# Post-Newtonian approximation of the maximum four-dimensional Yang-Mills gauge theory

Larry L. Smalley
*Space Sciences Laboratory, ES-63, George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Marshall Space Flight Center, Alabama 35812 and Department of Physics, University of Alabama in Huntsville, Huntsville, Alabama 35899*

(Received 30 September 1980; revised manuscript received 16 October 1981)

We have calculated the post-Newtonian approximation of the maximum four-dimensional Yang-Mills theory proposed by Hsu. The theory contains torsion; however, torsion is not active at the level of the post-Newtonian approximation of the metric. Depending on the nature of the approximation, we obtain the general-relativistic values for the classical Robertson parameters ($\gamma=\beta=1$), but deviations for the Nordtvedt effect and violations of post-Newtonian conservation laws. We conclude that in its present form the theory is not a viable theory of gravitation.

## I. INTRODUCTION

In this paper, we investigate the post-Newtonian limit of the maximum four-dimensional Yang-Mills gauge theory of Hsu.[1] This is a rather interesting theory in that the tetrads (and hence the metric) and torsion are postulated to be a function of the gauge fields which are related to the covariant derivative of the fermion fields. We have used the standard form of the post-Newtonian approximation[2] modified and extended slightly for use with tetrads and torsion terms. In our approach, we will choose the free parameters of the theory and thus attempt to align the theory as closely as possible for comparison with the known results for the general theory of relativity (GR). We conclude: (1) Torsion (i.e., spin energy-momentum) does not contribute to the metric in the post-Newtonian limit and (2) the theory is in disagreement with experiment, for example, with the Nordtvedt effect. There is also the question of the position of the spacetime metric $g_{\alpha\beta}$ in the theory. It is not clear in its formulation that the author intended to use the holonomic metric to raise and lower spacetime indices in the Lagrangian formulation of the theory and this thereby leads to some confusion in interpretation. In its present form the theory is not viable. However, this may not be altogether fatal to this approach since, in our opinion, the theory is not complete because of the absence of other fields which would contribute to the energy-momentum tensor and also the position of the metric in the theory.

In Sec. II we derive the field equations for the gauge fields following the formalism of Hsu. The nature of our approximations is listed in Sec. III, and we calculate the gauge fields in Sec. IV. Our conclusions are given in Sec. V.

## II. FIELD EQUATIONS

We follow the notation of Hsu (and references therein)[1] except we use the Minkowski metric $\eta_{ij}=(-1,1,1,1)$. The matrix representation of the generators of the de Sitter group $Z_A$ is given by

$$Z_A=(Z_i,Z_{jk})\equiv\left[\frac{\gamma_i}{2L},\frac{i\gamma_{[j}\gamma_{k]}}{2}\right], \tag{2.1}$$

where $\gamma^i$ are the constant Dirac matrices which satisfy $\gamma^i\gamma^j+\gamma^j\gamma^i=2\eta^{ij}$. The generators satisfy the commutation relations

$$[Z_B,Z_C]=if_{BC}{}^{A}Z_A\ . \tag{2.2}$$

The structure constants $f_{BC}{}^{A}$ then lead to the definition of the group metric

$$g_{AB}\equiv\tfrac{1}{6}f_{AC}{}^{D}f_{DB}{}^{C}\ . \tag{2.3}$$

The gauge covariant derivative of a Dirac spinor wave function in the theory is given by

$$D_\mu\psi=(\partial_\mu+iFh_\mu{}^{A}Z_A)\psi\ , \tag{2.4}$$

where $F$ is a real, dimensionless constant and

$$h_\mu{}^{A}=(h_\mu{}^{i},h_\mu{}^{jk}) \tag{2.5}$$

are the gauge fields. (Greek letters $\alpha,\beta\ldots$ refer to holonomic indices and Latin letters $j,k,\ldots$ refer to anholonomic indices.) The action of the theory is

$$S=\int d^4x\,(E_g L_\psi-\tfrac{1}{4}F_{\mu\nu}{}^{A}F_{\alpha\beta}{}^{B}g_{AB}E^{\mu\nu\alpha\beta}+L')\ , \tag{2.6}$$

where

$$L_\psi=-\tfrac{1}{2}i\bar{\psi}\gamma^\mu(\partial_\mu+iFh_\mu{}^{A}Z_A)\psi+\tfrac{1}{2}i\bar{\psi}(\overleftarrow{\partial}_\mu-iFh_\mu{}^{A}Z_A)\gamma^\mu\psi+m\bar{\psi}\psi+(\psi\to\psi_e,m\to m_e)\ , \tag{2.7}$$

where $\psi$ and $\psi_e$ are, respectively, the nucleon and electron wave functions,

$$E_g=(-\det g_{\mu\nu})^{1/2}\ , \tag{2.8}$$

with $g_{\mu\nu}$ the metric tensor which is related to the gauge fields through the tetrads $e_\mu{}^{i}$,

$$g_{\mu\nu}=e_\mu{}^{i}e_\nu{}^{j}\eta_{ij}\ , \tag{2.9}$$

by the definition

$$e_\mu{}^{i}=[\exp(h)]_\mu{}^{i}=\delta_\mu{}^{i}+h_\mu{}^{i}+\tfrac{1}{2}h_\mu{}^{k}h_k{}^{i}+\ldots \tag{2.10}$$

and

$$L'=-\tfrac{1}{2}\partial_\mu h_\nu{}^{A}\partial^\mu h^{\nu B}g_{AB}\ . \tag{2.11}$$

Note that the field Lagrangian $L'$ is the form used by Hsu in his calculation (cf. the errata[1]).

There are two problems which are worth mentioning about $L'$ at this time. The first is that we must necessarily take Eq. (2.11) literally in that $\partial^\mu h^{\nu B}=\partial^\mu(g^{\nu\beta}h_\beta{}^{B})$ and not $g^{\nu\beta}\partial^\mu h_\beta{}^{B}$. There is also the question of the covariance of the term itself without the $E_g$ factor. Both of these factors cause no problem in the Newtonian limit but do yield considerably different results in the post-Newtonian limit. Finally,

$$F_{\mu\nu}{}^{A}=\partial_\nu h_\mu{}^{A}-\partial_\mu h_\nu{}^{A}-Ff_{BC}{}^{A}h_\nu{}^{B}h_\mu{}^{C}\ . \tag{2.12}$$

Hsu also defines the torsion

$$S_\alpha{}^{\beta\gamma}=e_i{}^{\beta}e_j{}^{\gamma}h_\alpha{}^{ij} \tag{2.13}$$

and the matrices

$$\gamma^\mu=e_i{}^{\mu}\gamma^i\ . \tag{2.14}$$

For simplicity in our calculation of the post-Newtonian approximation (i.e., low-density small-velocity expansion of the metric), we will suppress in the equations the contribution to the metric due to $\psi_e$. This in effect introduces the parameter $m^*=\frac{1}{2}(m+m_e)\approx\frac{1}{2}m$ into the field equations (cf. Sec. III). In the end, we will discover that the ratio $F/m^*$ is fixed by the Newtonian limit of the gravitational field. This simplification may not be possible for the torsion terms since it results from the spin density. However, we ignore this consideration since in the end we will not specifically concern ourselves with the torsion but only with how $h_\mu{}^{ij}$ enters into the metric equation.

The variations of the action (2.6) with respect to $h_\nu{}^{k}$ yields the field equation

$$\begin{aligned}&\frac{\eta_{jk}}{2L^2}[-\partial_\mu g^{\mu\alpha}g^{\nu\beta}\partial_\alpha h_\beta{}^{j}-g^{\mu\alpha}\partial_\mu\partial_\alpha(g^{\nu\beta}h_\beta{}^{j})-g^{\mu\alpha}g^{\nu\sigma}\partial_\mu\partial_\alpha h_\sigma{}^{j}-g^{\nu\sigma}\partial_\alpha h_\sigma{}^{j}\partial_\mu g^{\mu\alpha}]\\&+\frac{\eta_{ij}}{2L^2}[-\partial_\mu h_\sigma{}^{i}\partial_\alpha h_\beta{}^{j}g^{\sigma\beta}(g^{\mu\nu}\delta_k{}^{\alpha}+g^{\alpha\nu}\delta_k{}^{\mu})-\partial_\mu\partial_\alpha h_\sigma{}^{i}g^{\mu\alpha}h_\beta{}^{j}(g^{\beta\nu}\delta_k{}^{\sigma}+g^{\sigma\nu}\delta_k^{\beta})]\\&-E_g\left\{\frac{i}{2}[\bar{\psi}\gamma^j\partial_\mu\psi-(\partial_\mu\bar{\psi})\gamma^j\psi]-\tfrac{1}{2}Fh_\mu{}^{A}\bar{\psi}(\gamma^jZ_A+Z_A\gamma^j)\psi\right\}[-\delta_k^\mu\delta_j^\nu+\tfrac{1}{2}(\delta_k^\mu\delta_j^\sigma h_\sigma{}^{\nu}+\delta_l^\mu\delta_j^\nu h_k{}^{l})]\\&+E_g\frac{F}{2}\bar{\psi}(\gamma^jZ_k+Z_k\gamma^j)\psi(\delta_j^\nu-\delta_l^\nu\delta_j^\alpha h_\alpha{}^{l})+(\psi\to\psi_e)=0\ ,\end{aligned} \tag{2.15}$$

where the tetrads have been expanded to second order in $h_\mu{}^{j}$, and in which we have used the identity $L_\psi=0$ which is obtained by varying the action with respect to $\psi$ and $\psi_e$ fields.

Varying (2.6) with respect to $h_\mu{}^{ij}$ gives

$$[g^{\nu\sigma}\partial_\mu\partial^\mu h_\sigma{}^{ij}+(\partial^\mu g^{\nu\sigma})(\partial_\mu h_\sigma{}^{ij})+\tfrac{1}{2}h_\sigma{}^{ij}\partial_\mu\partial^\mu g^{\nu\sigma}](\eta_{ik}\eta_{jl}-\eta_{il}\eta_{jk})+E_g\frac{F}{2}\bar{\psi}(\gamma^jZ_{kl}+Z_{kl}\gamma^j)e_j{}^{\nu}\psi+(\psi\to\psi_e)=0\ . \tag{2.16}$$

## III. NATURE OF THE APPROXIMATION

### A. Metric

From Eqs. (2.9) and (2.10), the post-Newtonian expansion of the metric is given by

$$g_{00}=-1-2h_0^{(2)0}-[2h_0^{(4)0}+2(h_0^{(2)0})^2]+\dots , \tag{3.1}$$

$$g_{0a}=h_0^{(3)a}-h_a^{(3)0}+\dots , \tag{3.2}$$

$$g_{ab}=\delta_{ab}+h_a^{(2)b}+h_b^{(2)a}+\dots . \tag{3.3}$$

For the purposes of comparison with GR, we assume the gauge conditions for the post-Newtonian approximation due to Chandrasekhar[2]

$$g^{(2)}_{00,a}+2g^{(2)}_{ab,b}-g^{(2)}_{bb,a}=0 , \tag{3.4}$$

$$2g^{(3)}_{0a,a}-g^{(2)}_{aa,0}=0 . \tag{3.5}$$

The number in parentheses, e.g., $h_0^{(N)0}$, refers to the order of $v^N$.

For the sake of comparison, we list the standard form of the parametrized post-Newtonian (PPN) approximation (see Ref. 3 for details):

$$g_{00}=-1-2U-2\beta U^2+4\Phi , \tag{3.6}$$

$$g_{ij}=\delta_{ij}-2\gamma\delta_{ij}U , \tag{3.7}$$

$$g_{0j}=-\tfrac{7}{2}\Delta_1 V_j-\tfrac{1}{2}\Delta_2 W_j , \tag{3.8}$$

where $U$ is the Newtonian potential,

$$\Phi=\int\frac{\rho(\vec{x}\,',t)\phi(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|}d^3x' \tag{3.9}$$

with

$$\phi=\beta_1 v^2-\beta_2 U , \tag{3.10}$$

$$V_j=\int\frac{\rho v^j}{|\vec{x}-\vec{x}\,'|}d^3x' \tag{3.11}$$

and

$$W_j=\int\frac{\rho v^k(x_k-x_k')(x_j-x_j')}{|\vec{x}-\vec{x}\,'|^3}d^3x' . \tag{3.12}$$

The parameters $\gamma$ and $\beta$ are the classical Robertson parameters. The general theory of relativity (GR) predicts the values for the parameters

$$\gamma=\beta=\beta_1=\beta_2=\Delta_1=\Delta_2=1 . \tag{3.13}$$

(Note that the internal energy fraction and the pressure terms related to the PPN parameters $\beta_3$ and $\beta_4$ have been omitted in this first approximation.)

The procedure we use is only equivalent to the PPN approximation if $\gamma$ can be made equal to 1. The metric is then gauged so that the time-space components satisfy the gauge conditions (3.5). For the PPN metric this requires $7\Delta_1-\Delta_2=6\gamma$. In effect we have chosen special values of $L$ and $F$ for the most convenient comparison with GR, the deviations, if any, then showing up in the fourth-order part of $g_{00}$.

### B. Densities

We assume (similar to Hsu) that we can replace to the order of the post-Newtonian approximation

$$\frac{\hbar}{i}\partial_\mu\psi\to P_\mu\psi=mu_\mu\psi , \tag{3.14}$$

where $u^\mu=u^0(1,v^j)$ is the four-velocity. In addition we assume a consistent scheme for replacing densities,

$$E_g(m\bar{\psi}\psi+m_e\bar{\psi}_e\psi_e)\to\rho(\vec{x},t) , \tag{3.15}$$

$$E_g(m\bar{\psi}\gamma_\nu\psi+m_e\bar{\psi}_e\gamma_\nu\psi_e)\to\rho u_\nu , \tag{3.16}$$

where $\rho(\vec{x},t)$ is the mass-energy density and then $\rho u_\nu$ is the momentum density. This is based upon an analogy with the Dirac equation for which $E_g\bar{\psi}\gamma^\mu\psi=j^\mu$ is the current density. The replacement (3.15) is justified in Sec. III C below.

Finally, we will encounter terms such as

$$\bar{\psi}(\gamma_j\gamma_{[k}\gamma_{l]}+\gamma_{[k}\gamma_{l]}\gamma_j)\psi=2\bar{\psi}\gamma_{[j}\gamma_k\gamma_{l]}\psi$$
$$=2\epsilon_{jklm}\bar{\psi}\gamma^5\gamma^m\psi , \tag{3.17}$$

where in the last step we have used an identity for Dirac matrices.[4] The axial vector $i\gamma^5\gamma^m=W^m$ is called the polarization vector and satisfies the following relations[5]:

$$W^m=\tfrac{1}{2}\epsilon^{mljk}\{\sigma_{lk},\gamma_k\}=i\gamma^5\gamma^m , \tag{3.18}$$

where the tensor $\sigma_{kl}\equiv\frac{1}{2}i[\gamma_l,\gamma_k]$ is proportional to the spin tensor operator $s_{jk}=\frac{1}{2}\hbar\sigma_{jk}$. Equation (3.18) is then consistent with the relation

$$s_m=\tfrac{1}{2}u^l s^{kj}\epsilon_{mlkj} , \tag{3.19}$$

providing we identify the spin vector operator

$$s^a=\hbar W^a=\hbar i\gamma^5\gamma^a . \tag{3.20}$$

Note that in the rest frame, $s_m$ only has nonzero spatial components (also, $\hbar=1$). We then define

$$\mathcal{S}^k\equiv E_g m i\bar{\psi}\gamma^5\gamma^k\psi \tag{3.21}$$

as the spin-density vector and correspondingly the spin potential

$$S^k = -\frac{1}{4\pi}\int \frac{\mathscr{S}^k d^3x'}{|\vec{x}-\vec{x}'|} . \tag{3.22}$$

It is then possible to identify $S^m$ as the intrinsic angular momentum vector.

### C. Energy-momentum tensor

For a general Dirac field $\psi$, it is possible to introduce a canonical energy-momentum tensor,[6]

$$T_\mu{}^\nu = \left[\partial_\mu \bar{\psi}\frac{\partial \mathscr{L}_\psi}{\partial \partial_\nu \bar{\psi}} + \frac{\partial \mathscr{L}_\psi}{\partial \partial_\nu \psi}\partial_\mu \psi\right] - \delta_\mu^\nu \mathscr{L}_\psi + (\psi \to \psi_e) , \tag{3.23}$$

where $L_\psi$ is the matter part of the total Lagrangian. We obtain from (2.6)

$$T_\mu{}^\nu = -E_g \frac{i}{2}[\bar{\psi}\gamma^\nu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\nu \psi] + (\psi \to \psi_e) , \tag{3.24}$$

where we have used explicitly $L_\psi = 0$ and dropped the $F^2$ term since it is of order $O(v^5)$ and therefore does not contribute to the post-Newtonian limit. We note that (3.24) is not necessarily the symmetrized form of the energy-momentum tensor, but in anticipation of our results below, the torsion does not contribute to the metric in the post-Newtonian limit and only the symmetric part of (3.24) is required. This would apparently not be a justified approximation in the post-post-Newtonian limit, since in general for theories with torsion, the canonical energy-momentum tensor is not symmetric.[7]

Note that if we take the trace of (3.24) and compare with the trace of (2.15), then to lowest order the density relation (3.15) is an identity. Furthermore, we find that the field equations give the same results to lowest order whether we use $T_\mu{}^\nu$ or the approximations (3.14)–(3.16).

## IV. CALCULATION OF GAUGE FIELDS

### A. Lowest-order metric gauge fields— Consistency with metric gauge conditions

From Eq. (2.15), to the first nontrivial order, we find

$$\nabla^2 h_l{}^\nu = -E_g L^2 \bar{\psi}\gamma^\nu P_l \psi + E_g \frac{F}{4} L \bar{\psi}(\gamma^\nu \gamma_l + \gamma_l \gamma^\nu)\psi + (\psi \to \psi_e) . \tag{4.1}$$

Utilizing the approximations of Sec. III, we obtain

$$\nabla^2 h_\nu{}^l = -L^2 \rho u^\nu u_l + \frac{1}{2}\frac{FL}{m^*}\delta_l^\nu \rho . \tag{4.2}$$

(At the level of this approximation, there is no longer any distinction between Greek and Latin indices.) The time component of (4.1) becomes

$$\nabla^2 h_0^{(2)0} = +\left[L^2 + \frac{FL}{2m^*}\right]\rho \tag{4.3}$$

so that

$$h_0^{(2)0} = +\frac{1}{4\pi}\left[L^2 + \frac{FL}{2m^*}\right] U(\vec{x},t) , \tag{4.4}$$

where we have used the definition of the Newtonian potential

$$U(\vec{x},t) = -\int \frac{\rho(\vec{x}',t)d^3x'}{|\vec{x}-\vec{x}'|} . \tag{4.5}$$

Similarly the spatial components of (4.1) are

$$\nabla^2 h_i^{(2)j} = \frac{FL}{2m^*}\delta_j^i \rho , \tag{4.6}$$

so that

$$h_j^{(2)i} = h_i^{(2)j} = \frac{1}{4\pi}\frac{FL}{2m^*}\delta_j^i U . \tag{4.7}$$

The time-space components of (4.1) are

$$\nabla^2 h_0^{(3)l} = -L^2 u^0 \rho u_l = -L^2 \rho v^l \tag{4.8}$$

and

$$\nabla^2 h_l^{(3)0} = -L^2 u^l \rho u_0 = -\nabla^2 h_0^{(3)l} , \tag{4.9}$$

so that

$$h_0^{(3)l} = -\frac{L^2}{4\pi} V^l . \tag{4.10}$$

From the definition of the metric (3.1)–(3.3), we then have

$$g_{00}^{(2)} = -\frac{1}{4\pi}\left[2L^2 + \frac{FL}{m^*}\right] U , \tag{4.11}$$

$$g_{ij}^{(2)} = \frac{1}{4\pi}\frac{FL}{m^*}\delta_{ij} U , \tag{4.12}$$

$$g_{0i}^{(3)} = -\frac{2L^2}{4\pi} V_i . \tag{4.13}$$

Substituting (4.11) and (4.12) into the gauge conditions (3.4) yields the consistency relation

$$F=-Lm^* . \quad (4.14)$$

Since the Newtonian limit requires that $g^{(2)}_{00} = -2U$, we obtain the further constraint from (4.11)

$$L^2=8\pi G . \quad (4.15)$$

[We have put $G$ into (4.15) so that its position in the theory is explicit even though in our units $G=1$. Also note that if we had followed the PPN prescription Eq. (4.12) would yield $\gamma=-FL/8\pi m^*$ and the Eq. (4.15) would read $L^2=4\pi G(1+\gamma)$. The interested reader can easily fill in the remaining steps.]

The gauge condition (3.5) is not satisfied by $g^{(3)}_{0i}=-4V_i$, where we have used (4.15) explicitly. This situation can be handled by the gauge transformation[8]

$$g^{+}_{\alpha\beta}=g_{\alpha\beta}-\xi_{\alpha;\beta}-\xi_{\beta;\alpha} . \quad (4.16)$$

For the particular choice

$$\xi_\beta=\tfrac{1}{2}(\partial_0\chi,0) , \quad (4.17)$$

where $\chi$ is the superpotential defined by

$$g^{(2)}_{00}\equiv\nabla^2\chi , \quad (4.18)$$

and noting from (4.11) that the changes to the metric occur at $O(v^3)$ or higher so that $g^{(2)+}_{00}=g^{(2)}_{00}$ and $g^{(2)+}_{ij}=g^{(2)}_{ij}$, then the time-space component of the metric

$$\begin{aligned} g^{(3)+}_{0i} &= g^{(3)}_{0i}-\tfrac{1}{2}\chi_{,0i} \\ &= -\tfrac{7}{2}V_i-\tfrac{1}{2}W_i , \end{aligned} \quad (4.19)$$

where the last step follows from the use of the continuity equation

$$\partial_0\rho+\partial_j(\rho v^j)=0 , \quad (4.20)$$

and parts integration. The second gauge condition (3.5) is now satisfied.

Similarly we find, using (4.16) and (4.17), that

$$\begin{aligned} g^{(4)+}_{00} &= g^{(4)}_{00}-\chi_{,00} \\ &= -2h^{(4)0}_{0}-2(h^{(2)0}_{0})^2+\partial_0\partial^0\chi . \end{aligned} \quad (4.21)$$

From here on we will refer only to the gauged metric and will therefore drop the ( + ) for convenience.

### B. Torsion gauge fields

From Eq. (2.16) to first nontrivial order, we find

$$\begin{aligned} \nabla^2 h_{\nu kl} &= -E_g i\frac{F}{4}\bar{\psi}\gamma_{[\nu}\gamma_k\gamma_{l]}\psi \\ &= -\frac{1}{4}\frac{F}{m}\epsilon_{m\nu kl}\mathcal{S}^m , \end{aligned} \quad (4.22)$$

where we have used (3.17) and (3.19)–(3.21) in going to the last step. Finally, upon inverting (4.22) and using (3.22), we obtain

$$h_{\nu kl}=-\frac{F}{4m}\epsilon_{m\nu kl}S^m . \quad (4.23)$$

If we are to identify $S^m$ as the intrinsic angular momentum vector, then $S^m=(0,\vec{S})$ in the rest frame.[5] This says that the lowest-order terms are of the form $h_{0kl}=-h_{lk0}$ are of $O(v^3)$ where $l,k$ are spatial indices. The purely spatial part $h_{jkl}$ is of $O(v^4)$. Consequently the torsion contribution to the metric gauge fields does not occur until $O(v^6)$ and therefore does not contribute to the post-Newtonian approximation Eq. (2.15) since only $S^2$-type terms occur in the metric gauge field.

### C. $O(v^4)$ Metric gauge fields—post-Newtonian approximation

First we expand Eq. (2.15) to terms of $O(v^4)$ and substitute the energy-momentum tensor (3.24) in the form

$$T_\mu{}^j=e_\nu{}^jT_\mu{}^\nu\approx(\delta_\nu{}^j+h_\nu{}^j)T_\mu{}^\nu ,$$

for the matter field terms, and because of our arguments of the last section, drop the contribution of the torsion-squared terms from the term containing $Z_A$. We get finally for the time-component equation

$$\begin{aligned} \frac{1}{2L^2}[ &-2\nabla^2h^{(4)0}_{0}-2\partial_0\partial^0h^{(2)0}_{0}-2g^{(2)}_{00}\nabla^2h^{(2)0}_{0}+2g^{(2)}_{ij}\partial_i\partial_jh^{(2)0}_{0}-h^{2(0)}_{0}\nabla^2g^{(2)}_{00} \\ &-2\partial_kg^{(2)kl}\partial_lh^{(2)0}_{0}+2h^{(2)0}_{0}\nabla^2h^{(2)0}_{0}-2(\nabla g^{(2)}_{00})\cdot(\nabla h^{(2)0}_{0})]-T^{(4)0}_{0}+\tfrac{3}{2}\rho h^{(2)0}_{0}=0 , \end{aligned} \quad (4.24)$$

where the order of each term is explicitly given and the energy-momentum tensor is expressed in terms of its holonomic coordinates. From Eqs. (3.1) and (3.3) for the metric and the gauge condition (4.21), the time component of the metric is given by

$$\nabla^2 g_{00}^{(4)} = \nabla^2[-2h_0^{(4)0} - 2(h_0^{(2)0})^2] + \nabla^2 \partial_0 \partial^0 \chi$$
$$= -2\nabla^2 U^2 - 16\pi\rho(v^2 + \tfrac{5}{2}U) , \qquad (4.25)$$

where in the last step Eq. (4.24) has been evaluated using the perfect fluid model of the energy-momentum (cf. Ref. 3 for details). Solving Eq. (4.25) we obtain

$$g_{00}^{(4)} = -2U^2 + 4\int \frac{\rho(v^2 + \frac{5}{2}U)}{|\vec{x} - \vec{x}'|} d^3x' . \qquad (4.26)$$

### D. The PPN parameters

By comparing Eqs. (4.11), (4.12), (4.19), and (4.26) with the definition of the PPN parameters given by Eqs. (3.6)–(3.10), we find that Hsu's theory predicts the following set of parameters:

$$\gamma = \beta = \beta_1 = \Delta_1 = \Delta_2 = 1 ,$$
$$\beta_2 = -\tfrac{5}{2} . \qquad (4.27)$$

We note that $\beta_2$ is considerably different from the GR value given by (3.13).

## V. CONCLUSIONS

### A. Theoretical considerations

We should point out that our treatment of densities is open to criticism. For this reason we have dropped consideration of the internal energy and pressure terms. We have been as explicit as possible in our approximations and calculations since others may also wish to calculate the post-Newtonian limit for tetrad-based gauge theories. Furthermore, we emphasize that the introduction of the energy-momentum tensor into the field equations is very sensitive to (1) the order of the approximation (spin-density contributions and symmetrization), (2) additional fields [their contribution to energy-momentum as seen in (3.23)], and (3) the use of the (pressureless) perfect fluid model. Each of the above considerations should be critically examined before attempting to determine the post-post-Newtonian approximation, at which level torsion is expected to play an important role in any tetrad-based gauge theory of gravitation. Also we have, through the use of the energy-momentum tensor, suppressed the distinction between torsion contributions from the nucleon wave function $\psi$ and the electronic wave function $\psi_e$.

### B. Consideration of alternate Lagrangians

The theory is extremely sensitive to the form of the Lagrangian $L'$ given by (2.11) which does not contain only the natural order of the indices for the gauge fields $h_\mu{}^A$. As a result uncertain terms in the variation due to the metric can occur and will result in different values for $\beta$ and $\beta_2$. In a more recent paper,[9] Hsu introduces a somewhat altered theory which partially alleviates this uncertainty by constructing the Lagrangian using only the Minkowski metric. We have not as yet investigated that theory but do consider here this approach, among others in what follows.

For this $L'$, we obtain the usual results[10,11] for light bending, perihelion shift, and red-shift. However, the parameter $\beta_2 = -\frac{5}{2}$ implies for the Nordtvedt effect,[12] described by the parameter $\eta$

$$\eta = 3\gamma + 4\beta - 7\Delta_1 - \tfrac{1}{3}(2\beta + 2\beta_2 - 3\gamma + \Delta_2 - 2)$$
$$= \begin{cases} \frac{7}{3}, & \text{Hsu} \\ 0, & \text{GR} \\ 0.00 \pm 0.03, & \text{lunar laser ranging (Ref. 13)} . \end{cases} \qquad (5.1)$$

The experimental limts on $\beta_2$ are very poorly known, so that $\eta$ itself, determined experimentally by lunar laser ranging, represents the best experimental test of $\beta_2$. One concludes that $\beta_2 \approx 1$. Among other effects there would be variations in the local gravitational constant and the ratio between the active and passive gravitational would be proportional to external gravitational potentials.

With these consequences in mind let us consider the appropriateness of the field Lagrangian $L'$ given by Eq. (2.11). There is of course the question of whether it is a scalar density unless one includes the $E_g$ factor, Eq. (2.8). In the Newtonian approximation there is no distinction but deviations do occur at the post-Newtonian level. Including $E_g$ in the calculation, we obtain $\beta = 3$ and $\beta_2 = -\frac{1}{2}$ which seems to give worse results than before. Furthermore Will has shown that for a Lagrangian-based theory[3] there are the constraints in particular,

$$2\beta+2\beta_2=3\gamma+1 \tag{5.2}$$

required for post-Newtonian conservation laws. This result was later modified and extended by Lee, Lightman, and Ni[14] who showed that almost all Lagrangian-based, generally convariant metric theories possess conservation laws which thereby lead to conserved quantities if constraints such as Eq. (5.2) are satisfied. Unfortunately neither of the field Lagrangians discussed above satisfy this constraint.

Because of these problems, we have investigated alternate field Lagrangians with the hope of discovering an acceptable choice. We summarize our results in Table I; no acceptable field Lagrangian was found.

It is interesting to note that the typical gauge field Lagrangian of the form $-\frac{1}{4}E_g F_{\mu\nu}{}^A F^{\mu\nu B} g_{AB}$ leads to inconsistent results. This may be because this form is not an overall SO(4,1) invariant action.[1] Lee, Lightman, and Ni[14] also point out that theories with absolute variables, such as the group metric $g_{AB}$, or those with a partial gauge group symmetry may not yield the same conserved quantities or constraints. Since the above field Lagrangians are not SO(4,1) invariants, they then will in general admit a partial gauge group symmetry. Unfortunately, the SO(4,1) invariant $-\frac{1}{4}F_{\mu\nu}{}^A \times F_{\alpha\beta}{}^B g_{AB} E^{\mu\nu\alpha\beta}$ also gives incompatible results.

We have also considered the case (last line in Table I) in which we treat the theory as a two-metric theory. The action is then constructed on a background, Minkowskian metric, and the particles then move on a spacetime metric built-up out of the gauge fields obtained from the action. We do not obtain a viable post-Newtonian limit in this case either.

In all cases considered, none satisfy the constraint, Eq. (5.2), required for post-Newtonian conservation laws. Although the theory seems to be Lagrangian-based theory, the relationship between the metric and the gauge fields is fixed (defined) outside of the action and hence the variation of the action as well. One possible solution to this problem could be the use of Lagrange undetermined multipliers which can introduce the metric-gauge relation as a constraint on the action itself.[15] This will ensure consistent relationships between the metric, gauge, and the fields. We will consider this approach elsewhere.

Finally we note that one of the referees has pointed out that since $h_\mu{}^A$ is a vector field on spacetime, then $\partial_\mu h_\nu{}^A$ is not generally convariant and therefore none of the field Lagrangians discussed above is generally covariant. He notes that $e_B{}^\mu h_\mu{}^A \equiv h_B{}^A$ is a spacetime scalar and thus the Lagrangian

$$L_R=-\tfrac{1}{2}E_g \partial_\mu h_B{}^A \partial^\mu h_C{}^D g^{BC} g_{AD} \tag{5.3}$$

is generally covariant. We have taken this suggestion and recalculated the post-Newtonian limit. We obtain $\gamma=1$, $\beta=3$, and $\beta_2=-\frac{1}{2}$ which unfortunately gives $2(\beta+\beta_2)=5$. Although this Lagrangian is manifestly covariant, the problem seems to persist. Thus we can only refer back to our original conjectures in this section.

TABLE I. Alternative field Lagrangians for Hsu's gauge theory of gravitation.

| Field Lagrangian | $\gamma$ | $\beta$ | $\beta_2$ | $2(\beta+\beta_2)$ |
|---|---|---|---|---|
| $-\frac{1}{2}\partial_\mu h_\nu{}^A \partial^\mu h^{\nu B} g_{AB} \equiv L'$ | 1 | 1 | $-\frac{5}{2}$ | $-3$ |
| $-\frac{1}{2}E_g \partial_\mu h_\nu{}^A \partial^\mu h^{\nu B} g_{AB}$ | 1 | 3 | $-\frac{1}{2}$ | 5 |
| $-\frac{1}{2}E_g g^{\mu\alpha} g^{\nu\beta} \partial_\mu h_\nu{}^A \partial_\alpha h_\beta{}^B g_{AB}$ | 1 | $\frac{5}{2}$ | 0 | 5 |
| $-\frac{1}{2}g^{\mu\alpha} g^{\nu\beta} \partial_\mu h_\nu{}^A \partial_\alpha h_\beta{}^B g_{AB}$ | 1 | $\frac{1}{2}$ | $-2$ | $-3$ |
| $-\frac{1}{2}E_g g^{\mu\beta} g^{\nu\alpha} \partial_\mu h_\nu{}^A \partial_\alpha h_\beta{}^B g_{AB}$ | No Newtonian limit | | | |
| $-\frac{1}{4}F_{\mu\nu}{}^A F_{\alpha\beta}{}^B g_{AB} E^{\mu\nu\alpha\beta}$ | Incompatible. Predicts $O(v^3)$ part for $g_{00}$ | | | |
| $-\frac{1}{4}E_g F_{\mu\nu}{}^A F^{\mu\nu B} g_{AB}$ | Inconsistent. Implies $U_{,0}=2V_{l,l}$ | | | |
| $-\frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta} \partial_\mu h_\nu{}^A \partial_\alpha h_\beta{}^B g_{AB}$ | 1 | 1 | $-\frac{3}{2}$ | 1 |

## ACKNOWLEDGMENTS

It is a pleasure to acknowledge the many fruitful discussions with Peter B. Eby, Helmut Rumpf, and Friedrich Hehl. Useful comments are gratefully acknowledged from C. M. Will who also pointed out the significance of the post-Newtonian conservation laws and their significance for Lagrangian-based theories.


---


[1]J. P. Hsu, Phys. Rev. Lett. 42, 934 (1980); 42, 1720 (1980).

[2]S. Chandrasekhar, Astrophys. J. 142, 1488 (1965).

[3]C. M. Will, in *Experimental Gravitation*, edited by B. Bertotti (Academic, New York, 1973), pp. 1—110.

[4]V. B. Berestetskiĭ, E. M. Lifschitz, and L. P. Pitaevskiĭ, *Relativistic Quantum Theory* (Pergamon, New York, 1971), Vol. 4, p. 70.

[5]H. Rumpf, in *Spin, Torsion, Rotation, and Supergravity*, edited by P. G. Bergmann and V. deSabbata (Plenum, New York, 1980), pp. 93—104.

[6]S. S. Schweber, *An Introduction to Relavistic Quantum Field Theory* (Harper and Row, New York, 1962), pp. 208 and 219.

[7]F. W. Hehl, Gen. Relativ. Gravit. 4, 333 (1973); 4, 491 (1973).

[8]C. M. Will, Astrophys. J. 163, 611 (1971).

[9]J. P. Hsu, Lett. Nuovo Cimento 28, 128 (1980).

[10]Compare the compilation of experimental data given by W. -T. Ni, Astrophys. J. 176, 769 (1972).

[11]Compare the review by K. S. Thorne, C. M. Will, and W. -T. Ni, in Proceedings of the Conference on Experimental Test of Gravitation Theories, edited by R. W. Davies, 1971 (NASA-JPL Technical Memo 33-449) (unpublished), pp. 10—13, and references therein.

[12]K. Nordtvedt, Jr., Phys. Rev. 169, 1014 (1968); 169, 1017 (1968); 170, 1186 (1968).

[13]J. G. Williams *et al.*, Phys. Rev. Lett. 36, 551 (1976); I. I. Shapiro *et al.*, *ibid.* 36, 555 (1976).

[14]D. L. Lee, A. P. Lightman, and W. -T. Ni, Phys. Rev. D 10, 1685 (1974).

[15]F. W. Hehl and G. D. Kerlick, Gen. Relativ. Gravit. 9, 691 (1978); F. W. Hehl, E. Lord, and L. L. Smalley, Gen. Relativ. Gravit. (to be published).