Space-time symmetries of confined quarks

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The first paragraph of Rotbart's recent paper is amplified. It is shown that the minimum symmetry for the space-time separation of the confined quarks is that of the O(3)-like little group of the Poincaré group for massive hadrons. It is then pointed out that the existence of additional coordinate dependences is determined from the way in which specific models are constructed.

In his recent paper,[1] Rotbart presented an interesting calculation on the complete set of relativistic harmonic-oscillator wave functions on the O(1,1) space which, because of the separability of the oscillator wave functions in the Cartesian coordinate system, can be trivially extended to the full O(3,1) Minkowskian space. Rotbart's Eq. (15) is correct, and reproduces the simpler calculation of Ruiz[2] and Kim *et al.*[3] if timelike excitations are suppressed in the Lorentz frame in which the hadron is at rest.

In the same paper, Rotbart stated that his calculation is useful in the relativistic oscillator models in which timelike excitations exist, and mentioned the model of Horwitz and Piron as an example.[4,5] He mentioned also the model discussed by Kim and Noz[6] as an example which does not contain timelike oscillations in the hadronic rest frame. While carefully avoiding the question of which model is consistent with the real world, Rotbart attempts to justify his calculation as a physically meaningful effort based on the requirement of "completeness" in the O(3,1) relativistic space-time.

Although Rotbart did not elaborate on the question of how much completeness is needed under what circumstances, his paper brings back to the surface a much broader question of whether imposing a constraint or subsidiary condition can be consistent with the accepted principles of quantum mechanics and special relativity. Indeed, since the appearance of Dirac's classic paper on "forms of relativistic dynamics",[7] this has been one of the most sensitive issues among model builders of relativistic quantum mechanics.[8,9]

The purpose of this paper is to set a criterion for determining how much completeness is needed. We attack this problem by examining the *minimum* space-time symmetry needed for relativistic dynamics of quarks confined inside a hadron with definite mass and spin. We then discuss what additional symmetries are needed to meet the requirements of specific models.

In order to see the minimum space-time symmetry needed for relativistic dynamics of quarks, let us consider a hadron consisting of two confined quarks whose space-time coordinates are x_1 and x_2, respectively. Then the standard procedure is to use the variables X and x defined as

$$X = (x_1 + x_2)/2 \tag{1}$$

and

$$x = (x_1 - x_2)/2\sqrt{2}\ . \tag{2}$$

The four-vector X specifies the space-time coordinate of the hadron. The variable x is the space-time separation between the quarks.

The basic space-time symmetry governing this two-quark system is of course that of the inhomogeneous Lorentz group or the Poincaré group.[8,10] This group consists of four space-time translations, three rotations, and three boosts. Let us consider first a translation on x_1 and x_2:

$$x_1 \to x_1 + a \ , \quad x_2 \to x_2 + a \ . \tag{3}$$

Then

$$X \to X + a \ . \tag{4}$$

However, the separation coordinate remains invariant under this translation:

$$x \to x \ . \tag{5}$$

We can therefore write the generators of the four translations as

$$P_\mu = i\partial/\partial X^\mu \ , \tag{6}$$

This is the four-momentum operator for the hadron.

Let us next consider rotations and boosts. The generators of these transformations are contained in the expression

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \ , \tag{7}$$

where

$$L_{\mu\nu} = i(X_\mu \partial/\partial X^\nu - X_\nu \partial/\partial X^\mu) \ ,$$

$$S_{\mu\nu} = i(x_\mu \partial/\partial x^\nu - x_\nu \partial/\partial x^\mu) \ .$$

It is quite clear from Eqs. (5) and (6) that the x coordinate is invariant under translation. This invariance leads us to a "temptation" to conclude that the space-time symmetry of the x coordinate is that of the homogeneous Lorentz group (without translation): O(3,1). We would like to point out here that this is not the case.[11]

In constructing representations of the Poincaré group generated by the operators given in Eqs. (6) and (7), the standard procedure is to find the state vectors which are diagonal in the Casimir operators[12]:

$$P^2 = P^\mu P_\mu \text{ and } W^2 = W^\mu W_\mu \ , \tag{8}$$

where

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} P^\nu M^{\alpha\beta} \ .$$

These operators commute with all of the ten generators given in Eqs. (6) and (7). The operator P^2 specifies (mass)^2 of the hadron, while W^2 is (mass $\times$spin)^2 for the hadronic system.

Strictly from the symmetry point of view, the eigenvalues of the above Casimir operators are the only Lorentz-invariant dynamical variables which can be extracted from the hadronic system.[11,12]

Let us then look at the expressions in Eq. (8) more closely. As was noted above, P^2 does not contain the x variable. As for W^2, it is determined from the expression for W_μ, which takes its simplest form in the Lorentz frame where the hadron is at rest. Then W^2 depends only on the rotation operator in the three-dimensional Euclidean space spanned by three spatial components of x, and does not depend on its time component. This O(3) space "perpendicular" to the hadronic four-momentum is indeed that of the *little group*[10] which leaves the hadronic momentum invariant.[13]

We can conclude therefore that the minimum space-time symmetry for the confined quarks is the *relativistic* O(3) as defined above. This probably is the reason why the calculations based on nonrelativistic quantum mechanics, which preserves the O(3) symmetry, are so successful in describing the real world,[14] in spite of the widespread misunderstanding that the O(3) symmetry is inherently nonrelativistic and that the basic relativistic space-time symmetry has to be O(3,1).

The above conclusion, however, does not prevent us from adding a dependence on the time component of the x variable. This is indeed the case when we require the covariance of the model.[4,6,15] The question then is how much time dependence is needed. In the harmonic-oscillator model discussed by Kim and Noz,[6,16] the time dependence is introduced through the ground-state wave function for the time-separation variable in the coordinate system in which the hadron is at rest. This procedure is consistent with the "c-number" nature of the time variable,[6,17] and with the experimental observation that there is a lower limit on the (mass)^2 spectrum. The existence of this ground-state—like time dependence has been experimentally confirmed in the parton phenomenon both qualitatively[18] and quantitatively.[19]

If the model discussed by Kim and Noz is so consistent with everything, why do we need Rotbart's calculation? Here are the reasons.

(a) Even in the standard approach of Kim and Noz, the absence of the timelike oscillation in one Lorentz frame does not mean its absence in all other frames. Those timelike states are linearly combined in such a way that there are no timelike excitations in that Lorentz frame in which the hadron is at rest.[3]

(b) Although Hussar's calculation[19] leads to a reasonably accurate proton structure function, it is quite possible that the real world is not exactly like a ground-state harmonic-oscillator wave function.

In this case, we have to consider a complete-orthonormal-set expansion, even if we impose the lower limit on the (mass)^2 spectrum. This is so even if we believe in Dirac's assertion that the time variable is a *c*-number, because the uncertainty principle does not specify the functional form.[17]

(c) The O(3) restriction discussed here is applicable only to the relative coordinate for the confined quarks. In constructing relativistic dynamics, we have to consider also other cases such as free relativistic particles with the Newton-Wigner problem.[4] In these general cases, where the complete Hilbert space on the IO(3,1) space is desired, the timelike oscillation should be taken into consideration.

(d) There is growing evidence that the present form of quantum field theory is not suitable for relativistic bound states.[20] For this reason, there are theoretical models in which confined quarks are described by relativistic wave functions, while hadrons are regarded as particles that can be explained by lines in Feynman diagrams.[21] In this case, we have to consider off-mass-shell hadrons. This may require timelike excitations.

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[2]M. J. Ruiz, Phys. Rev. D 10, 4306 (1974).

[3]Y. S. Kim, M. E. Noz, and S. H. Oh, J. Math. Phys. 21, 1224 (1980).

[4]L. P. Horwitz and C. Piron, Helv. Phys. Acta 46, 316 (1973).

[5]In Ref. 1, Rotbart quotes an unpublished paper by L. P. Horwitz and F. C. Rotbart [now published in Phys. Rev. D 24, 2127 (1981)]. However, prior to submittal of our paper, we did not have the benefit of seeing this work. For other papers mentioning timelike oscillations, see R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971); I. Sogami and H. Yabuki, Phys. Lett. 94, 157 (1980).

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[7]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[8]For some of the early papers dealing with the constraint or subsidiary conditions, see H. Yukawa, Phys. Rev. 91, 416 (1953); M. Markov, Nuovo Cimento Suppl. 3, 760 (1956); T. Takabayasi, Nuovo Cimento 33, 668 (1964); Prog. Theor. Phys. 34, 124 (1965); S. Ishida, *ibid.* 46, 1570 (1971); 46, 1905 (1971); K. Rafanelli, Phys. Rev. D 9, 2746 (1974); A. Hanson and T. Regge, Ann. Phys. (N.Y.) 87, 498 (1974). See also Ref. 6.

[9]For some of the recent articles which appeared in the Physical Review, see A. Komar, Phys. Rev. D 18, 1881 (1978); 18 1887 (1978); 18, 3617 (1978); H. Crater, *ibid.* 18, 2872 (1978); P. Droz-Vincent, *ibid.* 19, 702 (1979); F. Rohrlich, *ibid.* 23, 1305 (1981); N. Mukunda, A. P. Balachandran, J. S. Nilson, and E. C. G. Sudarshan, *ibid.* 23, 2189 (1981); A. Kihlberg, R. Marnelius, and N. Mukunda, *ibid.* 23, 2201 (1981); N. Mukunda and E. C. G. Sudarshan, *ibid.* 23, 2210 (1981); E. C. G. Sudarshan, N. Mukunda, and J. N. Goldberg, *ibid.* 23, 2218 (1981); J. N. Goldberg, E. C. G. Sudarshan and N. Mukunda, *ibid.* 23, 2231 (1981).

[10]E. P. Wigner, Ann. Math. 40, 149 (1939).

[11]For an early discussion of the point that temptations do not always lead to the truth in the case of the Poincaré group which is not a direct product of translation and Lorentz transformation, see L. Michel, in *Group Theoretical Concept and Methods in Elementary Particle Physics,* edited by F. Gursey (Gordon and Breach, New York, 1963). See also Y. S. Kim, M. E. Noz, and S. H. Oh, Am. J. Phys. 47, 892 (1979).

[12]Y. S. Kim, M. E. Noz, and S. H. Oh, J. Math. Phys. 20, 1341 (1979).

[13]The concept of "little group" is well known in connection with point particles with intrinsic angular momentum (Ref. 10). In this paper, we are discussing the case where the internal orbital motion of spinless quarks produces the hadronic spin.

[14]For some of the latest papers on this subject, see N. Isgur and G. Karl, Phys. Rev. D 19, 2653 (1978); D. P. Stanley and D. Robson, Phys. Rev. Lett. 45, 235 (1980). The subject of calculating the hadronic mass spectra using nonrelativistic bound-state quantum mechanics is by now a well-established scientific discipline, and is ready for classroom consumption in the traditional physics curriculum. For articles written for teaching purposes, see P. E. Hussar, Y. S. Kim, and M. E. Noz, Am. J. Phys. 48, 1038 (1980); 48, 1043 (1980). See also O. W. Greenberg, Am. J. Phys. (to be published).

[15]For an attempt to construct a covariant model without the time dependence, see A. L. Licht and A. Pagnamenta, Phys. Rev. D 2, 1150 (1970); 2, 1156 (1970).

[16]For a review-oriented article on the covariant oscillator model, see D. Han and Y. S. Kim, Am. J. Phys. 49, 1157 (1981).

[17]P. A. M. Dirac, Proc. R. Soc. London A114, 243 (1927); A114, 710 (1927). See also E. P. Wigner, in *Aspects of Quantum Theory, in Honour of P. A. M. Dirac's 70th Birthday,* edited by A. Salam and E. P. Wigner (Cambridge University Press, London, 1972). Dirac's point is very simple. There is an uncertainty relation between time and energy variables. However, neither variable is associated with the Hilbert space. This means that there are no excited states.

[18]Y. S. Kim and M. E. Noz, Phys. Rev. D 15, 335

(1977).
[19]P. E. Hussar, Phys. Rev. D 23, 2781 (1981).
[20]For instance, it is by now a well-accepted view that quantum chromodynamics, which is a form of field theory, can give corrections to the structure function of the proton, but does not give the structure function to which the QCD corrections are made. For a comprehensive discussion of this point, see R. C. Hwa, Phys. Rev. D 22, 759 (1980); R. C. Hwa and M. S. Zahir, *ibid.* 23, 2539 (1981).
[21]For some of the papers dealing with this problem, see I. Sogami, Prog. Theor. Phys. 50, 1729 (1973); Y. S. Kim, Phys. Rev. D 14, 273 (1976); T. J. Karr, Ph.D. Thesis, University of Maryland, 1976 (unpublished); A. Z. Capri and C. C. Chiang, Nuovo Cimento 36A, 191 (1976); Prog. Theor. Phys. 59, 996 (1978); G. Pocsik, Acta Phys. Austriaca 49, 47 (1978); Nuovo Cimento 54, 413 (1979); K. Mita, Phys. Rev. D 18, 4545 (1978).