Conserved charges from self-duality

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(Received 16 November 1981)

Given a simple self-dual quantum Hamiltonian $H=KB+\Gamma\tilde{B}$, where K and Γ are coupling constants, and the condition that $[B,[B,[B,\tilde{B}]]]=16[B,\tilde{B}]$, then we construct an infinite set of conserved charges Q_{2n}; $[H,Q_{2n}]=0$. In simple models, like the two-dimensional Ising or Baxter eight-vertex, these charges appear in the associated quantum theories and are equivalent to those which result from the transfer-matrix formulation and exact quantum integrability of the system. The power of our result is that it is an operator statement and does not refer to the number of dimensions or the nature of the space-time manifold: lattice, continuum, or loop space. It is suggested how the establishment of this link between duality and integrability could be used to exploit the Kramers-Wannier-type self-duality of the four-dimensional SU(N) gauge theory to find hidden symmetry.

I. INTRODUCTION

Kramers-Wannier duality is a far-reaching powerful property of certain statistical systems. It may be used to connect strong- and weak-coupling phases from model to model and in the case of self-duality within the same model.[1] It is striking that many of the theories possessing this property are also exactly integrable, i.e., interacting theories in which the number of constants of motion is equal to the number of degrees of freedom. The inverse-pro- blem method of describing completely integrable quantum systems has brought together the transfer-matrix formalism, the exact solution of the Ising and Baxter models, infinite sets of conserved charges, the Bethe-ansatz solution of quantum field theories, and the diagonalization of quantum Hamiltonians.[2,3]

In this paper we propose to adjoin self-duality to this remarkable collection, as a step in elucidating the hidden symmetry and subsequent integrability of gauge theories. The intriguing fact that closed strings have proven to be the appropriate choice for the fundamental excitation in descriptions of *both* the infinite set of conserved cur- rents[4,5] (or integrability) and the 't Hooft[6] self-duality relation led us further to develop the concept of a connection between these two properties.

Since two-dimensional models have provided a solid base for solvability, we begin in Sec. II with a discussion of hidden symmetry, integrability, and duality in some standard systems of statistical mechanics and field theory and how these concepts may be used in discussing integrability in gauge systems. In Sec. III, we prove the main result of this paper: Given a self-dual quantum Hamiltonian $H=KB+\Gamma\tilde{B}$ (where K and Γ are coupling constants and $\tilde{B}$ is the operator dual to B; $\tilde{\tilde{B}}=B$) and the one condition

$$[B,[B,[B,\tilde{B}]]]=16[B,\tilde{B}]$$

(conservation of the first charge), then there exists an infinite set of conserved commuting self-dual charges $Q_{2n}=K(W_{2n}-\tilde{W}_{2n-2})+\Gamma(\tilde{W}_{2n}-W_{2n-2})$ where $W_{2n}=-\frac{1}{8}[B,[\tilde{B},W_{2n-2}]]-\tilde{W}_{2n-2}$ and $W_0\equiv B$, $Q_0\equiv H$, $n=1,2,\ldots$.

The power of this result is that it is an operator statement and does not depend on the dimension of the system or the nature of the space-time manifold, i.e., lattice, continuum, or loop space. Exactly integrable systems are characterized by the number of constants of motion being equal to the number of degrees of freedom. In several specific models, the charges are seen to equal those found as a consequence of the exact integrability of the system.[7]

In Sec. IV, we outline how our result might be used to improve understanding of both the choice of the appropriate non-Abelian dual transformation and the explicit form of the hidden-symmetry conservation laws for the four-dimensional gauge system. In any case, the result proved in Sec. III establishes a rigorous connection between simple self-duality and an infinite set of constants of motion. It provides a framework in which self-duality may be exploited to uncover extra symmetry.

II. INTEGRABILITY AND DUALITY

Classical statistical mechanics in d dimensions is transparently connected with quantum field theory on a $(d-1)$-space-dimensional lattice via the transfer-matrix formalism.[8–11] This formalism has led through a rather circuitous path to the complete integrability of quantum systems.[2,3] Since the transfer matrix is perhaps more familiar to statistical mechanicians then to particle physicists, we review it here with the purpose of exposing those properties we believe may be useful for gluon dynamics.

In $d=1$ dimension, define a classical lattice partition function to be

$$Z_{\text{cl}}^{\text{lat}}(x,x')=\int_{-\infty}^{\infty}\prod_{k=1}^{n-1} dx_k e^{-\beta\mathscr{H}} , \tag{2.1}$$

where the fundamental variable x_k defined on site k of a one-dimensional lattice $k=0,1,\ldots,n$ of n points takes on a continuous range of values, $-\infty \leq x_k \leq \infty$, and let

$$\mathscr{H}\equiv\frac{\epsilon}{2}\sum_{j=1}^{n}\left[(x_j-x_{j-1})^2\frac{1}{\epsilon^2}+\omega^2 x_j{}^2\right] . \tag{2.2}$$

The initial and final configurations $x_0=x$ and $x_n=x'$ are not summed over. For comparision in the one-dimensional Ising model, the fundamental variable is $s(k)=\pm 1$ and the classical partition function is

$$Z=\sum_{s(1)=-1}^{1}\cdots\sum_{s(n)=-1}^{1} e^{-\beta\mathscr{H}} , \tag{2.3}$$

where

$$\mathscr{H}=-J\sum_{k=1}^{n} s(k)s(k-1) . \tag{2.4}$$

The transfer-matrix formalism follows from the definition of the measure of the Feynman path integral.[9] Equation (2.1) can be written in terms of a zero-dimensional operator $\hat{T}$, the transfer matrix:

$$\begin{aligned} Z_{\text{cl}}^{\text{lat}}(x,x')\,|_{\beta=1/\hbar}&=\int_{-\infty}^{\infty} dx_1\cdots dx_{n-1}\exp\left[-\frac{\epsilon}{4\hbar}\omega^2 x'^2\right]\\ &\qquad\times\langle x'\,|\,\hat{T}\,|\,x_{n-1}\rangle\langle x_{n-1}\,|\,\hat{T}\,|\,x_{n-2}\rangle\cdots\langle x_1\,|\,\hat{T}\,|\,x\rangle\exp\left[\frac{\epsilon}{4\hbar}\omega^2 x^2\right]\\ &=\exp\left[-\frac{\epsilon}{4\hbar}\omega^2(x'^2-x^2)\right]\langle x'\,|\,\hat{T}^n\,|\,x\rangle , \end{aligned} \tag{2.5}$$

where

$$\langle x_k\,|\,\hat{T}\,|\,x_{k-1}\rangle=\exp\left[-\frac{\epsilon}{2\hbar}\left[(x_k-x_{k-1})^2\frac{1}{\epsilon^2}+\frac{\omega^2}{2}(x_k{}^2+x_{k-1}{}^2)\right]\right] \tag{2.6}$$

and

$$\hat{T}=\exp\left[-\frac{\epsilon\omega^2}{4\hbar}\hat{x}^2\right]\exp\left[-\frac{\epsilon}{2\hbar}\hat{p}^2\right]\exp\left[-\frac{\epsilon\omega^2}{4\hbar}\hat{x}^2\right](2\pi\hbar\epsilon)^{1/2} . \tag{2.7}$$

Here the zero-dimensional operators $\hat{x}$ and $\hat{p}$ are canonical, $[\hat{x},\hat{p}]=i\hbar$, and act in the following space of complete states:

$$\begin{gathered}\int_{-\infty}^{\infty} dx\,|\,x\,\rangle\langle\,x\,|=1,\quad \hat{x}\,|\,x\,\rangle=x\,|\,x\,\rangle,\quad \langle x'\,|\,x\,\rangle=\delta(x-x') ,\\ \hat{p}\,|\,p\,\rangle=p\,|\,p\,\rangle,\quad \langle x\,|\,p\,\rangle=\frac{e^{ipx/\hbar}}{(2\pi\hbar)^{1/2}} ,\end{gathered} \tag{2.8}$$

so that

$$\left\langle x' \left| \exp\left[-\frac{\epsilon}{2\hbar}\hat{p}^2\right] \right| x \right\rangle = \left[\frac{1}{2\pi\hbar\epsilon}\right]^{1/2} \exp\left[-\frac{1}{2\hbar\epsilon}(x-x')^2\right] . \tag{2.9}$$

For periodic boundary conditions, $x'=x$, and we can sum over x to define

$$Z_{\text{cl}}^{\text{lat}}(\text{periodic b.c.}) \equiv \int_{-\infty}^{\infty} dx\, Z_{\text{cl}}^{\text{lat}}(x,x) = \text{Tr}\hat{T}^n . \tag{2.10}$$

The space of states described above is familar from the derivation of the path-integral representation of the kernel. For $\hat{H} \equiv \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$,

$$\begin{aligned}\langle x' | e^{-\hat{H}\tau/\hbar} | x \rangle &= \lim_{\substack{n\to\infty\\ \epsilon\to 0\\ n\epsilon=\tau}} (2\pi\hbar)^{-n} \int_{-\infty}^{\infty} \prod_{i=1}^{n} dp_i \int_{-\infty}^{\infty} \prod_{k=1}^{n-1} dx_k \exp\left\{\frac{1}{\hbar}\sum_{j=1}^{n}\left[ip_j(x_j-x_{j-1}) - \frac{\epsilon}{2}(p_j{}^2+\omega^2 x_j{}^2)\right]\right\} \\ &= \lim_{\substack{n\to\infty\\ \epsilon\to 0\\ n\epsilon=\tau}} (2\pi\hbar)^{-n} \left[\frac{2\pi\hbar}{\epsilon}\right]^{n/2} \int_{-\infty}^{\infty} \prod_{k=1}^{n-1} dx_k \exp\left\{-\frac{\epsilon}{2\hbar}\sum_{j=1}^{n}\left[(x_j-x_{j-1})^2\frac{1}{\epsilon^2}+\omega^2 x_j{}^2\right]\right\}\end{aligned} \tag{2.11a}$$

$$= N\int Dx(t) \exp\left[-\frac{1}{\hbar}\int_0^{\tau} dt(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2)\right] . \tag{2.11b}$$

Equation (2.11b) is the sum over paths [the normalization factor N depends on the normalization of the measure $Dx(t)$] and Eq. (2.11a) tells us how to compute it.

As an aside, we remark that the sum over paths (2.11) is also what appears in the proper-time[12] representation of the four-dimensional propagator. For free-particle amplitudes, in Euclidean space, we have, in analogy with (2.8),

$$\begin{aligned}&(-\Box+m^2)G(x-x') = \delta^4(x-x'),\quad G(x-x') = \langle 0 | T\varphi(x)\varphi(x') | 0 \rangle \equiv \langle x | \hat{G} | x' \rangle , \\ &\langle x | x' \rangle = \delta^4(x-x'),\quad \hat{x}_\mu | x \rangle = x_\mu | x \rangle,\quad \hat{p}_\mu | p \rangle = p_\mu | p \rangle, \\ &\langle x | \hat{p}_\mu | x' \rangle = \int d^4q\, \delta^4(q-x)\left[-\frac{i\partial}{\partial q^\mu}\delta^4(q-x')\right] = -i\frac{\partial}{\partial x^\mu}\delta(x-x') .\end{aligned} \tag{2.12}$$

Then

$$(\hat{p}_\mu\hat{p}_\mu + m^2)\hat{G} = \hat{I} . \tag{2.13}$$

Equation (2.13) follows from

$$\begin{aligned}\langle x | (\hat{p}^2+m^2)\hat{G} | x' \rangle &= \int d^4y \langle x | \hat{p}^2+m^2 | y \rangle G(y-x') \\ &= \int d^4y(-\partial_\mu\partial_\mu+m^2)\delta^4(x-y)G(y-x') \\ &= (-\Box_x+m^2)G(x-x') = \delta^4(x-x') .\end{aligned} \tag{2.14}$$

From (2.13),

$$\hat{G} = \tfrac{1}{2}\int_0^{\infty} d\tau \exp\left[-\frac{\tau}{2}(\hat{p}^2+m^2)\right] \tag{2.15}$$

so that

$$\langle x | \hat{G} | y \rangle = G(x-y) = \tfrac{1}{2}\int_0^{\infty} d\tau \langle x | e^{-\hat{H}\tau} | y \rangle$$

$$=\frac{1}{2}\int_0^\infty d\tau\, e^{-m^2\tau/2} N \int D\xi_\mu(t) \exp\left[-\frac{1}{2}\int_0^\tau dt\, \dot{\xi}^2\right], \qquad (2.16)$$

where $\hat{H}\equiv\frac{1}{2}\hat{p}^2+\frac{1}{2}m^2$ and $\xi_\mu(t)$ is a path through four-dimensional space-time connecting x with y, $\xi_\mu(0)=x_\mu$, $\xi_\mu(\tau)=y_\mu$.

For gauge systems it has been suggested[4,13] that the analogy of this proper-time expression be used to define the string propagator

$$\left\langle 0\left|\mathrm{tr}P\exp\left[\oint A\cdot d\xi\right]\right|0\right\rangle=G[\xi]$$

as a sum over not paths but surfaces. Thus the appropriate transfer matrix for gauge theories will operate not between points or rows of spins, etc., but between closed paths.

To return to our previous discussion of the transfer matrix, from Eqs. (2.11) and (2.1), we see that

$$\langle x' | e^{-\hat{H}T/\hbar} | x \rangle = \lim_{\substack{n\to\infty\\ \epsilon\to 0\\ n\epsilon=T}} (2\pi\hbar)^{-n} \left[\frac{2\pi\hbar}{\epsilon}\right]^{n/2} Z_{\mathrm{cl}}^{\mathrm{lat}}(x,x')\Bigg|_{\beta=1/\hbar} . \qquad (2.17)$$

Equation (2.17) gives the connection between the one-dimensional classical statistical system [with $\mathscr{H}$ given by (2.2)] and the lattice approximation to the zero-space-dimension quantum field theory (one-particle quantum mechanics) governed by

$$\hat{H}=\frac{1}{2}\hat{p}^2+\frac{1}{2}\omega^2\hat{x}^2 . \qquad (2.18)$$

The transfer matrix (2.7) is therefore seen to be essentially the time evolution operator between adjacent discrete times on a Euclidean lattice.

In any number of dimensions, we can associate a $(d-1)$-space-dimensional quantum Hamiltonian on the lattice with a d-dimensional statistical system. It turns out that although $\hat{T}$ is a Hermitian operator, it is not in general local; however, $\ln\hat{T}$ satisfies the locality requirement.[9] Furthermore, for certain problems, the τ-continuum $(\tau\to 0)$ limit[10] of $\hat{H}_\tau\equiv-(1/\tau)\ln\hat{T}$ (where τ is a function of the couplings) is a less complicated Hamiltonian than $\hat{H}_\tau$ itself, but can still be used to discuss various properties of the d-dimensional partition function such as its critical phenomena. For the one-dimensional example above [Eq. (2.7)] consider the operator

$$\hat{H}_\tau=-\left[\frac{\hbar}{\epsilon}\right]\ln\hat{T} \text{ or } \hat{T}=e^{-\tau\hat{H}_\tau},\quad \tau\equiv\frac{\epsilon}{\hbar} .$$

Since $\hat{x}$ and $\hat{p}$ do not commute, even with the use of the Baker-Hausdorff identity[14] it is difficult to find a closed form for $\ln\hat{T}$. The τ-continuum limit, however, is simple and is in fact (2.18):

$$\lim_{\epsilon\to 0}-\frac{\hbar}{\epsilon}\ln\hat{T}=\frac{\hat{p}^2}{2}+\frac{\omega^2}{2}\hat{x}^2=\hat{H} . \qquad (2.19)$$

Clearly, if $\hat{T}$ or $\ln\hat{T}$ or any other operator which commutes with $\hat{T}$ can be diagonalized, then the classical partition function $Z_{\mathrm{cl}}^{\mathrm{lat}}\equiv\mathrm{Tr}\hat{T}^n$ can be solved. Onsager,[15] and later Shultz, Mattis, and Lieb,[8] used this observation to solve the statistical mechanics of the two-dimensional Ising model. Baxter[16] identified a one-parameter family of commuting transfer matrices in the (two-dimensional) eight-vertex model which enabled him to solve it along with the eigenvectors of the associated one-dimensional (lattice) XYZ Hamiltonian. Thus for statistical mechanics, the transfer matrix is a powerful tool for solving the partition function. For lattice quantum field theory, it can lead to the derivation of the eigenstates of the Hamiltonian and eventually the quantum inverse method for exact integrability of the quantum system.[2]

Before discussing these two models with regards to their solvability and duality transformations, we make a brief digression on complete integrability[2,3] and the infinite set of conservation laws in $(1+1)$-dimensional systems: classical, quantum, and on the lattice. These three cases differ in the questions answered and subsequently in the roles of the various components such as the conservation laws. The classical inverse scattering method[17] (CISM) solves the following problem: Given specified initial-value data $\phi(x,0)$ and a nonlinear evolution equation for $\phi(x,t)$, the CISM solves for that state at a later time. It presents a linear eigenvalue problem (the scattering problem) whose integrability condition is the equation of motion of the nonlinear system. The CISM (1) maps the initial-value data $\phi(x,0)$ into the scattering data (the spectrum and asymptotic behavior of the eigenfunctions of the linear problem) (direct problem), (2) computes the evolution of the scattering data, and (3) at any later time, inverts the mapping to give the solution $\phi(x,t)$ (inverse problem). This mapping is a canonical transformation from $\phi(x,t)$ and its conjugate $\pi(x,t)$ to action-angle variables $P(k,t)$ and $Q(k,t)$.[2] This is to say, the Hamiltonian (whose equation of motion is the evolution equation) can be expressed as a function of only $P(k,t)$, so that $\dot{P}(k,t)=\delta H/\delta Q(k,t)=0$. Therefore, a power-series expansion in k of a function of $P(k,t)$ gives rise to an infinite set of commuting constants of the motion.

As an example, we outline the classical (repulsive, $c>0$) nonlinear Schrödinger equation (NLSE)

$$i\partial_t\phi(x,t)=-\partial_x{}^2\phi(x,t)+2c\phi^*\phi^2 . \tag{2.20}$$

This has the associated scattering problem

$$L_{\rm op}\Psi=-\frac{\xi}{2}\Psi , \tag{2.21a}$$

$$\frac{dL_{\rm op}}{dt}=[L_{\rm op},M_{\rm op}],\ \ \partial_t\Psi=-M_{\rm op}\Psi , \tag{2.21b}$$

where

$$L_{op}=\begin{bmatrix} i\partial_x & \sqrt{c}\,\phi(x,t) \\ \sqrt{c}\,\phi^*(x,t) & -i\partial_x \end{bmatrix} ,$$

$$\Psi=\begin{bmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{bmatrix}$$

and $M_{\rm op}$ is defined such that the eigenvalues have $\dot{\xi}=0$. Then the scattering data of (2.21) depends only on $\phi(x,0)$ not $\phi(x,t)$. The scattering data (at time $t=0$) are the functions $a(k,0)$ and $b(k,0)$ where

$$\lim_{x\to-\infty}\begin{Bmatrix} \psi_1(x,0;\xi=k \text{ real}) \\ \psi_2(x,0;k) \end{Bmatrix}=\begin{Bmatrix} e^{ikx/2} , \\ 0 \end{Bmatrix} \tag{2.22a}$$

and

$$\lim_{x\to\infty}\begin{Bmatrix} \psi_1(x,0;k) \\ \psi_2(x,0;k) \end{Bmatrix}=\begin{Bmatrix} a(k,0)e^{ikx/2} , \\ b(k,0)e^{-ikx/2} . \end{Bmatrix} \tag{2.22b}$$

Given $\phi(x,0)$ and (2.22a), we can determine (2.22b) from (2.21). Since $\dot{k}=0$, we can also determine $a(k,t)$ and $b(k,t)$ from $\phi(x,0)$ only. It is found that

$$\begin{aligned} a(k,t)&=a(k,0) , \\ b(k,t)&=e^{-k^2t}b(k,0) . \end{aligned} \tag{2.23}$$

The action-angle variables are

$$\begin{aligned} P(k,t)&=\frac{1}{\sqrt{c}}\ln|a(k,t)| , \\ Q(k,t)&=\frac{1}{\pi\sqrt{c}}\arg b(k,t) . \end{aligned} \tag{2.24}$$

The NLSE has

$$H=\int_{-\infty}^{\infty}dx[\partial_x\phi\partial_x\phi^*+c(\phi^*\phi)^2] . \tag{2.25}$$

In order to write the equation of motion (2.20) as a set of Hamilton's equations, define the Poisson brackets as

$$\{A,B\}=i\int_{-\infty}^{\infty}dy\left[\frac{\delta A}{\delta\phi(y,t)}\frac{\delta B}{\delta\phi^*(y,t)}-\frac{\delta B}{\delta\phi(y,t)}\frac{\delta A}{\delta\phi^*(y,t)}\right] . \tag{2.26}$$

Then, for $\pi\equiv i\phi^*$,

$$\begin{aligned} \dot{\phi}&=\{H,\phi\}=\frac{\delta H}{\delta\pi} , \\ \dot{\pi}&=\{H,\pi\}=-\frac{\delta H}{\delta\phi} , \end{aligned} \tag{2.27}$$

and (2.27) is equivalent to (2.20). The transformation from $(\phi(x,t),\pi(x,t))$ to $(Q(k,t),P(k,t))$ is canonical:

$$\{\pi(x,t),\phi(x,t)\}=\delta(x-y)$$

and

$$\{P(k,t),Q(k',t)\}=\delta(k-k') .$$

The equations of motion (2.27) in terms of (P,Q) are

$$\begin{aligned} \dot{Q}(k,t)&=\frac{\delta H}{\delta P(k,t))} , \\ \dot{P}(k,t)&=0 . \end{aligned} \tag{2.28}$$

The infinite set of conservation laws is given by M_l, the moments of P:

$$\ln a(k,t) \equiv -ic \sum_{l=0}^{\infty} M_l \frac{1}{k^{l+1}}$$

or (2.29)

$$M_l = \frac{1}{\pi\sqrt{c}} \int_{-\infty}^{\infty} dk\, P(k,t) k^l .$$

Note that $M_2 = H$. Since $\dot{a}(k,t)=0$, all $\dot{M}_l = 0$. Since $\{P(k,t),P(k',t)\}=0$, all $\{M_l, M_n\}=0$.

We remark that the classical nonlinear σ model (NLσM), although integrable in the sense that there exists a linear problem[18]

$$\left[\partial_\mu - \frac{1}{\lambda}\epsilon_{\mu\nu}(\partial^\nu + A^\nu)\right]\psi(x;\lambda)=0$$

whose integrability condition ($\partial_\mu\partial_\nu\psi = \partial_\nu\partial_\mu\psi$) is $\partial\mu A^\mu = 0$ (the equations of motion for $\mathscr{L} = \mathrm{tr}\partial_\mu g \partial^\mu g^{-1}$, $A^\mu \equiv g^{-1}\partial^\mu g$), has not been solved in the sense that given $g(x,0)$ we can compute $g(x,t)$. This is due to the fact that the scattering data $a(k,t)$, $b(k,t)$ for this linear problem remain constant for all time. The quantum NLσM may be easier to solve.[19]

The quantum inverse scattering method[2,3] answers two questions: the direct problem diagonalizes the Hamiltonian and the inverse problem solves for the operators and Green's functions. In the example of the quantum NLSE, ϕ and Ψ become operators. Although the scattering data $a(k,t)$, $b(k,t)$ are complicated functions of ϕ and ϕ^*, a and b have simple commutation relations among themselves and the Hamiltonian: $[H,a]=0$ and $[H,b]\sim b$. Thus $a(k,t)$ generates an infinite set of quantum charges and $b(k,t)$ creates energy eigenstates which turn out to be the same as those derived from Bethe's ansatz. [The NLSE equation is the second-quantized form of the N-body quantum-mechanics problem with a δ-function potential:

$$H = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^{\,2}} + c\sum_{i<j}\delta(x_i - x_j) . \tag{2.30}$$

Bethe's guess was that the N-particle wave function $\psi(x_1,\ldots,x_N)$, where $H\psi = E\psi$, took the form

$$\psi \sim \exp\left[-\sum_{i<j} |x_i - x_j|\right] . \tag{2.31}$$

This solution was extended to the field theory by Thacker.[20]] In the NLSE, the solution to the appropriate linear eigenvalue problem

$$\partial_x \Psi = -i :\tilde{Q}\Psi: ,$$

where

$$\tilde{Q} \equiv \begin{bmatrix} -\xi/2 & -\sqrt{c}\,\phi \\ \sqrt{c}\,\phi^* & \xi/2 \end{bmatrix} \text{ and } \Psi \equiv \begin{bmatrix} \psi_1 & \psi_2^* \\ \psi_2 & \psi_1^* \end{bmatrix} , \tag{2.32}$$

can be written as a path-ordered exponential

$$\Psi(x,t) = :P\exp\left[-i\int_y^x dz\,\tilde{Q}(z,t;\xi=k)\right]\Psi(y,t):$$
$$=: \lim_{\substack{N\to\infty \\ \epsilon\to 0 \\ N\epsilon = x-y}} \prod_{n=1}^{N+1} L_n(k)\Psi(y,t): , \tag{2.33}$$

where

$$L_n(k) = \exp[-i\epsilon\tilde{Q}(y-(n-1)\epsilon,t;k)] . \tag{2.34}$$

As Thacker[3] has pointed out, Eq. (2.33) motivates the lattice quantum inverse method and its relation with the transfer matrix. For, on the lattice, the classical partition function is expressed in terms of the transfer matrix $\hat{T}(Z \equiv \mathrm{Tr}\hat{T}^m)$ and $\hat{T}$ can sometimes itself be written as the trace of a product of matrices. That is to say, if we define from (2.33) the 2×2 matrix

$$\mathscr{T}(k) \equiv \prod_{n=1}^{N+1} L_n(k) \equiv \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix} , \tag{2.35}$$

then, on the lattice, $\hat{T}$ is the trace of $\mathscr{T}(k)$ a function of the scattering data: $\hat{T} \sim A + D$.

For example, in the XYZ model

$$\hat{T}(V) = \mathrm{tr}\prod_{n=1}^{N} \hat{\mathscr{L}}_n(V) . \tag{2.36}$$

Here $\hat{\mathscr{L}}_n(V)$ is a 2×2 matrix with operator entries and the trace is now in the 2×2 space:

$$\hat{\mathscr{L}}_n(V) = \sum_{j=1}^{4} w_j \hat{\sigma}_j(n)\sigma_j . \tag{2.37}$$

This comes from the underlying ($d=2$)-dimensional classical statistical mechanics of the symmetric eight-vertex model.[16] A comprehensive treatment of dual transformations, commuting charges, quantum lattice integrability, and the transfer matrix can be made in this model.[2] The partition function is

$$Z = \sum_{\text{all } N_j} e^{-\mathscr{H}} , \tag{2.38}$$

where each site (n_0, n_1) of a rectangular lattice has an associated energy ϵ_j depending on the vertex configuration (see Fig. 1 and Ref. 16 for more detail). Assume periodic boundary conditions. Now

$$\mathscr{H} = \sum_{j=1}^{8} N_j \epsilon_j \ . \tag{2.39}$$

Here N_j is the number of vertices of type j in an allowed configuration:

$$\sum_{j=1}^{8} N_j = NM \ . \tag{2.40}$$

Also define

$$\vec{\alpha}_{n_0} \equiv (\alpha_{n_0 1}, \ldots, \alpha_{n_0 N}) \ , \tag{2.41}$$

where $\alpha_{n_0 n_1} = \pm 1$ represents a single vertical lattice link. From (2.38), Z can be expanded in terms of the direct-product vector states

$$| \vec{\alpha}_{n_0} \rangle = | \alpha_{n_0 1} \rangle \otimes \cdots \otimes | \alpha_{n_0 N} \rangle \ , \tag{2.42}$$

where

$$| \alpha_{n_0 n_1} \rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ . \tag{2.43}$$

Then

$$\begin{aligned} Z &= \sum_{\vec{\alpha}_1 \cdots \vec{\alpha}_M} \langle \vec{\alpha}_1 | \hat{T} | \vec{\alpha}_2 \rangle \cdots \langle \vec{\alpha}_M | \hat{T} | \vec{\alpha}_1 \rangle \\ &= \hat{T}^M \ , \end{aligned} \tag{2.44}$$

where $\hat{T}$ is given by (2.36) and w_j are functions of ϵ_j.[16]

Baxter[16] found a parameterization of w_j in terms of three parameters V, ζ, l such that for fixed ζ and l,

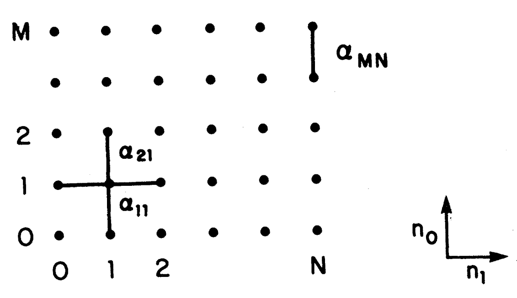

FIG. 1. Rectangular lattice for eight-vertex model.

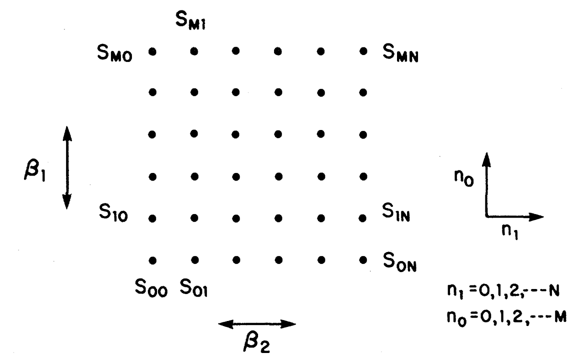

FIG. 2. Rectangular lattice for two-dimensional Ising model.

$$[\hat{T}(V), \hat{T}(V')] = 0 \ . \tag{2.45}$$

The remarkable existence of a one-parameter family of commuting transfer matrices led to the diagonalization of $\hat{T}$, a set of conserved commuting charges[21] ($\hat{C}_n$ for $\hat{H}_{XYZ}$), and the diagonalization of $\hat{H}_{XYZ}$ since

$$\hat{C}_n = \left. \frac{\partial^{n+1}}{\partial V^{n+1}} \ln \hat{T}(V) \right|_{V=\zeta} \tag{2.46}$$

and

$$\hat{C}_0 = -\frac{1}{J_z \operatorname{sn}(2\zeta, l)} \hat{H}_{XYZ} + \text{const} \ , \tag{2.47}$$

where

$$\hat{H}_{XYZ} = -\tfrac{1}{2} \sum_{n=1}^{N} \sum_{a=1}^{3} J_a \hat{\sigma}_a(n) \hat{\sigma}_a(n+1) \tag{2.48}$$

and

$$\frac{J_x}{J_z} = \operatorname{cn}(2\zeta, l), \quad \frac{J_y}{J_z} = \operatorname{sn}(2\zeta, l) \ . \tag{2.49}$$

Also

$$\begin{gathered} [\hat{\sigma}_a(n), \hat{\sigma}_b(m)] = 2i \epsilon_{abc} \hat{\sigma}_c(n) \delta_{nm} \ , \\ \hat{\sigma}_a(n) \hat{\sigma}_b(n) = \delta_{ab} + i \epsilon_{abc} \hat{\sigma}_c(n) \ . \end{gathered} \tag{2.50}$$

The integrability of (2.48) via the transfer matrix in a way follows the standard inverse scattering method. The equations of motion are

$$\dot{\hat{\sigma}}_a(n)=i[\hat{H},\hat{\sigma}_a(n)]\ .$$

We then wish to find action-angle variables so that the equations of motion can be linearized and explicitly solved. The transfer matrix $\hat{T}(V)$ is like the action variable since $[\hat{H}_{XYZ},\hat{T}(V)]=0$, $\hat{H}_{XYZ}$ can be reexpressed in terms of $\hat{T}(V)$ [(2.47)], and $\hat{T}(V)$ generates the set of conserved charges. In some sense we could say we are lead to the appropriate variable by the transfer-matrix formulation. In a loop-space formulation of gauge systems, a transfer matrix which acts between loops might be used to generate the functional symmetry currents.

A Kramers-Wannier dual transformation in the eight-vertex model is reflected in $\hat{H}_{XYZ}$ as follows: Define a dual transformation on the operators $\hat{\sigma}_a(n)$:

$$\mu_1(n)\equiv\tilde{\sigma}_1(n)=\sigma_3(n)\ ,$$
$$\mu_2(n)\equiv\tilde{\sigma}_2(n)=-\sigma_2(n)\ , \qquad (2.51)$$
$$\mu_3(n)\equiv\tilde{\sigma}_3(n)=\sigma_1(n)\ .$$

Then

$$\hat{H}_{XYZ}(J_1,J_2,J_3;\hat{\sigma})=\hat{H}_{XYZ}(J_3,J_2,J_1;\hat{\mu}) \qquad (2.52)$$

and $\hat{H}_{XYZ}$ is self-dual under (2.51). The interchange of high and low temperature, or here the exchange of couplings, is the general feature of duality transformations, and it is one of the properties that make them so useful. Self-duality is not a general feature, but occurs only for certain theories.

It is the purpose of this paper to exploit this property in the search for hidden symmetry. To do so, we first describe one more theory, the somewhat simpler Ising model[8] which has all the features of H_{XYZ} necessary for this pursuit.

In $d=2$ dimensions, we define the classical (lattice) partition function of the anisotropic Ising model to be

$$Z=\sum_{\{S_{n_0n_1}\}=-1}^{1} e^{-\mathscr{H}}\ , \qquad (2.53)$$

where the fundamental variable $s_{n_0n_1}$, defined on site (n_0,n_1) of a two-dimensional square lattice (see Fig. 2), takes on the discrete values $s_{n_0n_1}=\pm1$ and

$$\mathscr{H}=-\sum_{n_0=1}^{M}\sum_{n_1=1}^{N}(\beta_2 s_{n_0n_1}s_{n_0n_1-1}+\beta_1 s_{n_0n_1}s_{n_0-1n_1})\ . \qquad (2.54)$$

For periodic boundary conditions ($s_{0n_1}=s_{Mn_1}$, $s_{n_00}=s_{n_0N}$, $s_{00}=s_{M0}=s_{0N}=s_{MN}$) the lattice is a torus and all nearest-neighbor interactions are accounted for in $\mathscr{H}$ above. Then (2.53) can be written as

$$Z=\sum_{\vec{s}_1\cdots\vec{s}_M}\langle\vec{s}_M|\hat{T}|\vec{s}_1\rangle\cdots\langle\vec{s}_{M-1}|\hat{T}|\vec{s}_M\rangle$$
$$=\mathrm{Tr}\hat{T}^M\ , \qquad (2.55)$$

where

$$|\vec{s}_{n_0}\rangle=|s_{n_01},\ldots,s_{n_0N}\rangle$$
$$\equiv|s_{n_01}\rangle\otimes\cdots\otimes|s_{n_0N}\rangle \qquad (2.56)$$

is the state which describes a set of spins for one row (n_0). $|\vec{s}_{n_0}\rangle$ is a direct-product vector space of the basis states $|s_{n_0n_1}\rangle$ which describe a spin for one site, i.e., either $\binom{1}{0}$ or $\binom{0}{1}$. The operator $\hat{T}$ which is the transfer matrix of the Ising model is a function of the couplings β_1 and β_2:

$$\hat{T}(\bar{\beta}_1,\beta_2)=(2\sinh(2\beta_1))^{N/2}\exp\left[\bar{\beta}_1\sum_{n=1}^{N}\hat{\sigma}_1(n)\right]\exp\left[\beta_2\sum_{n-1}^{N}\hat{\sigma}_3(n)\hat{\sigma}_3(n+1)\right]\ . \qquad (2.57)$$

Here $\hat{\sigma}_a(0)\equiv\hat{\sigma}_a(N)$ and $\tanh\bar{\beta}_1=\exp(-2\beta_1)$ or equivalently $\tanh\beta_1=\exp(-2\bar{\beta}_1)$ or $\sinh(2\beta_1)\sinh(2\bar{\beta}_1)=1$. Equation (2.57) has the form $\hat{T}=c\exp(\bar{\beta}_1\hat{A})\exp(\beta_2\hat{B})$. Clearly any symmetrized version [e.g., $\hat{T}'=c\exp(\beta_2\frac{1}{2}\hat{B})\exp(\bar{\beta}_1\hat{A})\exp(\frac{1}{2}\beta_2\hat{B})$] gives rise to the same Z [see (2.55)]. Given the form of $\hat{A}$ and $\hat{B}$ from (2.57), however, it is difficult to find a one-parameter family of commuting transfer matrices,[14] which could then be used to construct a set of commuting charges.

Nevertheless, first observe the duality transformation in this model. On the partition function it is

$$Z(\bar{\beta}_1,\beta_2)=\mathrm{Tr}(\hat{T}'(\bar{\beta}_1,\beta_2))^M$$
$$\downarrow \text{ dual transformation}$$
$$=\tfrac{1}{2}(\sinh(2\beta_2))^{M/2}(\sinh(2\beta_1))^{N/2}Z(\beta_2,\bar{\beta}_1)\ . \qquad (2.58)$$

From the self-duality property of Z, it follows that if we define an operator dual transformation (away from the end points[3]):

$$\mu_1(n)\equiv\tilde{\sigma}_1(n)\equiv\sigma_3(n)\sigma_3(n+1),\quad \mu_3(n)\equiv\tilde{\sigma}_3(n)\equiv\sigma_1(1)\cdots\sigma_1(n)\,, \tag{2.59}$$

then the dual transform on the transfer matrix $\hat{T}'$ is

$$\hat{T}'_\mu(\bar{\beta}_1,\beta_2)=(2\sinh(2\beta_1))^{N/2}e^{(\beta_2/2)\hat{A}}e^{\bar{\beta}_1\hat{B}}e^{(\beta_2/2)\hat{A}}\,. \tag{2.60}$$

That is to say, $\hat{T}'_\mu(\bar{\beta}_1,\beta_2)$ gives the same Z as $\hat{T}'(\beta_2,\bar{\beta}_1)$, apart from an overall spin-independent factor.

To associate a $(d=1)$-dimensional simple quantum Hamiltonian with T, consider the $\tau=e^{-2\beta_1}=\lambda^{-1}\beta_2\to 0$ continuum limit. Then,

$$\lim_{\tau\to 0}-\frac{1}{\tau}\ln\hat{T}=\hat{H}_{\rm IM}=-\sum_{n=1}^{N}[\lambda\hat{\sigma}_3(n)\hat{\sigma}_3(n+1)+\hat{\sigma}_1(n)]\,. \tag{2.61}$$

So that, although diagonalizing $\hat{H}_{\rm IM}$ does not lead to a full solution of Z since $[\hat{T},\hat{H}]_{\rm IM}\neq 0$, we can discuss, within the context of $\hat{H}_{\rm IM}$ itself, duality, and commuting charges.

In Sec. III, we prove our main result, that for certain self-dual quantum Hamiltonians $\hat{H}=K\hat{B}+\Gamma\hat{\tilde{B}}$, there exists a set of conserved charges $[\hat{Q}_{2n},\hat{H}]=0$, by the explicit construction of $\hat{Q}_{2n}$. As stated in the Introduction, the power of this result is that it is an operator statement and it does not refer to number of dimensions or the nature of space-time manifold, i.e., lattice, continuum, local, or loop space; Q_{2n} is thus a form useful for gauge systems.

To make our result more familiar, however, two specific examples of it are (1) the Ising model where

$$B=\sum_{n=1}^{N}\hat{\sigma}_3(n)\hat{\sigma}_3(n+1)\ \text{and}\ \tilde{B}=\sum_{n=1}^{N}\hat{\sigma}_1(n)\,, \tag{2.62}$$

and (2) the XY model where

$$B=\sum_{n=1}^{N}\hat{\sigma}_3(n)\hat{\sigma}_3(n+1) \tag{2.63}$$

and the dual transformation is

$$\tilde{B}=\sum_{n=1}^{N}\hat{\sigma}_1(n)\hat{\sigma}_1(n+1)\,. \tag{2.64}$$

In both cases, our extra condition (the conservation of the first charge, $[B,[B,[B,\tilde{B}]]]=16[B,\tilde{B}]$) is satisfied and the set of commuting charges Q_{2n} are those which result from the exact integrability of the system. This holds since the charges in both models can be obtained from C_n of the XYZ model.[7]

Although the full XYZ model does not fit conveniently into our calculation, it is reasonable to assume that just as integrability itself is a property of certain interacting systems shared with free systems (the set of conserved integrals says that the interactions in some sense preserve the symmetry of the free system) so too the connection between self-duality and integrability will hold in the fully interacting system.

We make this statement since it is well known that $H_{\rm IM}$ and H_{XZ} are equivalent to free-fermion lattice theory via a Jordan-Wigner transformation.[8] Also it has been suggested[22] that the three-dimensional Ising model is equivalent to a free-fermionic string theory. The three-dimensional loop currents for Z(2) gauge theory might be used to linearize the loop-space equations of motion (via the Kramers-Wannier transformation) of the Ising model.

Thus although the sample models are in some sense expected to be integrable, we believe the ideas developed here will be applicable to more complicated theories and may open the way to hidden symmetries in four-dimensional gauge systems.

III. CONSTRUCTION OF AN INFINITE SET OF CONSERVED CHARGES

In this section we shall construct an infinite set of charges for a specific class of self-dual theories, namely those whose Hamiltonians can be written in the form

$$H=KB+\Gamma\tilde{B}\,, \tag{3.1}$$

where K and Γ are coupling constants and B is some operator. The form of the dual transformation need not be specified beyond that it is a linear operation which changes B to $\tilde{B}$ and $\tilde{B}$ to B. Only one additional condition is needed to guarantee the existence of an infinite set of charges. The charges are given by

$$Q_{2n} \equiv K(W_{2n} - \widetilde{W}_{2n-2}) + \Gamma(\widetilde{W}_{2n} - W_{2n-2}) \,, n = 1, 2, 3, \ldots \tag{3.2}$$

where

$$W_{2n+2} \equiv -\tfrac{1}{8}[B,[\widetilde{B},W_{2n}]] - \widetilde{W}_{2n} \tag{3.3}$$

and $W_0 \equiv B$, $n=1,2,3,\ldots$. The charges have been labeled with positive even integers in order to match as closely as possible the notation of Ref. 7. In order for Q_{2N} to be a conserved charge,

$$[Q_{2N},H]=0 \,, \tag{3.4}$$

it is sufficient to show that

$$[B,W_{2n}]=[B,\widetilde{W}_{2n-2}] \,, \tag{3.5a}$$

$$[\widetilde{B},W_{2n}]=-[B,\widetilde{W}_{2n}] \tag{3.5b}$$

for $1 \le n \le N$. We shall prove (3.5) inductively for all N. In order to perform the induction in N it is easier to consider (3.5b) with $n \to n-1$ as the natural partner to (3.5a), i.e., we shall prove the following set inductively for all N:

$$[B,W_{2n}]=[B,\widetilde{W}_{2n-2}] \,, \tag{3.6a}$$

$$[\widetilde{B},W_{2n-2}]=-[B,\widetilde{W}_{2n-2}] \tag{3.6b}$$

for $1 \le n \le N$. First, consider $n=1$:

$$[B,W_2]=-\tfrac{1}{8}[B,[B,[\widetilde{B},B]]]-[B,\widetilde{B}] \,. \tag{3.7}$$

We see that (3.6a) will be true only if

$$[B,[B,[B,\widetilde{B}]]]=16[B,\widetilde{B}] \,. \tag{3.8}$$

This we shall have to assume as an auxiliary condition to (3.1). Equation (3.6b) is seen to be trivially satisfied for $n=1$. Going on to $n=2$, we observe that

$$\begin{aligned}
[B,W_4] &= -\tfrac{1}{8}[B,[B,[\widetilde{B},W_2]]]-[B,\widetilde{W}_2] \\
&= \tfrac{1}{64}[B,[B,[\widetilde{B},[B,[\widetilde{B},B]]]]]-[B,\widetilde{W}_2] \\
&= \tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},B]]]]+\tfrac{1}{64}[B,[\widetilde{B},[B,[B,[\widetilde{B},B]]]]]-[B,\widetilde{W}_2] \\
&= \tfrac{1}{64}[[B,\widetilde{B}],[B,[B,[\widetilde{B},B]]]]+\tfrac{1}{64}[B,[\widetilde{B},[B,[B,[\widetilde{B},B]]]]]-[B,\widetilde{W}_2] \,.
\end{aligned} \tag{3.9}$$

Using (3.8), we find

$$[B,W_4]=\tfrac{1}{4}[B,[\widetilde{B},[\widetilde{B},B]]]-[B,\widetilde{W}_2]=[B,\widetilde{W}_2] \tag{3.10}$$

so (3.6a) is valid for $n=2$ and

$$[\widetilde{B},W_2]=-\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{B},B]]]=-\tfrac{1}{8}[B,[\widetilde{B},[\widetilde{B},B]]]=-[B,\widetilde{W}_2] \,, \tag{3.11}$$

so (3.6b) is also valid for $n=2$.

It will be useful to consider an alternative formula for W_{2n+2}, valid when (3.6) is valid for a given n. From the definition (3.3),

$$W_{2n+2}=-\tfrac{1}{8}[B,[\widetilde{B},W_{2n}]]-\widetilde{W}_{2n}=-\tfrac{1}{8}[[B,\widetilde{B}],W_{2n}]-\tfrac{1}{8}[\widetilde{B},[B,W_{2n}]]-\widetilde{W}_{2n} \,. \tag{3.12}$$

Applying (3.6), we obtain

$$W_{2n+2}= -\tfrac{1}{8}[[B,\widetilde{B}],W_{2n}] -\tfrac{1}{8}[\widetilde{B},[B,\widetilde{W}_{2n-2}]]-\widetilde{W}_{2n} \,, \tag{3.13}$$

$$W_{2n+2}=-\tfrac{1}{8}[[B,\widetilde{B}],W_{2n}]+W_{2n-2} \,. \tag{3.14}$$

Equation (3.14) can be extended to $n=0$ by defining

$$W_{-2} \equiv -\widetilde{W}_0 \,. \tag{3.15}$$

It can be further extended to $n<0$ by defining

$$W_{-2r} \equiv -\widetilde{W}_{2r-2} \tag{3.16}$$

for $r=1,2,3,\ldots$, as can be seen by taking the dual of (3.14). Thus if (3.6) is valid in the range $1 \le n \le N$, (3.14) will be valid in the range $-N-1 \le n \le N$.

Instead of proving only Eqs. (3.6), it will be conceptually easier to prove a more general equation (3.17). Furthermore, proof of (3.17) results in not only $[H,Q_{2n}]=0$ but also $[Q_{2n},Q_{2m}]=0$, i.e., the conserved charges all commute, a hallmark of exact integrability. We will prove

$$[\widetilde{W}_{2l},W_{2n-2l-2}]=[\widetilde{W}_{2l-2},W_{2n-2l}] \tag{3.17}$$

for all $n\geq 0$, $l\geq 0$. The equation then follows for all l, since if we let $l=n+p$ in (3.17), $p=1,2,\ldots,$ and take the dual of the equation we obtain

$$\begin{aligned}&[W_{2n+2p},\widetilde{W}_{-2p-2}]\\&\quad=[W_{2n+2p-2},\widetilde{W}_{-2p}]\ ,\quad p=1,2,3,\ldots\end{aligned} \tag{3.18}$$

which is (3.17) with $l=-p$. It then also follows for $n<0$ since with use of (3.16), (3.17) is equivalent to

$$[W_{-2l-2},\widetilde{W}_{-2n+2l}]=[W_{-2l},\widetilde{W}_{-2n+2l-2}]\ . \tag{3.19}$$

Letting $n'=-n$, $l'=-l$, and taking the dual of (3.19), we have

$$[\widetilde{W}_{2l'},W_{2n'-2l'-2}]=[\widetilde{W}_{2l'-2},W_{2n'-2l'}]\ , \tag{3.20}$$

which is valid for $n'\leq 0$, $-\infty\leq l'\leq\infty$. Thus if (3.17) is proven for $n\geq 0$, $l\geq 0$ it will also be true for all integral n and l. If we set $l=0$ in (3.17), we obtain

$$[\widetilde{B},W_{2n-2}]=-[B,W_{2n}]\ , \tag{3.21}$$

which is (3.6a) plus (3.6b). When (3.17) is added together for $l=1$, $l=2,\ldots,$ $l=n-1$, the result is

$$[\widetilde{W}_{2n-2},B]=[\widetilde{B},W_{2n-2}]\ , \tag{3.22}$$

which is (3.6b). Thus it is sufficient to prove the more general equation (3.17).

For $n=0$, (3.17) is trivially valid for $-\infty\leq l\leq\infty$. The case ($n=1$, $l=0,1$) follows from (3.8) as shown earlier. We also have proven the case ($n=2$, $l=0,1,2$) explicitly: (3.10), (3.11). We now prove (3.17) by induction. Assume (3.17) to be true for

$$\begin{aligned}&n=N,\ \ 0\leq l\leq N\ ,\\&n=N-1,\ \ 0\leq l\leq N-1\ ,\\&n=N-2,\ \ 0\leq l\leq N-1\ ,\\&n=N,\ \ 0\leq l\leq N\ ,\\&n=N-4,\ \ 0\leq l\leq N-2\ ,\\&\ldots\quad\ldots\\&n=N-k,\ \ 0\leq l\leq N-\tfrac{1}{2}k=\tfrac{1}{2}(N+n)\ ,\\&\ldots\quad\ldots\\&n=2,\ \ 0\leq l\leq\tfrac{1}{2}N+1\ ,\\&n=1,\ \ 0\leq l\leq\tfrac{1}{2}N+\tfrac{1}{2}\ ,\\&n=0,\ \ 0\leq l\leq\tfrac{1}{2}N\ \ (n\text{ and }l\text{ are integers})\ .\end{aligned} \tag{3.23}$$

Equation (3.17) has already been shown for $N=2$. This will serve as the base level for the induction. We now must prove (3.23) for $N\rightarrow N+1$, i.e., we must show (3.17) to be true when

$$\begin{aligned}&n=N+1,\ \ 0\leq l\leq N+1\ ,\\&n=N,\ \ 0\leq l\leq N\ ,\\&n=N-1,\ \ 0\leq l\leq N\ ,\\&n=N-2,\ \ 0\leq l\leq N-1\ ,\\&n=N+1,\ \ 0\leq l\leq N+1\ ,\\&n=N-4,\ \ 0\leq l\leq N-2\ ,\\&\ldots\quad\ldots\\&n=N-k,\ \ 0\leq l\leq N-\tfrac{1}{2}k+\tfrac{1}{2}=\tfrac{1}{2}(N+n+1)\ ,\\&\ldots\quad\ldots\\&n=2,\ \ 0\leq l\leq\tfrac{1}{2}N+\tfrac{3}{2}\ ,\\&n=1,\ \ 0\leq l\leq\tfrac{1}{2}N+1\ ,\\&n=0,\ \ 0\leq l\leq\tfrac{1}{2}N+\tfrac{1}{2}\ .\end{aligned} \tag{3.24}$$

Note that in addition to adding a new level to the induction [the first equation of (3.24), $n=N+1$] we must also raise the limit on l on half of the previous levels. The proof will consist of three parts. First we will prove (3.17) for $n=N+1$, $1\leq l\leq N$. Then we will extend this to $l=0$ and $l=N+1$. Finally we will raise the limit on l for the previous levels, $n<N$.

Part I. We now begin with the first part of the proof. Commute $-\frac{1}{8}[B,\tilde{B}]$ with (3.17) for $n=N$:

$$-\tfrac{1}{8}[[B,\tilde{B}],[\tilde{W}_{2l},W_{2N-2l-2}]]=-\tfrac{1}{8}[[B,\tilde{B}],[\tilde{W}_{2l-2},W_{2N-2l}]] \,. \tag{3.25}$$

Then

$$-\tfrac{1}{8}[[[B,\tilde{B}],\tilde{W}_{2l}],W_{2N-2l-2}]-\tfrac{1}{8}[\tilde{W}_{2l},[[B,\tilde{B}],W_{2N-2l-2}]]$$
$$=-\tfrac{1}{8}[[[B,\tilde{B}],\tilde{W}_{2l-2}],W_{2N-2l}]-\tfrac{1}{8}[\tilde{W}_{2l-2},[[B,\tilde{B}],W_{2N-2l}]] \,. \tag{3.26}$$

For $0\le l\le N$, (3.14) allows us to write

$$-[\tilde{W}_{2l+2},W_{2N-2l-2}]+[\tilde{W}_{2l-2},W_{2N-2l-2}]+[\tilde{W}_{2l},W_{2N-2l}]-[\tilde{W}_{2l},W_{2N-2l-4}]$$
$$=-[\tilde{W}_{2l},W_{2N-2l}]+[\tilde{W}_{2l-4},W_{2N-2l}]+[\tilde{W}_{2l-2},W_{2N-2l+2}]-[\tilde{W}_{2l-2},W_{2N-2l-2}] \,. \tag{3.27}$$

With the assumption (3.23) for $n=N-1$, Eq. (3.27) gives, for $1\le l\le N-1$,

$$2[\tilde{W}_{2l},W_{2N-2l}]=[\tilde{W}_{2l+2},W_{2N-2l-2}]+[\tilde{W}_{2l-2},W_{2N-2l+2}] \,. \tag{3.28}$$

Each of these equations is the sum of two adjacent equations in the desired set [(3.17) with $n=N+1$]. Equation (3.17) with $n=N+1$ and $1\le l\le N$ is a system of N equations in $N+1$ unknowns (it states that all of the considered commutators are equal). Equation (3.28) is a set of only $N-1$ equations in the same $N+1$ unknowns. The former set implies the latter set. To prove (3.17) (with $n=N+1$, $1\le l\le N$) from (3.28) we therefore need one more condition. There are two cases, depending upon whether N is odd or even.

Case 1. N is odd, $N\ge 3$. We now prove the additional equation $[\tilde{W}_{N+1},W_{N-1}]=[\tilde{B},W_{2N}]$. From the assumption (3.23) with $n=N$,

$$[\tilde{B},W_{2N-2}]=[\tilde{W}_{N-1},W_{N-1}] \,. \tag{3.29}$$

Then,

$$-\tfrac{1}{8}[\tilde{B},[B,[\tilde{B},W_{2N-2}]]]=-\tfrac{1}{8}[\tilde{B},[B,[\tilde{W}_{N-1},W_{N-1}]]] \,, \tag{3.30}$$

$$[\tilde{B},W_{2N}]+[\tilde{B},\tilde{W}_{2N-2}]=-\tfrac{1}{8}[\tilde{B},[[B,\tilde{W}_{N-1}],W_{N-1}]]-\tfrac{1}{8}[\tilde{B},[\tilde{W}_{N-1},[B,W_{N-1}]]] \,, \tag{3.31}$$

$$[\tilde{B},W_{2N}]+[\tilde{B},\tilde{W}_{2N-2}]=[\tilde{W}_{N+1},W_{N-1}]-\tfrac{1}{8}[[B,\tilde{W}_{N-1}],[\tilde{B},W_{N-1}]]$$
$$-\tfrac{1}{8}[[\tilde{B},W_{N-3}],[B,\tilde{W}_{N-3}]]+[\tilde{W}_{N-1},W_{N-3}] \,, \tag{3.32}$$

where we have made use of the Jacobi identity, definition (3.3), and the assumption (3.23) with $n=\frac{1}{2}(N-1)$. Using this same assumption again we obtain

$$[\tilde{B},W_{2N}]+[\tilde{B},\tilde{W}_{2N-2}]=[\tilde{W}_{N+1},W_{N-1}]+[\tilde{W}_{N-1},W_{N-3}] \,. \tag{3.33}$$

Finally using the assumption (3.23) with $n=N-1$, $l=\frac{1}{2}(N-1)$, we get

$$[\tilde{B},W_{2N}]=[\tilde{W}_{N+1},W_{N-1}] \,, \tag{3.34}$$

which is an equation derivable from (3.17) with $n=N+1$, $1\le l\le N$, and independent of (3.28). Together (3.28) and (3.34) prove (3.17) with $n=N+1$, $1\le l\le N$, N odd. The crucial step in this proof is the cancellation of the second term on the right-hand side of (3.32). If we had chosen an arbitrary equation from (3.17) as our starting point, instead of (3.29), then the corresponding term would not have canceled.

Case 2. N is even, $N\ge 2$. For this case, the extra condition derived is $[\tilde{B},W_{2N}]=-[B,\tilde{W}_{2N}]$. To prove this we first show an intermediate result, based on assumptions (3.23):

$$\tfrac{1}{8}[[B,\tilde{W}_{2p}],[\tilde{B},W_{2m-2p-2}]]=(p+1)([B,\tilde{W}_{2m}]+[\tilde{B},W_{2m}]) \tag{3.35}$$

for integral p, $0\le 2p\le m-1$ and $1\le m\le N$. First we show (3.35) for $p=0$:

$$\tfrac{1}{8}[[B,\widetilde{B}],[\widetilde{B},W_{2m-2}]]=-\tfrac{1}{8}[B,[\widetilde{B},[B,\widetilde{W}_{2m-2}]]]-\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{B},W_{2m-2}]]]$$
$$=[B,\widetilde{W}_{2m}]+[\widetilde{B},W_{2m}]\ , \tag{3.36}$$

where we have used the fact that $[B,W_{2m-2}]$ is anti-self-dual (ASD) as a consequence of the assumption (3.23) with $n=m-1$. Since (3.35) is true for $p=0$ and all m, $1\le m\le N$, the desired result is true for $m=1$ and $m=2$ (for these cases p is restricted to the value zero). We shall now prove (3.35) inductively in m. Assume (3.35) for $1\le m\le K-1$ for some K, $3\le K\le N$ and show it is true for $m=K$. We assume $p>0$, since the result for $p=0$ has already been shown, and also that the restriction on p following (3.35) holds:

$$\tfrac{1}{8}[[B,\widetilde{W}_{2p}],[\widetilde{B},W_{2K-2p-2}]]=-\tfrac{1}{64}[[B,[\widetilde{B},[B,\widetilde{W}_{2p-2}]]],[\widetilde{B},W_{2K-2p-2}]]$$
$$-\tfrac{1}{8}[[B,\widetilde{W}_{2p-4}],[\widetilde{B},W_{2K-2p-2}]]$$
$$=\tfrac{1}{64}[[\widetilde{B},[B,\widetilde{W}_{2p-2}]],[B,[\widetilde{B},W_{2K-2p-2}]]]$$
$$-\tfrac{1}{64}[B,[[\widetilde{B},[B,\widetilde{W}_{2p-2}]],[\widetilde{B},W_{2K-2p-2}]]]$$
$$-\tfrac{1}{8}[[B,\widetilde{W}_{2p-4}],[\widetilde{B},W_{2K-2p-2}]]\ . \tag{3.37}$$

If $p\ge 2$, then we can apply (3.35) with $m=K-2$, to the last term in (3.37). It can then be shown to be zero by the appropriate assumption in (3.23) [specifically in the form (3.6b) with $n=K-1$]. If $p=1$ this term is zero, since $\widetilde{W}-2=-B$. Therefore,

$$\tfrac{1}{8}[[B,\widetilde{W}_{2p}],[\widetilde{B},W_{2K-2p-2}]]=\tfrac{1}{8}[[B,\widetilde{W}_{2p-2}],[\widetilde{B},W_{2K-2p}]]+\tfrac{1}{8}[[B,\widetilde{W}_{2p-2}],[\widetilde{B},W_{2K-2p-4}]]$$
$$+\tfrac{1}{64}[\widetilde{B},[[B,\widetilde{W}_{2p-2}],[B,[\widetilde{B},W_{2K-2p-2}]]]]$$
$$-\tfrac{1}{64}[B,[\widetilde{B},[[B,\widetilde{W}_{2p-2}],[\widetilde{B},W_{2K-2p-2}]]]]$$
$$+\tfrac{1}{64}[B,[[\widetilde{B},W_{2p-2}],[\widetilde{B},[B,\widetilde{W}_{2K-2p-2}]]]]\ . \tag{3.38}$$

We can apply (3.35) with $m=K-2$ to the second term, since $2p-2\le K-3$, the condition for applicability, follows from the restriction on p for $m=K$: $2p\le K-1$. The second term is then zero by (3.23). Similarly, the fourth term is zero by (3.35) with $m=K-1$ and (3.23) with $n=K$. Also note that the third and fifth terms are the duals of each other. So we can write

$$\tfrac{1}{8}[[B,\widetilde{W}_{2p}],[\widetilde{B},W_{2K-2p-2}]]-\tfrac{1}{8}[[B,\widetilde{W}_{2p-2}],[\widetilde{B},W_{2K-2p}]]$$
$$=-\tfrac{1}{8}[\widetilde{B},[[B,\widetilde{W}_{2p-2}],W_{2K-2p}]]-\tfrac{1}{8}[\widetilde{B},[[B,\widetilde{W}_{2p-2}],\widetilde{W}_{2K-2p-2}]]+\text{dual}\ , \tag{3.39}$$

where dual indicates the dual of the expression. Consider the right-hand side (RHS) of (3.39):

$$\text{RHS}=\tfrac{1}{8}[\widetilde{B},[\widetilde{W}_{2p-2},[B,W_{2K-2p}]]]-\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{W}_{2p-2},W_{2K-2p}]]]$$
$$-\tfrac{1}{8}[[B,\widetilde{W}_{2p-2}],[\widetilde{B},W_{2K-2p-4}]]-\tfrac{1}{8}[[\widetilde{B},[B,\widetilde{W}_{2p-2}]]\ ,\widetilde{W}_{2K-2p-2}]+\text{dual}\ . \tag{3.40}$$

As before the third term is zero:

$$\text{RHS}=\tfrac{1}{8}[[B,\widetilde{W}_{2p-4}],[\widetilde{B}\ ,W_{2K-2p-2}]]+\tfrac{1}{8}[\widetilde{W}_{2p-2},[\widetilde{B},[B,\widetilde{W}_{2K-2p-2}]]]$$
$$-\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{B},W_{2K-2}]]]+[\widetilde{W}_{2p},\widetilde{W}_{2K-2p-2}]+[W_{2p-2},\widetilde{W}_{2K-2p-2}]+\text{dual}\ . \tag{3.41}$$

Again, the first term vanishes. Consider the fourth term:

$$[\widetilde{W}_{2p},\widetilde{W}_{2K-2p-2}]=[\widetilde{W}_{2K-2p-2},W_{-2p-2}]\ . \tag{3.42}$$

Let $l=K-p-1$, $n=K-2p-1$. Then

$$[\widetilde{W}_{2p},\widetilde{W}_{2K-2p-2}]=[\widetilde{W}_{2l},W_{2n-2l-2}]\ . \tag{3.43}$$

If we eliminate p, then

$$l=\tfrac{1}{2}(K+n-1)\leq\tfrac{1}{2}(N+n-1)\ . \tag{3.44}$$

Since $n<N$ (3.43) is therefore covered by the assumption (3.23) which implies that it is ASD:

$$[\widetilde{W}_{2p},\widetilde{W}_{2K-2p-2}]+\text{dual}=0\ . \tag{3.45}$$

The fifth term of (3.41) is also ASD by (3.23) with $n=K-1$. So

$$\text{RHS}=-[\widetilde{W}_{2p-2},\widetilde{W}_{2K-2p}]-[\widetilde{W}_{2p-2},W_{2K-2p-2}]+[\widetilde{B},W_{2K}]+[\widetilde{B},W_{2K-4}]+\text{dual}\ . \tag{3.46}$$

The first term can be disposed of in the same fashion as (3.42). The second and fourth are ASD by assumption (3.23). Therefore, recalling the left-hand side (LHS) (3.39) we have

$$\tfrac{1}{8}[[B,\widetilde{W}_{2p}],[\widetilde{B},W_{2K-2p-2}]]-\tfrac{1}{8}[[B,\widetilde{W}_{2p-2}],[\widetilde{B}W_{2K-2p}]]=[\widetilde{B},W_{2K}]+[B,\widetilde{W}_{2K}]\ . \tag{3.47}$$

If we add this equation for successive p's: $p=1$, $p=2$, etc. up to $p=q$ and use (3.35) for $p=0$, $m=K$, we get

$$\tfrac{1}{8}[[B,\widetilde{W}_{2q}],[\widetilde{B},W_{2K-2q-2}]]=(q+1)([\widetilde{B},W_{2K}]+[B,\widetilde{W}_{2K}])\ , \tag{3.48}$$

which is (3.35) for $m=K$. Thus the proof by induction of (3.35) is complete.

Returning to the main proof, we are considering the case when N is even. As the additional equation from the set (3.17) with $n=N+1$, for the present case we choose to prove

$$[\widetilde{B},W_{2N}]=-[B,\widetilde{W}_{2N}]\ , \tag{3.49}$$

i.e., that $[\widetilde{B},W_{2N}]$ is ASD. This is linearly independent of the equations (3.28). Consider the quantity $[\widetilde{W}_N,W_N]$, which is ASD by construction:

$$[\widetilde{W}_N,W_N]=-\tfrac{1}{8}[[\widetilde{B},[B,\widetilde{W}_{N-2}]],W_N]-[W_{N-2},W_N]\ . \tag{3.50}$$

The second term can be reduced to $-[\widetilde{B},B]$ by assumption (3.23) with $n=1$, $l=\tfrac{1}{2}N$, $\tfrac{1}{2}N-1,\ldots,1$. We can therefore write

$$-\tfrac{1}{8}[[\widetilde{B},[B,\widetilde{W}_{N-2}]],W_N]=\text{ASD}\ , \tag{3.51}$$

$$\tfrac{1}{8}[[B,\widetilde{W}_{N-2}],[\widetilde{B},W_N]]-\tfrac{1}{8}[\widetilde{B},[[B,\widetilde{W}_{N-2}],W_N]]=\text{ASD}\ . \tag{3.52}$$

The second term is

$$\begin{aligned}-\tfrac{1}{8}[\widetilde{B},[[B,\widetilde{W}_{N-2}],W_N]] &= -\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{W}_{N-2},W_N]]]+\tfrac{1}{8}[\widetilde{B},[\widetilde{W}_{N-2},[B,W_N]]]\\ &= -\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{B},W_{2N-2}]]]\\ &\quad+\tfrac{1}{8}[[\widetilde{B},\widetilde{W}_{N-2}],[B,W_N]]+\tfrac{1}{8}[\widetilde{W}_{N-2},[\widetilde{B},[B,W_N]]]\\ &= [\widetilde{B},W_{2N}]+[\widetilde{B},\widetilde{W}_{2N-2}]+\tfrac{1}{8}[[\widetilde{B},W_{N-4}],[B,\widetilde{W}_{N-2}]]\\ &\quad-[\widetilde{W}_{N-2},\widetilde{W}_N]-[\widetilde{W}_{N-2},W_{N-2}]\ .\end{aligned} \tag{3.53}$$

With use of (3.35) with $2p=N-4$, $m=N-2$, on the third term of (3.53) and (3.23) again on the last two, (3.53), except for the first term, can be shown to be ASD. Thus, returning to (3.52), we find

$$\tfrac{1}{8}[[B,\widetilde{W}_{N-2}],[\widetilde{B},W_N]]+[\widetilde{B},W_{2N}]=\text{ASD}\ . \tag{3.54}$$

The first term can be reduced with the help of (3.35) with $2p=N-2$, $m=N$:

$$(\tfrac{1}{2}N)([B,\widetilde{W}_{2N}]+[\widetilde{B},W_{2N}])+[\widetilde{B},W_{2N}]=\text{ASD}\ . \tag{3.55}$$

Adding to (3.55) its dual, we obtain

$$(N+1)([B,\widetilde{W}_{2N}]+[\widetilde{B},W_{2N}])=0\ . \tag{3.56}$$

Therefore, we have proven (3.49). Together with (3.28), (3.49) implies (3.17) with $n=N+1$, $1\le l\le N$, N even. Part I is now complete.

Part II. We now extend this result for $n=N+1$ to include $l=0$ and $l=N+1$. Since (3.17) with $l=N+1$ is just the dual of (3.17) with $l=0$, there is only one equation to prove, namely $n=N+1$, $l=0$, i.e.,

$$[B,W_{2N+2}]=-[\widetilde{B},W_{2N}] \tag{3.57}$$

or, with the result just established

$$[B,W_{2N+2}]=[B,\widetilde{W}_{2N}]\ . \tag{3.58}$$

We expand $[B,W_{2N+2}]$ in the by now familiar manner:

$$\begin{aligned}[B,W_{2N+2}] &= -\tfrac{1}{8}[B,[B,[\widetilde{B},W_{2N}]]]-[B,\widetilde{W}_{2N}]\\ &= -\tfrac{1}{8}[B,[[B,\widetilde{B}],W_{2N}]]-\tfrac{1}{8}[B,[\widetilde{B},[B,W_{2N}]]]-[B,\widetilde{W}_{2N}]\\ &= \tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]+\tfrac{1}{8}[B,[[B,\widetilde{B}],\widetilde{W}_{2N-2}]]+[B,W_{2N-2}]\ ,\end{aligned} \tag{3.59}$$

where (3.23) has been used for $n=N$, $l=0$, i.e.,

$$[B,W_{2N}]=[B,\widetilde{W}_{2N-2}]\ . \tag{3.60}$$

The first term on the right-hand side of (3.59) is

$$\begin{aligned}\tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]] &= \tfrac{1}{64}[[B,[B,\widetilde{B}]],[B,[\widetilde{B},W_{2N-2}]]]+\tfrac{1}{64}[[B,\widetilde{B}],[B,[B,[\widetilde{B},W_{2N-2}]]]]\\ &= -\tfrac{1}{64}[[B,[B,[B,\widetilde{B}]]],[\widetilde{B},W_{2N-2}]]+\tfrac{1}{64}[B,[[B,[B,\widetilde{B}]],[\widetilde{B},W_{2N-2}]]]\\ &\quad -\tfrac{1}{8}[[B,\widetilde{B}],[B,W_{2N}]]-\tfrac{1}{8}[[B,\widetilde{B}],[B,\widetilde{W}_{2N-2}]]\\ &= -\tfrac{1}{4}[[B,\widetilde{B}],[\widetilde{B},W_{2N-2}]]+\tfrac{1}{64}[B,[B,[[B,\widetilde{B}],[\widetilde{B},W_{2N-2}]]]]\\ &\quad -\tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]-\tfrac{1}{4}[[B,\widetilde{B}],[B,\widetilde{W}_{2N-2}]]\\ &= \tfrac{1}{64}[B,[B,(-[B,[\widetilde{B},[B,\widetilde{W}_{2N-2}]]]-[\widetilde{B},[B,[\widetilde{B},W_{2N-2}]]])]]\\ &\quad -\tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]\\ &= \tfrac{1}{8}[B,[B,([B,\widetilde{W}_{2N}]+[B,W_{2N-2}]+[\widetilde{B},W_{2N}]+[\widetilde{B},\widetilde{W}_{2N-2}])]]\\ &\quad -\tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]\ .\end{aligned} \tag{3.61}$$

The first term in (3.61) is zero due to the assumptions (3.23) and (3.56). Therefore, we have shown that the expression on the left-hand side of (3.61) is equal to its negative and is therefore zero. Returning to (3.59) we may now write

$$\begin{aligned}[B,W_{2N+2}] &= \tfrac{1}{8}[B,[B,[\widetilde{B},\widetilde{W}_{2N-2}]]]-\tfrac{1}{8}[B,[\widetilde{B},[B,\widetilde{W}_{2N-2}]]]+[B,W_{2N-2}]\\ &= -[B,\widetilde{W}_{2N-4}]+[B,\widetilde{W}_{2N}]+[B,W_{2N-2}]=[B,\widetilde{W}_{2N}]\ .\end{aligned} \tag{3.62}$$

Thus (3.58) is true and (3.17) holds for $n=N+1$ and $l=0$ or $l=N+1$.

Part III. The final step involved is to raise the limit on l for $n=N-1$, $n=N-3,\ldots$. Commute $-\frac{1}{8}[B,\widetilde{B}]$ with (3.17) with $n=N$, $l=N$ to obtain (3.27) with $l=N$:

$$\begin{aligned}&[\widetilde{W}_{2N+2},\widetilde{B}]-[\widetilde{W}_{2N-2},\widetilde{B}]+[\widetilde{W}_{2N},B]+[\widetilde{W}_{2N},\widetilde{W}_2]\\ &\qquad\qquad =-[\widetilde{W}_{2N},B]+[\widetilde{W}_{2N-4},B]+[\widetilde{W}_{2N-2},W_2]+[\widetilde{W}_{2N-2},\widetilde{B}]\ .\end{aligned} \tag{3.63}$$

Using the results of the previous sections [(3.17) with $n=N+1$, $l=N+1$, and $l=N$] and the assumption

(3.23) with $n=N-1$, $l=N-1$, we find

$$[\widetilde{W}_{2N},\widetilde{W}_2]=[\widetilde{W}_{2N-2},\widetilde{B}] \tag{3.64}$$

which is (3.17) with $n=N-1$, $l=N$. Thus we have obtained the desired result for $n=N-1$. The same procedure can be applied to the lower levels, namely, if we commute $-\frac{1}{8}[B,\widetilde{B}]$ with (3.17) with $n=k\equiv N-2j$ for integral j, $0\le 2j\le N-1$, and for $l=\frac{1}{2}(N+k)=N-j$ we get (3.27) with $N\to k$, $l\to\frac{1}{2}(N+k)$:

$$-[\widetilde{W}_{N+k+2},W_{k-N-2}]+[\widetilde{W}_{N+k-2},W_{k-N-2}]+[\widetilde{W}_{N+k},W_{k-N}]-[\widetilde{W}_{N+k},W_{k-N-4}]$$
$$=-[\widetilde{W}_{N+k},W_{k-N}]+[\widetilde{W}_{N+k-4},W_{k-N}]+[\widetilde{W}_{N+k-2},W_{k-N+2}]-[\widetilde{W}_{N+k-2},W_{k-N-2}]\,. \tag{3.65}$$

The right-hand side can be seen to be zero from the assumption (3.23) with $n=k+1$, $l=N-j$ and $n=k-1$, $l=N-j-1$, leaving

$$[\widetilde{W}_{N+k+2},W_{k-N-2}]-[\widetilde{W}_{N+k},W_{k-N}]=-[\widetilde{W}_{N+k},W_{k-N-4}]-[\widetilde{W}_{N+k-2},W_{k-N-2}]\,. \tag{3.66}$$

The left-hand side [of (3.66)] is zero if (3.17) with $n=k+1$, $l=\frac{1}{2}(N+k)+1=\frac{1}{2}[N+1+(k+1)]$ is true. The right-hand side is zero if (3.17) with $n=k-1$, $l=\frac{1}{2}(N+k)=\frac{1}{2}[N+1+(k-1)]$ is true. If we add Eq. (3.66) times $(-1)^j$ for $j=0,1,2,\ldots,q$ successively (recall $k=N-2j$) we get

$$-[\widetilde{W}_{2N+2},\widetilde{B}]-[\widetilde{W}_{2N},B] = (-1)^q(-[\widetilde{W}_{2N-2q},W_{-2q-4}] + [\widetilde{W}_{2N-2q-2},W_{-2q-2}]) \tag{3.67}$$

for $0\le q\le\frac{1}{2}(N-1)$. The left-hand side is zero by (3.62). Thus

$$[\widetilde{W}_{2N-2q},W_{-2q-4}]=[\widetilde{W}_{2N-2q-2},W_{-2q-2}] \tag{3.68}$$

for $0\le q\le\frac{1}{2}(N-1)$. Setting $n=N-2q-1$, $l=N-q$ in (3.68) we get (3.17) for $n=N-2q-1$, $l=\frac{1}{2}(N+n+1)$:

$$[\widetilde{W}_{2l},W_{2n-2l-2}]=[\widetilde{W}_{2l-2},W_{2n-2l}]\,. \tag{3.69}$$

Looking at (3.24) we can see we have achieved the desired result, i.e., raising the integral limit on l by one for $n=N-1, N-3, \ldots$ (0 or 1). This is equivalent to raising the half-integral limit on l given by $l=\frac{1}{2}(N+n)$ by $\frac{1}{2}$ for all $n\le N$. All entries in (3.24) have now been derived. The proof by induction of (3.17) is complete.

Thus, (3.17) holds for all n and l and charges Q_{2n} defined in (3.2) are conserved for all n $(n=1,2,3,\ldots,)$. This result requires only two assumptions. The first is that the Hamiltonian can be written in the self-dual form (3.1). The second assumption is (3.8), which states that the *first* charge in the set, i.e. Q_2, is conserved.

In addition to the conservation of charges $[Q_{2n},H]=0$, it is also possible to show that the charges Q_{2n} commute with each other. $[Q_{2n},Q_{2m}]$ will vanish if

$$[(W_{2n}-\widetilde{W}_{2n-2}),(W_{2m}-\widetilde{W}_{2m-2})]=0 \tag{3.70}$$

and

$$[(W_{2n}-\widetilde{W}_{2n-2}),(\widetilde{W}_{2m}-W_{2m-2})]=\text{ASD}\,. \tag{3.71}$$

Consider $[W_{2n},W_{2m}]$. By (3.16) and (3.17)

$$[W_{2n},W_{2m}] = -[W_{2n},\widetilde{W}_{-2m-2}] = -[B,\widetilde{W}_{2n-2m-2}]\,. \tag{3.72}$$

Also,

$$[\widetilde{W}_{2n-2},\widetilde{W}_{2m-2}] = -[\widetilde{W}_{2n-2},W_{-2m}] = -[\widetilde{B},W_{2n-2m-2}] = [B,\widetilde{W}_{2n-2m-2}]\,. \tag{3.73}$$

So,

$$[W_{2n},W_{2m}]=-[\widetilde{W}_{2n-2},\widetilde{W}_{2m-2}]\,. \tag{3.74}$$

By a similar argument (3.17) shows that

$$[W_{2n},\widetilde{W}_{2m-2}] = [W_{2n-2},\widetilde{W}_{2m}] = -[\widetilde{W}_{2n-2},W_{2m}]\,. \tag{3.75}$$

Together (3.74) and (3.75) prove (3.70). It can easily be checked that (3.17) implies that all four of the commutators in (3.71) are ASD, thus (3.71) is satisfied. Therefore,

$$[Q_{2n},Q_{2m}]=0 \tag{3.76}$$

for all n,m, $1\le n\le\infty$, $1\le m\le\infty$.

IV. GAUGE THEORIES

The relevant framework for studying nonperturbative effects in non-Abelian gauge systems appears to be a nonlocal reformulation. Instead of the local Lagrangian $\mathscr{L}(x)=\frac{1}{4}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)$, the theory is described by a functional formalism whose fundamental variable is the element of the holonomy group

$$\psi[\xi]=P\exp\left[\oint A\cdot d\xi\right] .$$

Loop-space Green's functions can then be defined as

$$\langle 0\,|\,\mathrm{tr}\psi[\xi_1]\mathrm{tr}\psi[\xi_2]\cdots\,|\,0\rangle$$

and expressed in terms of the "string propagator"

$$G[\xi]=\langle 0\,|\,\mathrm{tr}\psi[\xi]\,|\,0\rangle .$$

It has been suggested[4,13] that once the propagation of free strings is described, knowledge of the hidden symmetry of interacting strings could be used to restrict the Green's functions and thus solve gluon dynamics, in analogy with this chain of events in the nonlinear σ model.

So far, the only example of interacting string symmetry is in three dimensions. In this case, the equations are similar to the chiral model, where it is known that a Kac-Moody infinite-parameter subalgebra is responsible for the nonlocal currents.[5] It is not unreasonable to speculate that this abstract Lie algebra can also be realized in the four-dimensional theory, but the explicit form of the conserved charges has remained well hidden.

The ultimate goal of our investigation is to exploit the suggested self-duality of the four-dimensional gauge system in pursuit of the conservation laws. 't Hooft[6] introduced path-dependent operators for the gauge system in order to concentrate directly on the formation of flux tubes like magnetic monopoles which could drive a magnetic Higgs mechanism resulting in electric confinement:

$$A(C)\equiv\frac{1}{2}\,\mathrm{tr}P\exp\left[i\oint\vec{\mathrm{A}}\cdot d\vec{\mathrm{x}}\right] \equiv e^{i\phi_B} . \tag{4.1}$$

The dual operator $B(C)$ measures electric flux and was defined by its commutation relation with A:

$$A(C)B(C')=B(C')A(C)e^{2\pi in/N} , \tag{4.2}$$

where N is from SU(N) and $n=1,2,\ldots,$ is the number to times the closed path C' encircles C. Furthermore, the operator $B(C)$ can be derived from $A(C)$ also by a partial Kramers-Wannier Z(N) dual transformation. That is to say the operator $A(C)$, an element of SU(N), can be expressed in terms of matrices of the factor group SU(N)/Z(N) and of the group Z(N). A transformation similar to Kramers-Wannier duality for Z(N) gauge theories then results in the disorder parameter $B(C)$.[23] Also, in treatment of the whole theory, the four-dimensional SU(N)-gauge-theory partition function has been noted to reflect a self-duality property similar to the two-dimensional Ising model.[6]

In order to generate conserved charges with our result, the fundamental operators must satisfy the extra condition $[B,[B,[B,\widetilde{B}]]]=16\,[B,\widetilde{B}]$. If we define $B=A(C)$ and $\widetilde{B}=B(C')$, then

$$[A(C),[A(C),[A(C),B(C')]]] =A^2(C)(1-e^{2\pi in/N})^2[A(C),B(C')] . \tag{4.3}$$

That is to say, with this choice of B and $\widetilde{B}$, the first charge is not conserved and we cannot generate the infinite set.

This may be due to the fact that $B(C')$ is only a partial dual of $A(C)$. In this way, we can think of the extra condition as a guide in choosing the correct duality transformation. It of course must also have the property that $\widetilde{\widetilde{A}}=A$.

V. CONCLUSION

In this paper we have developed a rigorous connection between simple self-duality and a set of commuting conserved charges. For a finite lattice, the set is finite. For an infinite system, the set is infinite. The number of charges equals the number of degrees of freedom of the system, the characteristic of exact integrability. In the XZ and Ising models, our charges coincide with known results. For the family[24]

$$\hat{H}_k=K\sum_{n=1}^{N}\sigma_1(n)\sigma_2(n+1)\cdots\sigma_2(n+k)\sigma_1(n+k+1) +\Gamma\sum_{n=1}^{N}(-1)^k\sigma_3(n)\sigma_2(n+1)\cdots\sigma_2(n+k)\sigma_3(n+k+1) \tag{5.1}$$

the first charge is conserved and we can generate a previously unknown set of conserved charges.

The advantage of our result is that it does not depend on the lattice formulation or the number of dimensions. Although the self-dual formulation of four-dimensional SU(N) gauge theories in terms of $A(C)$ and $B(C')$ operators does not fit exactly into our calculation, the ideas in this paper seem valuable for gauge theories as well in that they establish a connection in simpler models which may be reflected in Yang-Mills theory, and thus provide a new tool with which to search for symmetry.

ACKNOWLEDGMENT

This work was supported in part by the U. S. Department of Energy under Contract No. DE-AC02-81ER40033. B000.

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