Relativistic Newtonian mechanics for particles with spin

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Conditions for Lorentz invariance are derived for Newtonian equations of motion for particles with positive mass and nonzero spin. Additional equations are obtained for two-particle forces when it is assumed that sums of two-particle forces satisfy the Lorentz-invariance conditions for three-particle systems. These imply that the two-particle forces satisfy separately the nonlinear and linear parts of the Lorentz-invariance conditions. Lorentz transformations justified by the linear parts are used to obtain the general forms of the two-particle forces, as functions of the momentum of the source particle, in terms of the forces in the rest frame of the source particle. These have to satisfy the remaining nonlinear equations.

INTRODUCTION

One of the more interesting characteristics of relativistic Newtonian mechanics is that in a system of more than two particles the force on a particle generally cannot be the sum of two-particle forces. The reason is that the conditions for Lorentz invariance of Newtonian equations of motion are nonlinear equations for the forces.[1–3] As soon as these conditions were formulated[1,2] it was evident that the nonlinearity makes it impossible to use sums of two-particle forces,[2] but a proof was published only recently.[4] The proof[4] was for particles with positive mass and zero spin. It assumed that the two-particle forces satisfy the Lorentz-invariance conditions for systems of two particles and that sums of these two-particle forces satisfy the Lorentz-invariance conditions for systems of three particles. It showed that then a particle can be accelerated only by forces from particles that do not accelerate.

When one or more of the particles has zero mass, this statement is typically but not generally true. There are exceptional cases where these assumptions allow two-particle forces with acceleration of both particles; they are based on equations of motion for the momenta and the fact that for particles with zero mass there are more variables for momenta than for velocities.[4]

Are there any exceptions for particles with spin? Are there any two-particle forces that escape this implication of the nonlinearity so that forces in systems of more than two particles can be sums of these two-particle forces? As before, exceptions of only a technical nature, that are not particularly meaningful physically, might be interesting because they would indicate what could and could not be proved in more sophisticated formulations.[5] We shall find there are exceptions but they appear to be limited.

Here we first formulate relativistic Newtonian mechanics for particles with positive mass and nonzero spin. Leaving aside more difficult problems involving equations of motion for particles with spin,[6] we just consider Newtonian equations of motion for a simple but adequate choice of position and spin variables, and find the conditions for these equations of motion to the Lorentz invariant, assuming altogether that they are Poincaré invariant.

Then we find the equations two-particle forces have to satisfy, in addition to the Lorentz-invariance conditions for the two-particle systems, if sums of these two-particle forces are to satisfy the Lorentz-invariance conditions for three-particle systems. These imply the two-particle forces satisfy separately the nonlinear and linear parts of the Lorentz-invariance conditions. The linear parts are the simplified Lorentz-invariance conditions we would have for the force on one particle in a two-particle system if there were no force on the other particle. By making a Lorentz transformation from the rest frame of the other particle—calculated according to the simplified Lorentz-invariance conditions, as if there were no force on the other particle—we obtain the general forms of the two-particle forces, as functions of the momentum of the other particle, in terms of the forces in the rest frame of the other particle. These have to satisfy the remaining nonlinear equations. The situation that results is summarized at the end of the paper. Some limited solutions of the nonlinear equations will be found in a subsequent paper.

I. EQUATIONS OF MOTION

Consider a classical-mechanical system of N particles described by positions $\vec{x}^n$, velocities $\vec{v}^n = d\vec{x}^n/dt$, and spins $\vec{s}^n$, for $n=1,2,\ldots,N$. We assume each particle has positive mass m_n and use relativistic momenta

$$\vec{u}^n = m_n \vec{v}^n/[1-(\vec{v}^n)^2]^{1/2} , \tag{1.1}$$

so that $\vec{v}^n = \vec{u}^n/\vec{u}^n{}_0$, with

$$u^n{}_0 = [(\vec{u}^n)^2 + m_n{}^2]^{1/2} . \tag{1.2}$$

We choose units such that $c=1$.

The spin variables we use are those for which the Pauli-Lubanski four-vectors are[7,8]

$$w^n{}_0 = \vec{u}^n \cdot \vec{s}^n , \tag{1.3}$$

$$\vec{w}^n = (u^n{}_0 + m_n)^{-1}(\vec{u}^n \cdot \vec{s}^n)\vec{u}^n + m_n \vec{s}^n , \tag{1.4}$$

so that

$$\vec{s}^n = m_n{}^{-1}[\vec{w}^n - (u^n{}_0 + m_n)^{-1} w^n{}_0 \vec{u}^n] . \tag{1.5}$$

They correspond to the ordinary quantum-mechanical spin operators that have angular-momentum commutation relations and commute with the position and momentum operators.[8] We assume the length $|\vec{s}^n|$ is fixed, so only the direction of $\vec{s}^n$ is variable.

We consider equations of motion of the form

$$d\vec{u}^n/dt = \vec{F}^n(\vec{x}^1,\vec{x}^2,\ldots,\vec{x}^N,\vec{u}^1,\vec{u}^2,\ldots,\vec{u}^N,\vec{s}^1,\vec{s}^2,\ldots,\vec{s}^N) , \tag{1.6}$$

$$d\vec{s}^n/dt = \vec{G}^n(\vec{x}^1,\vec{x}^2,\ldots,\vec{x}^n,\vec{u}^1,\vec{u}^2,\ldots,\vec{u}^N,\vec{s}^1,\vec{s}^2,\ldots,\vec{s}^N) \tag{1.7}$$

for $n=1,2,\ldots,N$, where t is ordinary time and $\vec{F}^n$ and $\vec{G}^n$ are functions of the positions, momenta, and spins of the different particles at one time. To keep $|\vec{s}^n|$ fixed we must have

$$\vec{s}^n \cdot \vec{G}^n = 0 \tag{1.8}$$

for $n=1,2,\ldots,N$. We make the equations of motion invariant for time translations by not letting $\vec{F}^n$ or $\vec{G}^n$ depend explicitly on time, for space translations by letting $\vec{F}^n$ and $\vec{G}^n$ depend on the positions only through the relative positions $\vec{x}^n - \vec{x}^m$, and for rotations by letting $\vec{F}^n$ and $\vec{G}^n$ be vector functions (that rotate when $\vec{x}^1,\vec{x}^2,\ldots,\vec{x}^N$, $\vec{u}^1,\vec{u}^2,\ldots,\vec{u}^N,\vec{s}^1,\vec{s}^2,\ldots,\vec{s}^N$ rotate). Lorentz invariance remains to be considered.

II. CONDITIONS FOR LORENTZ INVARIANCE

Consider a Lorentz transformation from time-space coordinates $x=(t,\vec{x})$ to $x'=(t',\vec{x}')$. Let Λ be the four-by-four matrix for this transformation of four-vector coordinates, so that

$$x' = \Lambda x . \tag{2.1}$$

We assume a point on the world line of a particle has coordinates

$$x^n(t) = (t, \vec{x}^n(t)) \tag{2.2}$$

and

$$x'^n(t') = (t', \vec{x}'^n(t')) \tag{2.3}$$

in the two reference frames, which are related by

$$x'^n(t') = \Lambda x^n(t) . \tag{2.4}$$

It follows that

$$u'^n(t') = \Lambda u^n(t) \tag{2.5}$$

for the energy-momentum four-vector $u^n = (u^n{}_0, \vec{u}^n)$. We assume the four-vector transformation

$$w'^n(t') = \Lambda w^n(t) \tag{2.6}$$

also for the Pauli-Lubanski four-vector $w^n = (w^n{}_0, \vec{w}^n)$.

The combination of position and spin variables we use is not one of those described by Pryce[9] for which

$$\vec{x}^n \times \vec{u}^n + \vec{s}^n \tag{2.7}$$

is the particle's total angular momentum corresponding to the quantum-mechanical operator that is the generator of unitary transformations of the particle's wave function for rotations. For that, the spin variables we use would have to be paired with the Newton-Wigner position[10] which does not transform as part of a four-vector.[10,11]

We assume the equations of motion (1.6) and (1.7) are Lorentz invariant. This means that after a Lorentz transformation the equations of motion are the same, with the same functions $\vec{F}^n$ and $\vec{G}^n$, in terms of the variables of the new reference frame at one time in that frame.

We assume the world line of every particle and the particle's spin at every point on the world line are determined by the equations of motion from the initial values of the positions, momenta, and spins. Then $\vec{x}'^n$, $\vec{u}'^n$, $\vec{s}'^n$ at $t'=0$ are functions of $\vec{x}^n$, $\vec{u}^n$, $\vec{s}^n$ at $t=0$. They are also functions of the parameters of the Lorentz transformation, the velocity $\vec{\beta}$ of the new frame relative to the old. We use a bracket-generator symbol $[\ ,\vec{K}]$ for the derivative with respect to $\vec{\beta}$ at $\vec{\beta}=0$, so to first order in $\vec{\beta}$ we write

$$f(\vec{x}'^1,\vec{x}'^2,\ldots,\vec{x}'^N,\vec{u}'^1,\vec{u}'^2,\ldots,\vec{u}'^N,\vec{s}'^1,\vec{s}'^2,\ldots,\vec{s}'^N)-f(\vec{x}^1,\vec{x}^2,\ldots,\vec{x}^N,\vec{u}^1,\vec{u}^2,\ldots,\vec{u}^N,\vec{s}^1,\vec{s}^2,\ldots,\vec{s}^N)$$
$$=\sum_{n=1}^{N}\sum_{j=1}^{3}\{(\partial f/\partial x^n{}_j)[x^n{}_j,\vec{\beta}\cdot\vec{K}]+(\partial f/\partial u^n{}_j)[u^n{}_j,\vec{\beta}\cdot\vec{K}]+(\partial f/\partial s^n{}_j)[s^n{}_j,\vec{\beta}\cdot\vec{K}]\} \tag{2.8}$$

for the change in a function f of the position, momentum, and spin variables from the old frame at $t=0$ to the new frame at $t'=0$. The building blocks are the particular cases where f is just one of the variables. From our assumptions of four-vector transformations it follows that

$$[x^n{}_j,K_k]=x^n{}_k v^n{}_j=x^n{}_k u^n{}_j/u^n{}_0\ , \tag{2.9}$$

$$[u^n{}_j,K_k]=x^n{}_k du^n{}_j/dt-\delta_{jk}u^n{}_0=x^n{}_k F^n{}_j-\delta_{jk}u^n{}_0\ , \tag{2.10}$$

$$[u^n{}_0,K_k]=x^n{}_k du^n{}_0/dt-u^n{}_k=x^n{}_k\vec{u}^n\cdot\vec{F}^n/u^n{}_0-u^n{}_k\ , \tag{2.11}$$

$$[w^n{}_j,K_k]=x^n{}_k dw^n{}_j/dt-\delta_{jk}w^n{}_0\ , \tag{2.12}$$

$$[w^n{}_0,K_k]=x^n{}_k dw^n{}_0/dt-w^n{}_k\ , \tag{2.13}$$

$$\begin{aligned}[s^n{}_j,K_k]&=x^n{}_k ds^n{}_j/dt+(u^n{}_0+m_n)^{-1}u^n{}_j s^n{}_k-(u^n{}_0+m_n)^{-1}(\vec{u}^n\cdot\vec{s}^n)\delta_{jk}\\&=x^n{}_k G^n{}_j+(u^n{}_0+m_n)^{-1}u^n{}_j s^n{}_k-(u^n{}_0+m_n)^{-1}(\vec{u}^n\cdot\vec{s}^n)\delta_{jk}\end{aligned} \tag{2.14}$$

for $n=1,2,\ldots,N$ and $j,k=1,2,3$. In each case the first term is the effect of changing from $t=0$ to $t'=0$ which on the world line of particle n means $t=\vec{\beta}\cdot\vec{x}^n$.[12,13,3]

Altogether we assume the equations of motion are Poincaré invariant. This means the Lorentz transformations must fit in with the other transformations of the Poincaré group. Using bracket-generator symbols $[\ ,H]$ and $[\ ,\vec{P}]$ for the time and space translations,[14] and using the Lie bracket relations of the Poincaré group,[14,7] we get

$$[[u^n{}_j,H],K_k]=[[u^n{}_j,K_k],H]-[u^n{}_j,[K_k,H]]=[[u^n{}_j,K_k],H]-[u^n{}_j,P_k] \tag{2.15}$$

and similarly

$$[[s^n{}_j,H],K_k]=[[s^n{}_j,K_k],H]-[s^n{}_j,P_k]\ . \tag{2.16}$$

Since the momenta and spins are invariant for space translations we have

$$[u^n{}_j,P_k]=0\ , \tag{2.17}$$

$$[s^n{}_j,P_k]=0\ . \tag{2.18}$$

Then for the changes Lorentz transformations make in the time derivatives of the momenta and spins we get

$$[du^n{}_j/dt, K_k] = d[u^n{}_j, K_k]/dt\ , \tag{2.19}$$

$$[ds^n{}_j/dt, K_k] = d[s^n{}_j, K_k]/dt \tag{2.20}$$

which we can calculate by taking the time derivatives of (2.10) and (2.14).

Using the latter to transform the left sides of the equations of motion, and using the former equations (2.9), (2.10), and (2.14) to transform the right sides as in (2.8), and requiring that the changes are the same for both sides, we find that the conditions for Lorentz invariance of the equations of motion (1.6) and (1.7) are

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{l=1}^{3}(x^n{}_k - x^m{}_k)(F^m{}_l \partial F^n{}_j/\partial u^m{}_l + G^m{}_l \partial F^n{}_j/\partial s^m{}_l) + \sum_{m=1}^{N}\sum_{l=1}^{3}(x^n{}_k - x^m{}_k)(u^m{}_l/u^m{}_0)\partial F^n{}_j/\partial x^m{}_l \\
&\qquad + \sum_{m=1}^{N} u^m{}_0 \partial F^n{}_j/\partial u^m{}_k \\
&\qquad + \sum_{m=1}^{N}(u^m{}_0 + m_m)^{-1}\left[(\vec{u}^m\cdot\vec{s}^m)\partial F^n{}_j/\partial s^m{}_k - \sum_{l=1}^{3} s^m{}_k u^m{}_l \partial F^n{}_j/\partial s^m{}_l\right] \\
&\qquad + F^n{}_j u^n{}_k/u^n{}_0 - \delta_{jk}(\vec{u}^n\cdot\vec{F}^n)/u^n{}_0 = 0\ ,
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{l=1}^{3}(x^n{}_k - x^m{}_k)(F^m{}_l \partial G^n{}_j/\partial u^m{}_l + G^m{}_l \partial G^n{}_j/\partial s^m{}_l) + \sum_{m=1}^{N}\sum_{l=1}^{3}(x^n{}_k - x^m{}_k)(u^m{}_l/u^m{}_0)\partial G^n{}_j/\partial x^m{}_l \\
&\quad + \sum_{m=1}^{N} u^m{}_0 \partial G^n{}_j/\partial u^m{}_k + \sum_{m=1}^{N}(u^m{}_0 + m_m)^{-1}\left[(\vec{u}^m\cdot\vec{s}^m)\partial G^n{}_j/\partial s^m{}_k - \sum_{l=1}^{3} s^m{}_k u^m{}_l \partial G^n{}_j/\partial s^m{}_l\right] \\
&\quad + G^n{}_j u^n{}_k/u^n{}_0 + F^n{}_j s^n{}_k/(u^n{}_0 + m_n) + G^n{}_k u^n{}_j/(u^n{}_0 + m_n) - u^n{}_j s^n{}_k(\vec{u}^n\cdot\vec{F}^n)/u^n{}_0(u^n{}_0 + m_n)^2 \\
&\quad - \delta_{jk}(\vec{s}^n\cdot\vec{F}^n)/(u^n{}_0 + m_n) + \delta_{jk}(\vec{u}^n\cdot\vec{s}^n)(\vec{u}^n\cdot\vec{F}^n)/u^n{}_0(u^n{}_0 + m_n)^2 - \delta_{jk}(\vec{u}^n\cdot\vec{G}^n)/(u^n{}_0 + m_n) = 0
\end{aligned} \tag{2.22}$$

for $n = 1, 2, \ldots, N$ and $j, k = 1, 2, 3$.

The fixed values of the $|\vec{s}^n|$ and the conditions (1.8) that keep them fixed are consistent with Lorentz invariance. Lorentz transformations do not change $|\vec{s}^n|$ because

$$|\vec{w}^n|^2 - (w^n{}_0)^2 = m_n{}^2 |\vec{s}^n|^2\ .$$

III. TWO-PARTICLE FORCES IN MANY-PARTICLE SYSTEMS

Suppose the force on each particle is a sum of Poincaré-covariant two-particle forces, so that

$$\begin{aligned}
\vec{F}^n &= \sum_{\substack{r=1\\ r\neq n}}^{N} \vec{F}^{nr}(\vec{x}^n - \vec{x}^r, \vec{u}^n, \vec{s}^n, \vec{u}^r, \vec{s}^r)\ , \\
\vec{G}^n &= \sum_{\substack{r=1\\ r\neq n}}^{N} \vec{G}^{nr}(\vec{x}^n - \vec{x}^r, \vec{u}^n, \vec{s}^n, \vec{u}^r, \vec{s}^r)\ ,
\end{aligned} \tag{3.1}$$

where $\vec{F}^{nr}, \vec{G}^{nr}$ and $\vec{F}^{rn}, \vec{G}^{rn}$ are rotational-vector functions that satisfy the Lorentz-invariance conditions (2.21) and (2.22) for a system of two particles. By comparing the Lorentz-invariance conditions for two-particle and three-particle systems, we find that

$$\begin{aligned}
&\sum_{l=1}^{3}(F^{rs}{}_l \partial\vec{F}^{nr}/\partial u^r{}_l + G^{rs}{}_l \partial\vec{F}^{nr}/\partial s^r{}_l) = 0\ , \\
&\sum_{l=1}^{3}(F^{rs}{}_l \partial\vec{G}^{nr}/\partial u^r{}_l + G^{rs}{}_l \partial\vec{G}^{nr}/\partial s^r{}_l) = 0
\end{aligned} \tag{3.2}$$

for $r\neq n$, $s\neq r$, and $s\neq n$. By considering three-particle systems in which particles n and s cause identical forces, and assuming $\vec{F}^{rs}$ and $\vec{G}^{rs}$ are continuous functions of $\vec{x}^s, \vec{u}^s, \vec{s}^s$, we conclude that these equations (3.2) hold also for $s = n$, so they hold for all $r\neq n$ and $s\neq r$.

Then $\vec{F}^{nr}$ and $\vec{G}^{nr}$ satisfy separately the nonlinear and linear parts of the Lorentz-invariance conditions (2.21) and (2.22). In particular, the linear parts satisfied by $\vec{F}^{nr}$ and $\vec{G}^{nr}$ are the simplified Lorentz-invariance conditions (2.21) and (2.22) that we would have for the system of two particles n and r if $\vec{F}^{rn}$ and $\vec{G}^{rn}$ were zero.

IV. CONSTRUCTING TWO-PARTICLE FORCES

The last observation suggests a way to construct $\vec{F}^{nr}$ and $\vec{G}^{nr}$ as functions of $\vec{u}^r$. Let ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ be $\vec{F}^{nr}$ and $\vec{G}^{nr}$ for $\vec{u}^r=0$. That is what we have in the rest frame of particle r. To go to other frames, for nonzero $\vec{u}^r$, we use Lorentz transformations with

$$\vec{\beta}=-\vec{u}^r/u^r{}_0$$

and get $\vec{F}^{nr}$ and $\vec{G}^{nr}$ as the Lorentz transforms of ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$. In calculating these Lorentz transforms we can let $\vec{F}^{rn}$ and $\vec{G}^{rn}$ be zero, and therefore let $\vec{u}^r$ and $\vec{s}^r$ be constant in time, since the Lorentz-invariance conditions for the system of two particles n and r are satisfied with zero $\vec{F}^{rn}$ and $\vec{G}^{rn}$. Lorentz invariance means the Lorentz transforms of ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ are the functions $\vec{F}^{nr}$ and $\vec{G}^{nr}$ of the variables $\vec{u}^n$, $\vec{s}^n$, $\vec{x}^n-\vec{x}^r$, $\vec{u}^r$, and $\vec{s}^r$ for the new frame.[15] Let ${}^0\vec{u}^n$, ${}^0\vec{s}^n$, ${}^0\vec{x}^n-{}^0\vec{x}^r$, ${}^0\vec{u}^r$, and ${}^0\vec{s}^r$ be the inverse Lorentz transforms of $\vec{u}^n$, $\vec{s}^n$, $\vec{x}^n-\vec{x}^r$, $\vec{u}^r$, and $\vec{s}^r$, also calculated as if $\vec{u}^r$ and $\vec{s}^r$ were constant in time. These are the variables for the rest frame of particle r on which ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ can depend. They are functions of $\vec{u}^n$, $\vec{s}^n$, $\vec{x}^n-\vec{x}^r$, $\vec{u}^r$, and $\vec{s}^r$. Of course ${}^0\vec{u}^r$ is zero.

From ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ given as functions of ${}^0\vec{u}^n$, ${}^0\vec{s}^n$, ${}^0\vec{x}^n-{}^0\vec{x}^r$, and ${}^0\vec{s}^r$ we get $\vec{F}^{nr}$ and $\vec{G}^{nr}$ as functions of $\vec{u}^n$, $\vec{s}^n$, $\vec{x}^n-\vec{x}^r$, $\vec{u}^r$, and $\vec{s}^r$ that satisfy the simplified Lorentz-invariance conditions (2.21) and (2.22) that we would have for the system of two particles n and r if $\vec{F}^{rn}$ and $\vec{G}^{rn}$ were zero. Conversely, any solution of these simplified Lorentz-invariance conditions must have this form for some functions ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$. This provides the general solution for the linear parts of the Lorentz-invariance conditions. Then from the Lorentz-invariance conditions only Eqs. (3.2) remain to be satisfied, since they imply that the nonlinear parts of the Lorentz-invariance conditions are satisfied for systems of either two or more particles.

More specifically, consider the Lorentz transformation (2.1) given by

$$t'=(t-\vec{x}\cdot\vec{\beta})/(1-\vec{\beta}^2)^{1/2}\,, \tag{4.1}$$

$$\vec{x}\,'=\vec{x}+\{(\vec{x}\cdot\vec{\beta}/\vec{\beta}^2)[1/(1-\vec{\beta}^2)^{1/2}-1]-t/(1-\vec{\beta}^2)^{1/2}\}\vec{\beta}\,. \tag{4.2}$$

At $t'=0$ on the world line of particle n we have

$$t=\vec{x}^n(t)\cdot\vec{\beta} \tag{4.3}$$

so for the position of particle n in the new frame at $t'=0$ we get

$$\vec{x}\,'^n=\vec{x}^n+(\vec{x}^n\cdot\vec{\beta}/\vec{\beta}^2)[(1-\vec{\beta}^2)^{1/2}-1]\vec{\beta}\,. \tag{4.4}$$

This is a consequence of the four-vector transformation (2.4) for the time-space coordinates of particle n. The four-vector transformation (2.5) gives

$$\begin{aligned} u'^n{}_0&=(u^n{}_0-\vec{u}^n\cdot\vec{\beta})/(1-\vec{\beta}^2)^{1/2}\,,\\ \vec{u}\,'^n&=\vec{u}^n+\{(\vec{u}^n\cdot\vec{\beta}/\vec{\beta}^2)[1/(1-\vec{\beta}^2)^{1/2}-1]-u^n{}_0/(1-\vec{\beta}^2)^{1/2}\}\vec{\beta} \end{aligned} \tag{4.5}$$

for the energy-momentum of particle n in the new frame at $t'=0$. The four-vector transformation (2.6) of the Pauli-Lubanski variables is the same as (4.5) with each u replaced by the corresponding w. From these and Eqs. (1.3)–(1.5) we can calculate the spin $\vec{s}\,'^n$ of particle n in the new frame at $t'=0$. Of course the same equations hold with n changed to r. In all these equations the position, momentum, and spin variables of the old frame are at time t corresponding to $t'=0$, as given by (4.3), but that is a complication we can avoid almost completely. We let the origin for the position coordinates be at $\vec{x}^n(t=0)$ for this calculation. We can do so without loss of generality because we only use differences of positions in translation-invariant functions. Then $t'=0$ on the world line of particle n is at $t=0$, so the variables of particle n are all at $t=0$. We treat $\vec{u}^r$ and $\vec{s}^r$ as constant in time. That leaves only $\vec{x}^r$ at $t\neq0$, and it is easy to relate $\vec{x}^r$ at the value of t corresponding to $t'=0$ to $\vec{x}^r$ at $t=0$, since the velocity $\vec{u}^r/u^r{}_0$ of particle r is constant. Thus, writing everything in terms of variables for the old frame at $t=0$, and substituting

$$\vec{\beta}=\vec{u}^r/u^r{}_0$$

for the Lorentz transformation to the rest frame of particle r, we get

$$ {}^0\vec{u}^n=\vec{u}^n-\vec{u}^r[u^n{}_0/m_r-\vec{u}^n\cdot\vec{u}^r/(u^r{}_0+m_r)m_r] , \tag{4.6}$$

$$ {}^0u^n{}_0=(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/m_r , \tag{4.7}$$

$$\begin{aligned} {}^0\vec{s}^n=\;&\vec{s}^n-\vec{u}^n(\vec{u}^n\cdot\vec{s}^n)(u^r{}_0-m_r)/(u^n{}_0+m_n)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &+\vec{u}^n(\vec{s}^n\cdot\vec{u}^r)/(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &-\vec{u}^r(\vec{u}^n\cdot\vec{s}^n)[(u^n{}_0+m_n)(u^r{}_0+m_r)-2\vec{u}^n\cdot\vec{u}^r]/(u^n{}_0+m_n)(u^r{}_0+m_r)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &-\vec{u}^r(\vec{s}^n\cdot\vec{u}^r)(u^n{}_0-m_n)/(u^r{}_0+m_r)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r) , \end{aligned}\tag{4.8}$$

$$ {}^0\vec{u}^n\cdot{}^0\vec{s}^n={}^0w^n{}_0=[u^r{}_0-\vec{u}^n\cdot\vec{u}^r/(u^n{}_0+m_n)]\vec{u}^n\cdot\vec{s}^n/m_r-\vec{s}^n\cdot\vec{u}^rm_n/m_r , \tag{4.9}$$

$$ {}^0\vec{x}^n-{}^0\vec{x}^r=\vec{x}^n-\vec{x}^r+\vec{u}^r(\vec{x}^n-\vec{x}^r)\cdot\vec{u}^r/(u^r{}_0+m_r)m_r , \tag{4.10}$$

$$ {}^0\vec{u}^r=0 , \tag{4.11}$$

$$ {}^0\vec{s}^r=\vec{s}^r . \tag{4.12}$$

One can check that

$$ ({}^0\vec{s}^n)^2=(\vec{s}^n)^2 . \tag{4.13}$$

To get $\vec{F}^{nr}$ from ${}^0\vec{F}^{nr}$ we calculate du'^n/dt' at $t'=0$ by taking the derivative of the right side of Eq. (4.5) with respect to t and dividing by

$$ dt'/dt=(1-\vec{\beta}\cdot\vec{u}^n/u^n{}_0)/(1-\vec{\beta}^2)^{1/2} \tag{4.14}$$

calculated from Eq. (4.1). We let the origin for the position coordinates be at $\vec{x}^n(t=0)$ again, so $t'=0$ on the world line of particle n is at $t=0$. Now we interchange the roles of the two frames. We let the old frame be the rest frame of particle r where we have ${}^0\vec{u}^n$ and ${}^0\vec{F}^{nr}$ for $\vec{u}^n$ and $d\vec{u}^n/dt$. We drop the "prime" label for the new frame and write $\vec{u}^n$ instead of $\vec{u}'^n$. Thus we get

$$\begin{aligned} d\vec{u}^n/dt={}&{}^0\vec{F}^{nr}(1-\vec{\beta}^2)^{1/2}/(1-\vec{\beta}\cdot{}^0\vec{u}^n/{}^0u^n{}_0)+\{({}^0\vec{F}^{nr}\cdot\vec{\beta}/\vec{\beta}^2)[1-(1-\vec{\beta}^2)^{1/2}]\\ &-{}^0\vec{u}^n\cdot{}^0\vec{F}^{nr}/{}^0u^n{}_0\}\vec{\beta}/(1-\vec{\beta}\cdot{}^0\vec{u}^n/{}^0u^n{}_0) . \end{aligned}\tag{4.15}$$

Then substituting $\vec{\beta}=-\vec{u}^r/u^r{}_0$ for the Lorentz transformation from the rest frame of particle r, we get

$$ \vec{F}^{nr}={}^0\vec{F}^{nr}(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0m_r+[{}^0\vec{F}^{nr}\cdot\vec{u}^n/u^n{}_0m_r-{}^0\vec{F}^{nr}\cdot\vec{u}^r/(u^r{}_0+m_r)m_r]\vec{u}^r . \tag{4.16}$$

A similar calculation for $d\vec{s}^n/dt$ yields

$$\begin{aligned} \vec{G}^{nr}={}&{}^0\vec{G}^{nr}(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0m_r+{}^0\vec{F}^{nr}[(\vec{u}^n\cdot\vec{s}^n)(u^r{}_0-m_r)/(u^n{}_0+m_n)-(\vec{s}^n\cdot\vec{u}^r)]\\ &\times(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0m_r(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &-\vec{u}^n(\vec{u}^n\cdot{}^0\vec{G}^{nr})(u^r{}_0-m_r)(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0(u^n{}_0+m_n)m_r(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &-\vec{u}^n(\vec{u}^r\cdot{}^0\vec{G}^{nr})[(u^n{}_0+m_n)(u^r{}_0+m_r)-2\vec{u}^n\cdot\vec{u}^r](u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)\\ &\times1/u^n{}_0m_r(u^n{}_0+m_n)(u^r{}_0+m_r)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &-\vec{u}^n(\vec{s}^n\cdot{}^0\vec{F}^{nr})(u^r{}_0-m_r)(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0(u^n{}_0+m_n)m_r(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &+\vec{u}^n(\vec{u}^n\cdot{}^0\vec{F}^{nr})(\vec{s}^n\cdot\vec{u}^r)/u^n{}_0(u^n{}_0+m_n)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &+\vec{u}^n(\vec{u}^r\cdot{}^0\vec{F}^{nr})(\vec{s}^n\cdot\vec{u}^r)[u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r-u^n{}_0m_r]\\ &\times1/u^n{}_0m_r(u^n{}_0+m_n)(u^r{}_0+m_r)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\ &+\vec{u}^r(\vec{u}^n\cdot{}^0\vec{G}^{nr})(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0m_r(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r) \end{aligned}$$

$$
\begin{aligned}
&-\vec{u}^r(\vec{u}^r\cdot{}^0\vec{G}^{nr})(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)(u^n{}_0-m_n)/u^n{}_0m_r(u^r{}_0+m_r)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\
&+\vec{u}^r(\vec{s}^n\cdot{}^0\vec{F}^{nr})(u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r)/u^n{}_0m_r(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\
&-\vec{u}^r(\vec{u}^n\cdot{}^0\vec{F}^{nr})(\vec{u}^n\cdot\vec{s}^n)/u^n{}_0(u^n{}_0+m_n)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\\
&-\vec{u}^r(\vec{u}^r\cdot{}^0\vec{F}^{nr})(\vec{u}^n\cdot\vec{s}^n)[u^n{}_0u^r{}_0-\vec{u}^n\cdot\vec{u}^r-u^n{}_0m_r]\\
&\times 1/u^n{}_0m_r(u^n{}_0+m_n)(u^r{}_0+m_r)(u^n{}_0u^r{}_0+m_nm_r-\vec{u}^n\cdot\vec{u}^r)\ . \qquad (4.17)
\end{aligned}
$$

One can check that G^{nr} satisfies the condition (1.8) provided

$$
{}^0\vec{s}^{\,n}\cdot{}^0\vec{G}^{nr}=0\ . \qquad (4.18)
$$

We have to specify ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ as functions of ${}^0\vec{u}^n$, ${}^0\vec{s}^{\,n}$, ${}^0\vec{x}^n-{}^0\vec{x}^r$, and ${}^0\vec{s}^{\,r}$ so as to satisfy this condition (4.18) and the remaining equations (3.2). Some limited solutions of these equations will be found in a subsequent paper.

V. SUMMARY

For Poincaré-invariant Newtonian equations of motion (1.6) and (1.7) for N particles with spin, the conditions for Lorentz invariance are Eqs. (2.21) and (2.22). If the force on each particle is a sum of Poincaré-covariant two-particle forces as in Eqs. (3.1), then the two-particle forces must satisfy Eqs. (3.2) for all $r\neq n$ and $s\neq r$. It follows that the two-particle forces have the forms (4.16) and (4.17) where ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ are functions of the variables ${}^0\vec{u}^n$, ${}^0\vec{s}^{\,n}$, ${}^0\vec{x}^n-{}^0\vec{x}^r$, and ${}^0\vec{s}^{\,r}$ given by Eqs. (4.6), (4.8), (4.10), and (4.12). The condition (1.8) will be satisfied by $\vec{G}^{nr}$ if ${}^0\vec{G}^{nr}$ satisfies the condition (4.18). Then the only equations that remain to be satisfied by ${}^0\vec{F}^{nr}$ and ${}^0\vec{G}^{nr}$ are Eqs. (3.2), since they imply that the two-particle forces (4.16) and (4.17) satisfy the Lorentz-invariance conditions for systems of two or more particles. Some limited solutions of these equations will be found in a subsequent paper.

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