studied separately and leads to a trivial result: from Eqs. (5.13), $\lambda_{m,M}=0$ imply $\xi_i^{(m),(M)}=0$ which leads to the unitarity bounds (5.7) and (5.8) for the differential cross sections. In almost all the cases investigated by us only the second alternative occurred, i.e., $\lambda_{m,M}\neq 0$, the optimal parameters $\xi_i^{(m),(M)}$ being situated on the boundary of their admissible domain. By introducing into the equality

$$G(\xi_i^{(m),(M)})=0 \tag{5.14}$$

the expressions (5.13) of $\xi_i^{(m),(M)}$, one is finally led to a nonlinear equation for the Lagrange multipliers λ_m and λ_M. The solutions of this equation were found with standard computer programs. Using relations (5.10) and (5.13) one can see that Eq. (5.14) has a unique positive solution λ_m which corresponds to the minimum of F. On the other hand, the number of negative solutions for λ is finite and therefore one may choose unambiguously the value λ_M corresponding to the maximum of F.

A last comment is now in order concerning the weight function w entering the coefficients a, b_i, and c written in Eqs. (5.2). The procedure described above has to be applied for every fixed w, an additional optimization upon w being necessary afterwards. We have taken again w of the form (3.36), repeated the algorithm for various values of α in the interval $(-1,1)$ and finally picked up the best results, i.e., the greatest lower bound and the smallest upper bound on F. As in the previous section, this procedure proved itself quite economical from the point of view of the computational time. The lower bounds varied quite smoothly with α around the best found values while, on the contrary, the upper bounds exhibited a drastic variation.

We have first computed upper and lower bounds on the differential cross section at some points below the threshold ν_0^2 previously investigated also in the framework of Refs. 6 and 7. In connection with the point $\omega = 80.9$ MeV, $\theta_{\text{c.m.}} = 95°$, we have used several values for the threshold ν_{in}^2. When ν_{in}^2 is set equal to ν_0^2, we are in the conditions of Ref. 7 and our results should essentially reproduce the values reported there. This was indeed the case, as it may be seen comparing the values from Table III for $\nu_1^2 = \nu_0^2$ with the corresponding ones from Table 6 of Ref. 7. The very good agreement found offered us an additional confirmation that the simple procedure regarding the weight function w adopted in this section works well. From Table III one sees that when the value taken for the threshold ν_{in}^2 is increased, i.e., when the information about the absorptive parts is enlarged, the lower bound slightly improves while the upper bound worsens severely (which is otherwise expected in view of the maximum modulus principle). The knowledge of the absorptive parts in the region $\nu_0^2 \leq \nu^2 \leq \nu_{\text{in}}^2$ is therefore of not much help in improving the upper bounds on the cross sections at interior points at least unless an additional physical information (for instance the specification of the cross section at several other points below ν_{in}^2) is not taken into account. In the actual formulation it is therefore reasonable to confine ourselves to the consideration of the lower bounds. In Table IV we display the computed lower bounds on the unpolarized cross section (with ν_{in}^2 taken such that the corresponding photon laboratory energy ω_{in} be 1210 MeV) at the laboratory photon energies ω and c.m. scattering angles $\theta_{\text{c.m.}}$ considered in Table 6 of Ref. 7. The present lower bounds are somewhat stronger than the corresponding ones from Ref. 7 (especially at large angles) but still remain too weak to be practically useful as seen from the last column of Table IV where the experimental values are listed.

We now start considering bounds on the differential cross sections in the interval $\nu_0^2 < \nu^2 < \nu_{\text{in}}^2$ above the pion photoproduction threshold. From the computational point of view, the only additional difficulty encountered now when $\nu_2^2 > \nu_0^2$ (i.e., $z_2 > x_0$) is the calculation of the principal value integrals appearing in Eqs. (5.2). We have applied a standard numerical procedure writing

$$\frac{\mathrm{P}}{\pi}\int_{x_0}^{1}\frac{g(x)}{x-z_2}dx=\frac{1}{\pi}g(z_2)\ln\frac{1-z_2}{z_2-x_0}+\frac{1}{\pi}\int_{x_0}^{1}\frac{g(x)-g(z_2)}{x-z_2}dx$$

and computing the last integral using quadrature programs. For the double integral entering the coefficient c in Eqs. (5.2) the above formula was applied twice.

Except for the bounds on the unpolarized differential cross section $d\sigma/d\Omega$ we have also calculated bounds on the differential cross sections for polarized photons for which some experimental data are reported in Ref. 28 and 29. The results of our calculations are presented at several angles $\theta_{\text{c.m.}}$ in Figs. 1–9. Only the lower bounds (computed with a threshold ν_{in}^2 corresponding to $\omega_{\text{in}} = 1210$ MeV) are graphically represented since the upper bounds turned up again to be by an order of magnitude too high. For reference, we have drawn in Figs. 1–6 and 9, together with the calculated lower bounds, experimental points taken from Refs. 25, 26, and 28. As a general feature one notices that the lower bounds are very close to the experimental values and, at the same time, very near to the semiphenomenologic bounds discussed in Sec. VIII of Ref. 7. However, unlike the latter bounds, the present ones are rigorous and resulted from a consistent exploitation of the fixed-t analyticity properties of the amplitudes and of the

TABLE III. Upper and lower bounds on $(d\sigma/d\Omega)_{\text{c.m.}}$ (units of 10^{-32} cm^2) at $\omega = 80.9$ MeV and $\theta_{\text{c.m.}} = 95°$, calculated with several values for the threshold ω_{in}.

ω_{in}	Lower bound	Upper bound
$\omega_0 = \mu + \mu^2/2m$	0.40	7.2
400 MeV	0.49	23.7
1210 MeV	0.51	323.7

TABLE IV. Lower bounds on $(d\sigma/d\Omega)_{\text{c.m.}}$ (units of 10^{-32} cm^2) at several energies below the threshold of single-pion photoproduction, obtained for $\omega_{\text{in}} = 1210$ MeV.

ω(MeV)	$\theta_{\text{c.m.}}$	Lower bound	Experimental value
80.9	95°	0.5	1.16 ± 0.06 (Ref. 30)
109.9	95.6°	0.5	1.03 ± 0.06 (Ref. 30)
97.0	150°	0.1	2.57 ± 0.51 (Ref. 31) ($\theta_{\text{c.m.}} = 138.7°$)
111.1	152.7°	0.1	1.70 ± 0.07 (Ref. 30)

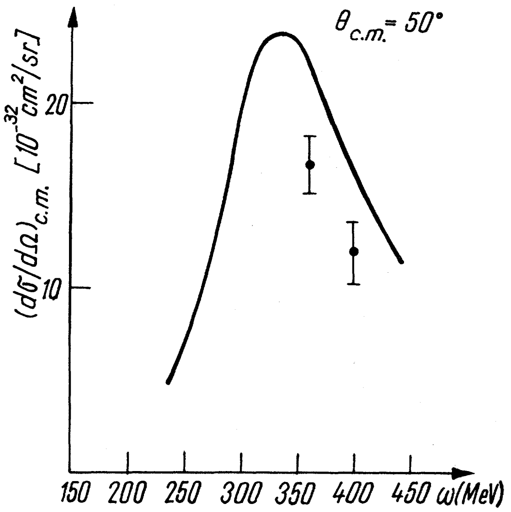

FIG. 1. Energy dependence of the c.m. unpolarized differential cross section at c.m.-scattering angle $\theta_{c.m.} = 50°$. The continuous line is the lower bound computed in the present paper; experimental data taken from Table III of Ref. 26.

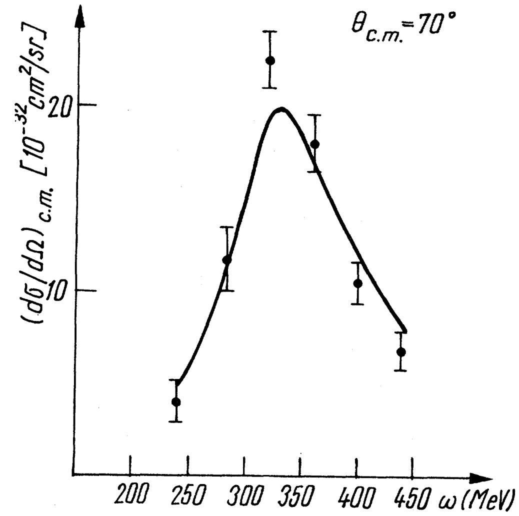

FIG. 3. The same as in Fig. 1 for $\theta_{c.m.} = 70°$.

consequences of the (two particle) unitarity condition.

An important output of our calculations as seen from Figs. 1, 3, and 4 is that in some cases ($\theta_{c.m.} = 50°$, $\omega = 360$ MeV, and $\omega = 400$ MeV; $\theta_{c.m.} = 70°$, $\omega = 400$ MeV, and $\omega = 440$ MeV; $\theta_{c.m.} = 90°$, $\omega = 214$ MeV, and $\omega = 320$ MeV) the lower bounds appear to lie definitely above the experimental points (with the quoted error bars included). These violations clearly reveal that the results of the pion-photoproduction multipole analyses used in the evaluation of the bounds and at least some of the experimental data for the Compton-scattering unpolarized differential cross section in the N^*-resonance region are at variance. This conclusion, previously reached in a phenomenological way in Ref. 27 on the basis of a purely unitarity bound and afterwards strengthened within the approach from Sec. VIII of Ref. 7 (where analyticity requirements were somehow taken care of but in a very crude way) is now reached rigorously. In order to make sure that the violations are really significant, one has, of course, to estimate the

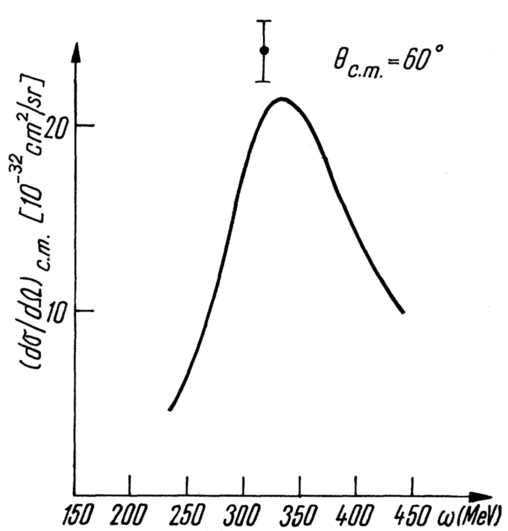

FIG. 2. The same as in Fig. 1 for $\theta_{c.m.} = 60°$.

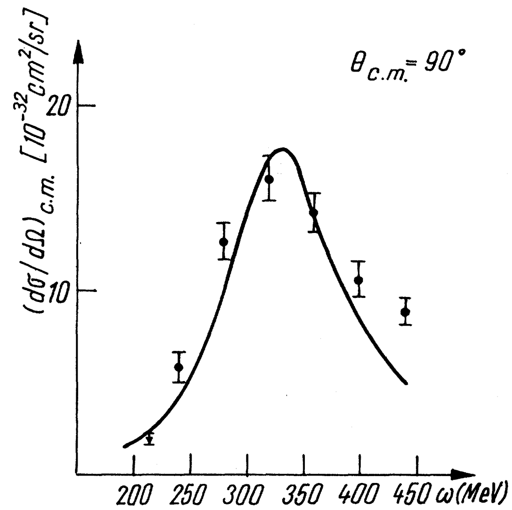

FIG. 4. The same as in Fig. 1 for $\theta_{c.m.} = 90°$, except the experimental point at 214 MeV which is taken from Ref. 25.

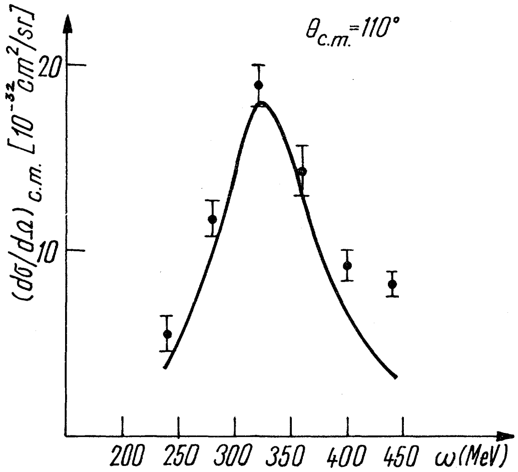

FIG. 5. The same as in Fig. 1 for $\theta_{c.m.}=110°$.

uncertainties affecting the calculation of the lower bounds. Keeping fixed the imaginary parts from Eq. (2.1) (which are being tested), the possible uncertainties originate mainly in the experimental errors on the cross section for $\nu^2 \geq \nu_{in}^2$ (the errors introduced by numerical integrations, etc., were estimated to be not greater than 2–3%). To get a corridor of variation for the computed lower bounds, we have varied the input cross-section data inside reasonable limits and established how this change affects the output. A variation with ±10% of the cross section given on the boundary ($\nu^2 \geq \nu_{in}^2$) had an effect not greater than ∓5%. An overall error of approximately 5–7% seems therefore to be a most pessimistic estimation. The discrepancies mentioned above still persist even if we decrease the lower bounds by such an amount. Also they are still present if one

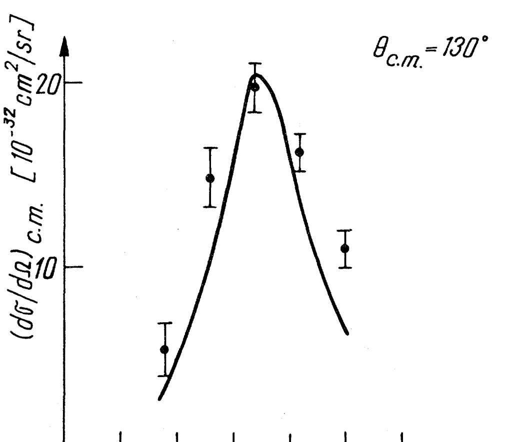

FIG. 6. The same as in Fig. 1 for $\theta_{c.m.}=130°$.

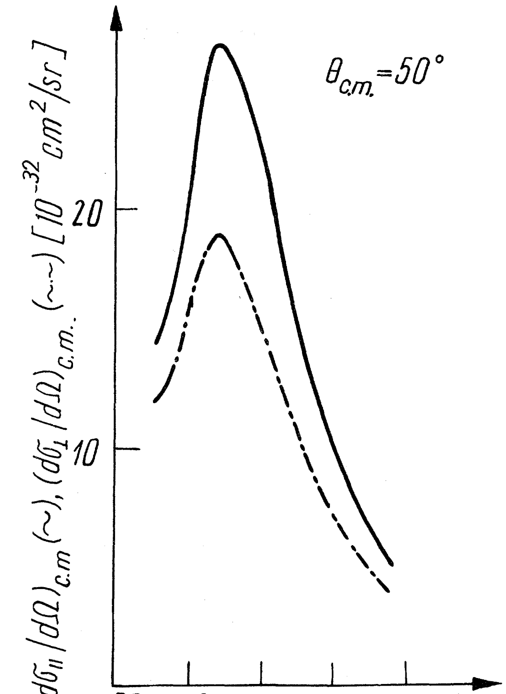

FIG. 7. Energy dependence of the c.m. polarized differential cross sections at c.m.-scattering angle $\theta_{c.m.}=50°$. The curves represent the lower bounds computed in the present paper.

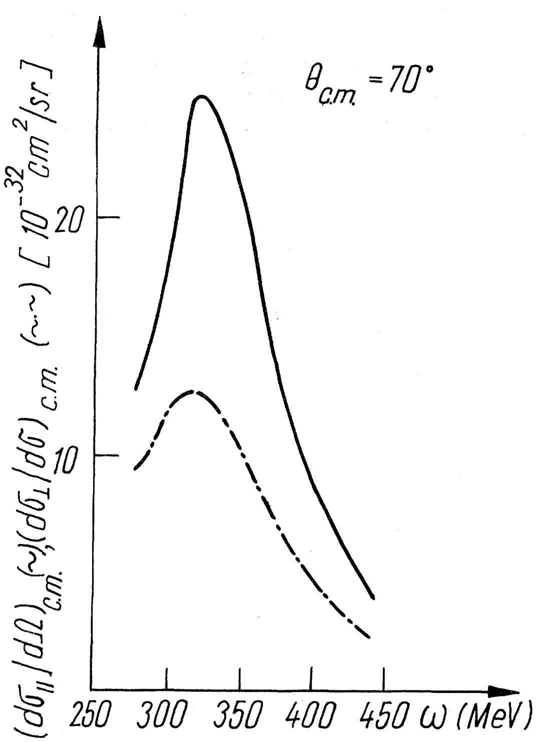

FIG. 8. The same as in Fig. 7 for $\theta_{c.m.}=70°$.

lowers the threshold ν_{in}^2 up to a value corresponding to the laboratory photon energy $\omega_{in} \simeq 600$ MeV. The conclusions are therefore quite stable against various sources of bias in the numerical consideration of the bounds.

We have found that an overall increase and decrease of 5% of the imaginary parts $\operatorname{Im}\overline{A}_i(\nu^2, t)$ led to a change of approximately ±10% in the lower bounds.

VI. SOME COMMENTS

In this paper we have presented a modern dispersion approach to proton Compton scattering able to yield results unaffected by the usual model dependence introduced by subtractions and the accompanying annihilation-channel contributions. The lower bounds on the (polarized or unpolarized) differential cross sections appear very restrictive and improve substantially previous constraints based on unitarity alone. In some cases the bounds are clearly violated and thereby show inconsistencies between results of single-pion-photoproduction multipole extractions and some Compton-scattering data. The upper bounds come out disappointingly poor but possible ways remain open to strengthen them through appropriate modifications of the initial statement of the problem so as to include additional physical information. For instance, one may try solving the problem of finding similar constraints (it is hoped, optimal) in the case in which apart from the knowledge of the six absorptive parts, along the interval $[\nu_0^2, \nu_{in}^2]$ of the unitarity cut, one takes as known the unpolarized differential cross section all along the interval $[\nu_0^2, \infty)$ and not only along $[\nu_{in}^2, \infty)$ as considered in this work.

We end with the remark that the techniques employed here in order to exploit optimally the fixed-t analyticity properties of the Compton amplitudes may, with various degrees of utility, be used in connection with other hadron reactions as well.

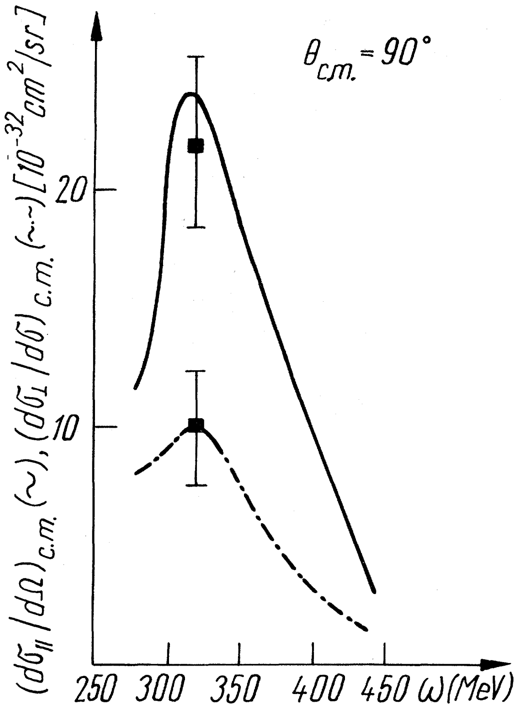

FIG. 9. The same as in Fig. 7 for $\theta_{c.m.} = 90°$; experimental values taken from Ref. 28.

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