Do quarks and leptons know a simple group?

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(Received 25 November 1981)

Without using the requirement of a single coupling constant for grand unified theory (GUT), we prove the uniqueness of SU(5) and SO(10). Hence, the way of having the family structure is $G = G(\text{family}) \otimes G(\text{GUT})$ only. There exist no simple or nontrivial semisimple groups which unify both the family structure and GUT.

In our last three papers,[1,2] we have shown the uniqueness of SU(5) and SO(10) grand unified theories (GUT's) *among simple groups* using only the ordinary quantum numbers of quarks and leptons under the electroweak group $SU_L(2) \otimes U(1)$.[3] The proof was done without assuming the color group to be SU(3). The same proof goes through and leads to SO(10), even if the left-right-symmetric group $SU_L(2) \otimes SU_R(2) \otimes U(1)_{B-L}$ (Ref. 4) is used. The reason is that the U(1) generator of $SU_L(2) \otimes U(1)$ corresponds to the sum of generators $(I_R)_3 + \frac{1}{2}(B-L)$ in the left-right-symmetric model. Since we can have only SU(5) with $\underline{\bar{5}} + \underline{10}$ or SO(10) with $\underline{16}$, it is impossible to have a simple group which also unifies the family structure. We just have repetitions of one family.

In the case of the Pati-Salam-type grand unification [i.e., grand unification of particles (GUP), not grand unification of particles and antiparticles (GUPA)], we have shown that it is possible to have models with three generations,[5] although groups are *semisimple (two or more coupling constants possibly), not simple (single coupling constant)*. Therefore, it is natural to try semisimple groups for GUPA. Of course, if $G = G(\text{family}) \otimes G(\text{GUT})$,[6] it is obvious that one has the family structure. Here, we look for a nontrivial way, i.e., $G = G_1 \otimes G_2 \otimes \cdots \otimes G_N \otimes U(1)$, where none of G_j is a family group. This is the topic of this paper.

I. ASSUMPTIONS

Using the fact that our multiplet is finite dimensional and eigenvalues are real, the group must be of the form[7]

$$G = G_1 \otimes G_2 \otimes \cdots \otimes U(1) \otimes U(1) \otimes \cdots \otimes U(1)\,, \quad (1.1)$$

where G_j are simple groups. We assume that G is

$$G = G_1 \otimes G_2 \otimes \cdots \otimes G_N \otimes U(1)\,. \quad (1.2)$$

The quantum numbers of quarks, U and D, and leptons, N and E, under $SU_L(2) \otimes U(1)$ are as follows:

	G_C	I	I_3	Y	No.
N_L	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	n_l
E_L	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	n_l
N_L^C	1	0	0	0	n_0
E_L^C	1	0	0	1	n_l
U_L	m	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	mn_q
D_L	m	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	mn_q
U_L^C	$\bar{m}$	0	0	$-\frac{2}{3}$	mn_q
D_L^C	$\bar{m}$	0	0	$\frac{1}{3}$	mn_q

where m denotes the multiplicity under the color group G_C. Electric charges are $(Q_N, Q_E) = (0, -1)$ and $(Q_U, Q_D) = (\frac{2}{3}, -\frac{1}{3})$. The numbers n_l and n_q denote the numbers of lepton and quark families, respectively. We assume the anomaly-free condition for GUT.[2]

II. UNIQUENESS OF SU(5) AND SO(10)

A. Lemmas

We develop a few lemmas for our purpose.[8]

Lemma 1. For $G = G_1 \otimes G_2 \otimes \cdots \otimes G_N \otimes U(1)$ where each G_j has n_j distinct eigenvalues, we have at least

$$\sum_{j=1}^{N} n_j - N + 1$$

distinct eigenvalues as a whole.

Proof. We denote n_j distinct eigenvalues as b_j^i $(1 \le i \le n_j)$ where $b_j^1 > b_j^2 > \cdots > b_j^{n_j}$. Then, we have the following ordered sequence of eigenvalues:

$$\sum_{j=1}^{N} b_j^1 > b_1^2 + \sum_{j=2}^{N} b_j^1 > b_1^3 + \sum_{j=2}^{N} b_j^1 > \cdots > b_1^{n_1} + \sum_{j=2}^{N} b_j^1 > b_1^{n_1} + b_2^2 + \sum_{j=3}^{N} b_j^1 > b_1^{n_1} + b_2^3 + \sum_{j=3}^{N} b_j^1 > \cdots > \sum_{j=1}^{N} b_j^{n_j}$$

The total number of these eigenvalues is

$$\sum_{j=1}^{N} n_j - N + 1 .$$

Lemma 2. The group G is the same as Lemma 1. The smallest number of eigenvalues m as a whole is

$$m = N + 1 .$$

Proof. Use the fact that $n_j \ge 2$, since $\mathrm{tr} X_j = 0$ where $X_i \in G_j$.

Lemma 3. The group G is the same as Lemma 1. The largest number of eigenvalues as a whole is

$$\prod_{j=1}^{N} n_j \quad (\ge 2^N) .$$

Therefore, in order to have m distinct eigenvalues, we must satisfy

$$N + 1 \le \sum_{j=1}^{N} n_j - N + 1 \le m \le \prod_{j=1}^{N} n_j . \qquad (2.1)$$

For the case where there are six eigenvalues, we have

$$N + 1 \le \sum_{j=1}^{N} n_j - N + 1 \le 6 \le \prod_{j=1}^{N} n_j . \qquad (2.2)$$

From this equation, we see

$$N \le 5 . \qquad (2.3)$$

Hereafter, we assume without loss of generality that

$$n_j \ge n_{j+1} \quad (j = 1, 2, \ldots, N-1) . \qquad (2.4)$$

Lemma 4. The group G is the same as Lemma 1. If we have only m eigenvalues where

$$m = \sum_{j=1}^{N} n_j - N + 1 \quad (N \ne 1) ,$$

these eigenvalues are equally spaced.

Proof. Consider the following two sequences:

$$\sum_{j=1}^{N} b_j^1 > b_1^2 + \sum_{j=2}^{N} b_j^1 > \cdots ,$$

$$\sum_{j=1}^{N} b_j^1 > b_2^2 + b_1^1 + \sum_{j=3}^{N} b_j^1 > \cdots ,$$

where each has $m = \sum_{j=1}^{N} n_j - N + 1$ eigenvalues. Hence, we must have

$$b_1^1 - b_1^2 = b_2^1 - b_2^2 .$$

Similarly, we can show that $b_j^i - b_j^{i+1}$ is the same for all j.

Since eigenvalues of Y are six in number and not equally spaced, we discuss only the following groups for G_Y which contain Y as a generator:

$$N = 4: \; n_1 = n_2 = n_3 = n_4 = 2 , \qquad (2.5)$$

$$N = 3: \; n_1 = n_2 = n_3 = 2 , \qquad (2.6)$$

$$n_1 = 3, \; n_2 = n_3 = 2 , \qquad (2.7)$$

$$N = 2: \; n_1 = 3, \; n_2 = 2 , \qquad (2.8)$$

$$n_1 = 3, \; n_2 = 3 , \qquad (2.9)$$

$$n_1 = 4, \; n_2 = 2 , \qquad (2.10)$$

$$N = 1: \; n_1 = 6 , \qquad (2.11)$$

which are derived from Eq. (2.2) and Lemma 4.

B. Examination of G_Y

Here, we discuss the group G_Y which contains Y as a generator. For $G_Y = G_1 \otimes G_2 \otimes \cdots \otimes G_N \otimes \mathrm{U}(1)$, the generator Y is given by

$$Y = \sum Y_i + c ,$$

where $Y_j \in G_j$ and c comes from U(1).

1. $G_Y=G_1 \otimes G_2 \otimes U(1)$

For the case where $n_1=3$ and $n_2=2$, we have four distinct eigenvalues, $b_1^1+b_2^1>b_1^1+b_2^2>b_1^2+b_2^2>b_1^3+b_2^2$, and $b_1^2+b_2^1>b_1^3+b_2^1$. Eigenvalues of Y $(1, \frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{2}, -\frac{2}{3})$ are spaced as

$$\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{1}{6} . \quad (2.12)$$

Hence, the solution is

$$b_1^1-b_1^2=\frac{5}{6},\ b_1^2-b_1^3=\frac{1}{6},\ b_2^1-b_2^2=\frac{2}{3} . \quad (2.13)$$

Quantum-number assignments yield

$$n_1^1 n_2^1=n_l,\ n_1^1 n_2^2=mn_q,\ n_1^2 n_2^1=2mn_q ,$$
$$n_1^3 n_2^1=n_0,\ n_1^2 n_2^2=2n_l,\ n_1^3 n_2^2=mn_q , \quad (2.14)$$

where n_j^i denotes the dimension of eigenvalue b_j^i. From Eq. (2.14), we see

$$n_1^1=n_1^3=\frac{1}{2}n_1^2,\ n_1^1=n_2^2 , \quad (2.15)$$

and

$$n_l=mn_q,\ n_l=n_0 . \quad (2.16)$$

Using $\mathrm{tr}Y_j=0$ and Eqs. (2.15) and (2.16), we obtain

$$b_1^1=\frac{2}{3},\ b_1^2=-\frac{1}{6},\ b_1^3=-\frac{1}{3} ,$$
$$b_2^1=\frac{1}{3},\ b_2^2=-\frac{1}{3} . \quad (2.17)$$

It is easy to see that we do not need U(1) in this case. However, the solution is not anomaly-free, since $\mathrm{tr}Y_1^3\neq 0$.

For the case where $n_1=3$ and $n_2=3$, we denote $a=b_1^1-b_1^2$, $b=b_1^2-b_1^3$, $c=b_2^1-b_2^2$, $d=b_2^2-b_2^3$. Since we have at least five distinct eigenvalues ($b_1^1+b_2^1>b_1^2+b_2^1>b_1^3+b_2^1>b_1^3+b_2^2>b_1^3+b_2^3$) and the spacing is given by Eq. (2.12), four cases are examined:

(1) $a=\frac{5}{6}$, $b=\frac{1}{6}$, $c=\frac{1}{2}$, $d=\frac{1}{6}$;

(2) $a=\frac{2}{3}$, $b=\frac{1}{3}$, $c=\frac{1}{2}$, $d=\frac{1}{6}$;

(3) $a=\frac{2}{3}$, $b=\frac{1}{6}$, $c=\frac{2}{3}$, $d=\frac{1}{6}$;

(4) $a=\frac{2}{3}$, $b=\frac{1}{6}$, $c=\frac{1}{6}$, $d=\frac{2}{3}$.

However, it turns out that in each case, we have seven or more eigenvalues.

For the case where $n_1=4$ and $n_2=2$, we have at least five distinct eigenvalues ($b_2^1+b_2^1>b_1^2+b_2^1>b_1^3+b_2^1>b_1^4+b_2^1>b_1^4+b_2^2$). In the same way as above, we can show that we have seven or more eigenvalues.

2. $G_Y=G_1 \otimes G_2 \otimes G_3 \otimes U(1)$

For the case where $n_1=n_2=n_3=2$, we have at least four distinct eigenvalues ($b_1^1+b_2^1+b_3^1>b_1^2+b_2^1+b_3^1>b_1^2+b_2^2+b_3^1>b_1^2+b_2^2+b_3^2$). Hence, we examine six cases:

(1) $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{6}$;

(2) $a=\frac{2}{3}$, $b=\frac{5}{6}$, $c=\frac{1}{6}$;

(3) $a=\frac{2}{3}$, $b=\frac{1}{6}$, $c=\frac{5}{6}$;

(4) $a=\frac{5}{6}$, $b=\frac{2}{3}$, $c=\frac{1}{6}$;

(5) $a=\frac{5}{6}$, $b=\frac{1}{6}$, $c=\frac{2}{3}$;

(6) $a=\frac{2}{3}$, $b=\frac{1}{3}$, $c=\frac{2}{3}$;

where $a=b_1^1-b_1^2$, $b=b_2^1-b_2^2$, $c=b_3^1-b_3^2$. We can show that in each case, we have seven or more eigenvalues.

For the case where $n_1=n_2=n_3=2$, we can show that it is impossible to have six eigenvalues in the same way as above.

3. $G_Y=G_1 \otimes G_2 \otimes G_3 \otimes G_4 \otimes U(1)$

It is impossible to have six distinct eigenvalues, using the same argument as above.

4. $G_Y=G_1 \otimes U(1)$

The trace identity, $\mathrm{tr}Y_1=0$, yields

$$0=(-\tfrac{1}{2}+x)2n_l+xn_0+(1+x)n_l$$
$$+(\tfrac{1}{6}+x)2mn_q+(-\tfrac{2}{3}+x)mn_q+(\tfrac{1}{3}+x)mn_q ,$$
$$0=x(4mn_q+3n_l+n_0) .$$

Hence, $x=0$, i.e., there is no U(1).

We have shown that G_Y must be *a simple group* where $G_Y \in Y$: Hypercharges of quarks and leptons pick G_Y as a simple group.

C. Incompatibility of $G \supset G_l \otimes G_Y$

In the case of GUP, we can have the possibility of unifying the family structure, using either

$G \supset G_0$ where G_0 is simple and both I_3 and Y are contained in G_0 or $G \supset G_I \otimes G_Y$ where G_I and G_Y are simple and $I_3 \in G_I$ and $Y \in G_Y$.[5] In GUPA, we can show the incompatibility of $G \supset G_I \otimes G_Y$ easily: If so, we would have the electric charges $\frac{3}{2}$, $-\frac{1}{6}$, $\frac{5}{6}$, etc. Therefore, only the case where a simple group G_0 contains both I_3 and Y as generators is allowed in GUPA.

D. Incompatibility of $G \supset G_C \otimes G_0$

In subsection C, we have shown that G_0 contains both I_3 and Y as generators. It is easy to see that G_0 must contain G_C (color group) as its subgroup, since otherwise we would have colored leptons. Hence, *a simple group G_0 can be regarded as a grand unification group*. Now, we can prove the uniqueness of SU(5) and SO(10), using Ref. 1.

III. CONCLUSIONS

We have shown the uniqueness of SU(5) and SO(10), without using the requirement of a single coupling constant in GUT. The use of a simple group for grand unification is a consequence of hypercharges of quarks and leptons. Therefore, *the way of having the family structure is $G = G$ (family) $\otimes G$(GUT) only.*[6] We cannot have G as a simple or semisimple group, using just ordinary quarks and leptons, except the form above. Various attempts for G as a simple group *necessarily* lead to the introduction of unfamiliar quantum numbers in $SU_L(2) \otimes U(1)$.[9] Although this fact has been known by practices, we have proved it here. In the proof, we have used the value of Y for quark doublets, which yields $(Q_U, Q_D) = (\frac{2}{3}, -\frac{1}{3})$. Once we find G is simple, then we need not assume eigenvalues of Y for quarks as was done in Ref. 1.

Our results may imply the compositeness of quarks and leptons, if we believe in G as a simple group and believe that G must produce the family structure. This is one of the ways we can take here.

Note added in proof. For the case where $n_0 = 0$, i.e., no N_L^c, the proof holds except for $N = 2$, $n_1 = 3$, $n_z = 2$ with $b_1^1 - b_1^2 = \frac{4}{6}$, $b_1^2 - b_1^3 = \frac{1}{6}$, $b_2^1 - b_2^2 = \frac{5}{6}$. However, in this case, we have $n_q \neq n_l$.

ACKNOWLEDGMENTS

I would like to thank Professor S. Okubo and Professor R. E. Marshak for discussions. Deep appreciation goes to Professor J. Iizuka for his continuous encouragement. This work was supported in part by the Department of Energy under Contract No. DE-AS05-80ER10713.

[1]Y. Tosa and S. Okubo, Phys. Rev. D 23, 2486 (1981); 23, 3058 (1981).
[2]S. Okubo, Had. J. (to be published).
[3]S. L. Glashow, Nucl. Phys. 22, 579 (1961); S. Weinberg, Phys. Rev. Lett. 27, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium, No. 8)* edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.
[4]G. Senjanović and R. N. Mohapatra, Phys. Rev. D 12, 1502 (1975); R. E. Marshak and R. N. Mohapatra, Phys. Lett. 91B, 222 (1980); R. N. Mohapatra and R. E. Marshak, Phys. Rev. Lett. 44, 1316 (1980); T. G. Rizzo and G. Senjanović, Phys. Rev. Lett. 46, 1315 (1981); Phys. Rev. D 24, 704 (1981); 25, 235 (1982).
[5]Y. Tosa, Phys. Rev. D 25, 1150 (1982). The examination of possibilities in GUP have been done in Y. Tosa, *ibid.* 22, 3113 (1980); 24, 2772(E) (1981).
[6]F. Wilczek and A. Zee, Phys. Rev. Lett. 42, 421 (1979); A. Davidson, Phys. Lett. 90B, 87 (1980); 93B, 359 (1980); A. Davidson and K. C. Wali, Phys. Lett. 94B, 359 (1980); A. Davidson, P. D. Mannheim, and K. C. Wali, Phys. Rev. Lett. 47, 149 (1981).
[7]M. Hausner and J. T. Schwartz, *Lie Groups and Lie Algebras* (Gordon and Breach, New York, 1968).
[8]Y. Tosa, Ph.D. Thesis, University of Rochester Report No. UR-793, 1981 (unpublished); Phys. Rev. D 22, 3113 (1980); 24, 2772(E) (1981).
[9]H. Georgi, Nucl. Phys. B156, 126 (1979); P. H. Frampton, Phys. Lett. 88B, 299 (1979); P. H. Frampton and S. Nandi, Phys. Rev. Lett. 43, 1460 (1979); P. H. Frampton, Phys. Lett. 89B, 352 (1980); M. Claudson, A. Yildiz, and P. H. Cox (unpublished); J. Chakrabarti, M. Popovic, and R. N. Mohapatra, Phys. Rev. D 21, 3212 (1980); Y. Fujimoto, Nucl. Phys. B182, 242 (1981); T. L. Curtright and P. G. O. Freund, in *Supergravity*, edited by P. van Nieuwenhuizen and D. Z. Freedman (North-Holland, Amsterdam, 1979); M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity;* C. W. Kim and C. Roiesnel, Phys. Lett. 93B, 343 (1980); F. Wilczek and A. Zee, Phys. Rev. D 25, 553 (1982); M. Ida, Y. Kayama, and T. Kitazoe, Prog. Theor. Phys. 64, 1745 (1980); I. Bars and M. Günaydin, Phys. Rev. Lett. 45, 859 (1980).