Dynamical symmetry breaking due to radiative corrections in cosmology

L. H. Ford
Department of Physics, Tufts University, Medford, Massachusetts 02155

D. J. Toms
The Blackett Laboratory, Imperial College, London SW7 2BZ, England
(Received 16 October 1981)

Dynamical symmetry breaking in massless $\lambda\phi^4$ theory in curved spacetime with nonconformal coupling to the curvature is investigated. It is shown that in general the one-loop self-energy for such a field will involve a term of the form $R \ln|R\mu^{-2}|$, where R is the scalar curvature, and μ is a mass. This term can give rise to symmetry breaking. Two models, the Einstein universe and a spatially flat Robertson-Walker universe with a power-law expansion, are considered where this term is the sole contribution to the one-loop self-energy. In both cases a phase transition will occur at a critical value of the curvature. The form of the two-loop corrections to the self-energy and the limits of validity of the one-loop approximation are discussed.

I. INTRODUCTION

Recently, the problems of interacting quantum field theory in curved spacetime, including the effects of spacetime curvature upon symmetry breaking,[1–12] have begun to be treated. In particular, the generation of a mass as a result of one-loop quantum corrections in a homogeneous spacetime with nontrivial topology has been considered.[1–2] It has been found that this mass may be tachyonic and can, under certain circumstances, introduce an instability into the theory. This instability may be interpreted as resulting in dynamical symmetry breaking. For the case of a self-coupled scalar field, the vacuum state in which $\langle\phi\rangle=0$ ceases to be the stable ground state of the theory is replaced by a configuration in which $\langle\phi\rangle\neq0$. This is similar to the familiar Goldstone model,[13] but with the essential difference that the tachyonic mass is generated dynamically rather than being introduced by hand. The dynamical symmetry breaking with which we will be concerned differs from that treated by Coleman and Weinberg[14] in that the latter mechanism can occur in Minkowski spacetime, whereas we will be concerned with effects produced by a nonzero curvature or nontrivial topology. It also differs from the symmetry breaking considered by Grib and Mostepanenko[15] which is produced by a background gravitational field at the classical level, and that of Abbott[16] who added an $R\phi^2$ term onto a result for the effective potential computed in Minkowski spacetime. Our calculation involves the quantum corrections to the classical theory computed in curved spacetime.

Here we will consider dynamical symmetry breaking for an uncharged scalar field theory with a $\lambda\phi^4$ self-interaction which is produced by one-loop quantum corrections in a curved background spacetime. Our primary concern is with effects which arise when the scalar field is coupled nonconformally to gravitation. When the spacetime is not stationary, the analysis of stability and the characterization of the stable ground state is considerably complicated; examples in which this complication arises will be considered.

The outline of this paper is as follows. In Sec. II we develop some formalism which will be useful in discussing the stability of $\lambda\phi^4$ theory on a curved background. In Sec. III we consider the general form of $\langle\phi^2\rangle$ for a nonconformally coupled scalar field in a curved spacetime. This quantity governs the radiative effects upon vacuum stability. It is shown on the basis of general arguments that at the one-loop level a term of the form $R \ln|R\mu^{-2}|$, where R is the scalar curvature and μ is a mass, will arise, and its coefficient is calculated using an argument based upon the renormalization group. It is also argued that the two-loop corrections will introduce a term of the form $R \ln^2|R\mu^{-2}|$ and its coefficient is calculated. In Sec. IV some particular spacetimes are considered in which $\langle\phi^2\rangle$ may be explicitly calculated at the one-loop level and shown to be of the form determined in Sec. III. These results are then applied to the analysis of dynamical symmetry breaking. The two models considered are the Einstein universe and a spatially flat Robertson-Walker universe with a power-law expansion. In both cases it is found that the symmetric $\langle\phi\rangle=0$ vacuum state is stable for certain values of the scalar curvature and unstable for others. Consequently, a phase transition can occur between the symmetric and unsymmetric phases as the curvature varies. In Sec. V the limits of validity of the one-loop approximation are discussed, and it is found that higher-loop corrections can be neglected near the phase transition for certain choices of the parameters in the theory.

II. STABILITY OF $\lambda\phi^4$ THEORY IN CURVED SPACETIME

In this paper we will be concerned with the effects of spacetime curvature upon the stability of the vacuum state of a neutral scalar field with a $\lambda\phi^4$ self-coupling. We assume the field to be massless at the tree-graph level so that the Lagrangian density is taken to be

$$\mathscr{L}=(-g)^{1/2}(\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-\tfrac{1}{2}\xi R\phi^2-\tfrac{1}{12}\lambda\phi^4)\,, \qquad (2.1)$$

where R is the scalar curvature and ξ is a dimensionless parameter describing the coupling of the scalar field to the gravitational background. (The metric is taken to have signature -2.) The choice $\xi=\frac{1}{6}$ gives a conformally invariant theory. We assume $\lambda\geq 0$ so that the field energy is bounded from below. The equation of motion associated with (2.1) is

$$\Box\phi+\xi R\phi+\tfrac{1}{3}\lambda\phi^3=0\,. \qquad (2.2)$$

Let us now write

$$\phi=\phi_c+\phi_q\,, \qquad (2.3)$$

where ϕ_c is a classical field, and ϕ_q is a quantum field with a vanishing vacuum expectation value, so that

$$\langle\phi\rangle_0\equiv\langle 0\,|\,\phi\,|\,0\rangle=\phi_c\,. \qquad (2.4)$$

Because $\langle\phi_q{}^3\rangle=0$ (e.g., by Wick's theorem), if we take the vacuum expectation value of Eq. (2.2) the result is

$$\Box\phi_c+\xi R\phi_c+\lambda\langle\phi_q{}^2\rangle_0\phi_c+\tfrac{1}{3}\lambda\phi_c{}^3=0\,. \qquad (2.5)$$

If we subtract Eq. (2.5) from Eq. (2.2), the resulting equation satisfied by ϕ_q is

$$\Box\phi_q+\xi R\phi_q+\tfrac{1}{3}\lambda(\phi_q{}^3+3\phi_q{}^2\phi_c+3\phi_q\phi_c{}^2-3\langle\phi_q{}^2\rangle_0\phi_c)=0\,. \qquad (2.6)$$

The problem of characterizing the ground state of the field theory is that of finding a solution to Eq. (2.5) which is associated with a stable quantum field theory. The criterion for stability which will be adopted here is that introduced in Ref. 9, which may be formulated as follows. Let

$$\Phi=\langle z\,|\,\phi_q\,|\,z\rangle\,, \qquad (2.7)$$

where $|\,z\rangle$ is a coherent state for some particular mode. Because

$$\langle z\,|\,\phi_q{}^2\,|\,z\rangle=\Phi^2+\langle\phi_q{}^2\rangle_0 \qquad (2.8)$$

and

$$\langle z\,|\,\phi_q{}^3\,|\,z\rangle=\Phi^3+3\langle\phi_q{}^2\rangle_0\Phi\,, \qquad (2.9)$$

the equation for Φ is

$$\Box\Phi+\xi R\Phi+\lambda(\tfrac{1}{3}\Phi^3+\phi_c\Phi^2+\phi_c{}^2\Phi+\langle\phi_q{}^2\rangle_0\Phi)=0\,. \qquad (2.10)$$

If we assume that Φ is small and drop the nonlinear terms, then this equation becomes

$$\Box\Phi+\xi R\Phi+\lambda(\phi_c{}^2+\langle\phi_q{}^2\rangle_0)\Phi=0\,. \qquad (2.11)$$

The stability criterion will be that Eq. (2.11) does not possess any solutions which grow too rapidly, although it is not easy in general to give a precise statement of what constitutes too rapid a growth. In a stationary spacetime unstable solutions are characterized by exponential time dependence. In nonstationary spacetimes the time dependence (relative to some chosen set of observers) of solutions to Eq. (2.11) will in general be rather complicated. In Sec. IV a particular cosmological example will be considered in which growing solutions to Eq. (2.11) arise and which may be considered to be a case of instability.

This is a criterion for relative stability and not absolute stability. In situations where an effective potential may be defined (that is, where homogeneity in space and time imply $\phi_c=$constant), it is equivalent to requiring that the vacuum be associated wtih a local minimum of the effective potential. It does not eliminate the possibility of vacuum decay of the sort which arises when the vacuum is not associated with an absolute minimum of the effective potential. Another criterion for relative stability is the negative-eigenvalue method described in Ref. 10; this approach, the effective-potential method, and the procedure adopted in this paper are all equivalent to the extent that when more than one of them may be applied to the same problem the results are in agreement.

Although we have chosen to work directly with the equations of motion, the principal results of this section, Eqs. (2.5) and (2.11), can also be derived by use of the effective action.[17] Note that Eq. (2.11) is also the equation which would be satisfied by small perturbations of a solution to Eq. (2.5). Thus coherent-state expectation values of the quantum field are equivalent to small shifts in the classical field ϕ_c.

In this paper we will be primarily concerned with one-loop quantum effects, so that Eq. (2.6) may be replaced by the linear equation

$$\Box\phi_q+\xi R\phi_q+\lambda{\phi_c}^2\phi_q=0\ . \tag{2.12}$$

We will be particularly interested in studying the stability of the $\phi_c=0$ configuration. In this case ϕ_q is the usual massless free scalar field with a ξR coupling to the spacetime curvature. For cases in which $\langle{\phi_q}^2\rangle_0<0$, the one-loop quantum effects can destabilize the $\phi_c=0$ configuration, although a negative $\langle{\phi_q}^2\rangle_0$ does not necessarily imply instability.

III. THE FORM OF $\langle\phi^2\rangle_0$

In this section we wish to make some general observations on the characteristics of $\langle\phi^2\rangle_0$. At the one-loop level we take $\phi=\phi_q$ to be a free field satisfying Eq. (2.12) with $\phi_c=0$. It has been implicitly assumed in the above discussion that the singular parts of $\langle\phi^2\rangle_0$ have been absorbed by renormalizations of the bare parameters entering the theory and that the finite remainder is what appears in Sec. II along with renormalized values for ξ and λ. If the expectation value of ϕ^2 is calculated in a given quantum state in curved spacetime using dimensional regularization, the result is found to be

$$\langle\phi^2\rangle_{\text{reg}}=\frac{1}{8\pi^2}(n-4)^{-1}(\xi-\tfrac{1}{6})R+\langle\phi^2\rangle_0\ . \tag{3.1}$$

The pole term is independent of the choice of state and takes the same form in all spacetimes.[18] The finite part $\langle\phi^2\rangle_0$ contains all information concerning the details of the state. If the pole term is removed by a renormalization of ξ, there is also the freedom to perform additional finite renormalizations which redefine $\langle\phi^2\rangle_0$. We assume that all terms proportional to R are removed by renormalization to the extent to which this is unambiguously possible.

In general the calculation of $\langle\phi^2\rangle_0$ in a given spacetime is a difficult task; however, some explicit cases in which this calculation may be performed will be discussed in Sec. IV. In these examples a term of the form $R\ln|R\mu^{-2}|$ is found to appear in $\langle\phi^2\rangle_0$, where μ is an arbitrary mass. Here we wish to argue that such a term is expected in general. Assume for a moment that $\langle\phi^2\rangle_0$ is a local functional of R and μ only; that is, assume it is geometrical and does not depend on invariants such as $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. Then it must be of the form

$$\langle\phi^2\rangle_0=RF(R\mu^{-2})\ , \tag{3.2}$$

where F is a function which may be determined by the following argument. In dimensional regularization, μ is introduced to keep the correct dimensionality of $\langle\phi^2\rangle_{\text{reg}}$ in n spacetime dimensions and appears as a multiplicative factor μ^{4-n}. As $n\to4$,

$$\mu^{4-n}=1+(4-n)\ln\mu+O((n-4)^2) \tag{3.3}$$

so that μ will appear in $\langle\phi^2\rangle_0$ only in the combination $\ln\mu$. This allows the form of $\langle\phi^2\rangle_0$ in (3.2) to be deduced on dimensional grounds. Furthermore, at the one-loop level, μ^{4-n} multiplies a simple pole at $n=4$ in $\langle\phi^2\rangle_{\text{reg}}$ and so F must be linear in $\ln\mu$. We must have

$$F(x)=C(\xi)\ln x\ , \tag{3.4}$$

where $C(\xi)$ is a function of ξ to be determined. (F is only determined up to a constant which may be absorbed by a redefinition of μ.)

From Eq. (3.2) we have

$$\langle\phi^2\rangle_0=C(\xi)R\ln|R\mu^{-2}|\ . \tag{3.5}$$

Equation (2.11) becomes ($\phi_c=0$)

$$\Box\Phi+\xi R\Phi+\lambda C(\xi)R(\ln|R\mu^{-2}|)\Phi=0\ . \tag{3.6}$$

Now consider a redefinition of the parameters μ, ξ, and λ (the renormalization-group transformation):

$$\mu\to\mu'\ , \tag{3.7}$$

$$\xi\to\xi'\ , \tag{3.8}$$

$$\lambda\to\lambda'\ , \tag{3.9}$$

where $\lambda'=\lambda+O(\lambda^2)$ and $\xi'=\xi+O(\lambda)$. If we work only to order λ and write

$$\lambda C(\xi')=\lambda C(\xi)+O(\lambda^2)\,, \tag{3.10}$$

then Eq. (3.6) is left invariant by this rescaling provided that

$$\xi=\xi'+\lambda C(\xi')\ln\left[\frac{\mu}{\mu'}\right]^2 . \tag{3.11}$$

The renormalization-group equations may now be used to determine $C(\xi)$. Let the bare quantity ξ_B be expressed in terms of the renormalized one ξ by

$$\xi_B=\xi+\sum_{\nu=1}^{\infty}(n-4)^{-\nu}d_\nu(\xi,\lambda)\,, \tag{3.12}$$

and define the function

$$\beta_\xi=\mu\frac{\partial\xi}{\partial\mu}\,. \tag{3.13}$$

It may be shown that [19–21]

$$\beta_\xi=-\lambda\frac{\partial d_1}{\partial\lambda} \tag{3.14}$$

and that[22]

$$d_1=-\frac{\lambda}{8\pi^2}(\xi-\tfrac{1}{6})+O(\lambda^2)\,. \tag{3.15}$$

Equations (3.13)–(3.15) yield, to order λ,

$$\mu\frac{\partial\xi}{\partial\mu}=\frac{\lambda}{8\pi^2}(\xi-\tfrac{1}{6}) \tag{3.16}$$

which has the solution (for $\xi\neq\frac{1}{6}$),

$$\ln\left[\frac{\xi-\frac{1}{6}}{\xi'-\frac{1}{6}}\right]=\frac{\lambda}{16\pi^2}\ln\left[\frac{\mu}{\mu'}\right]^2 . \tag{3.17}$$

If we take $\xi-\xi'=O(\lambda)$, then we may write

$$\xi=\xi'+\frac{\lambda}{16\pi^2}(\xi'-\tfrac{1}{6})\ln\left[\frac{\mu}{\mu'}\right]^2+O(\lambda^2)\,. \tag{3.18}$$

Comparison with Eq. (3.11) leads to

$$C(\xi)=\frac{1}{16\pi^2}(\xi-\tfrac{1}{6})\,. \tag{3.19}$$

We conclude from Eqs. (3.5) and (3.19) that a geometrical state-independent contribution to $\langle\phi^2\rangle_0$ of

$$\langle\phi^2\rangle_0=\frac{1}{16\pi^2}(\xi-\tfrac{1}{6})R\ln|R\mu^{-2}| \tag{3.20}$$

is expected on general grounds. This term may be regarded as a finite residual of the renormalization process which is not of a form which can be removed by an additional finite renormalization. In this sense it is analogous to the anomalous trace of the stress tensor. In general there will be additional state-dependent contributions to $\langle\phi^2\rangle_0$. However, as will be illustrated in Sec. IV, it is often the only contribution.

The arguments used above to obtain Eq. (3.5) may be extended to find the corresponding two-loop contribution to $\langle\phi^2\rangle_0$, which will be of order λ. Although there are nonlocal divergences which arise at the two-loop level, these cancel among each other and the theory remains finite,[2,22–25] Bunch[26] has shown that this is true to all orders. To two-loop order, Eq. (3.15) is replaced by[22]

$$d_1=-\frac{\lambda}{8\pi^2}(\xi-\tfrac{1}{6})+\frac{5\lambda^2}{3(16\pi^2)^2}(\xi-\tfrac{7}{30})\,. \tag{3.21}$$

The solution of Eq. (3.14) is now

$$\xi=\xi'+\frac{\lambda'}{8\pi^2}(\xi'-\tfrac{1}{6})\ln\left[\frac{\mu}{\mu'}\right]-\frac{5\lambda'^2}{6(8\pi^2)^2}(\xi'-\tfrac{7}{30})\ln\left[\frac{\mu}{\mu'}\right]+\frac{2\lambda'^2}{(8\pi^2)^2}(\xi'-\tfrac{1}{6})\ln^2\left[\frac{\mu}{\mu'}\right]. \tag{3.22}$$

Here λ and λ' are related by

$$\lambda'=\lambda+\frac{3\lambda^2}{8\pi^2}\ln\left[\frac{\mu'}{\mu}\right]+O(\lambda^3)\,. \tag{3.23}$$

Note that to order λ, the conformal choice $\xi=\frac{1}{6}$ is a fixed point of the renormalization group, but that this is no longer true to order λ^2.

The appearance of the term $\ln^2(\mu/\mu')$ in Eq. (3.22) means that the two-loop contribution to $\langle\phi^2\rangle_0$ must contain a term of the form $R\ln^2|R\mu^{-2}|$ in order to maintain invariance under the redefinition of μ, ξ, and λ. If we write

$$\langle\phi^2\rangle_0=R\,[C(\xi)\ln|R\mu^{-2}|+D(\xi)\ln^2|R\mu^{-2}|]\,, \tag{3.24}$$

then Eq. (2.11) will be invariant under the redefinitions in Eqs. (3.22) and (3.23) provided that

$$C(\xi)=\frac{1}{16\pi^2}(\xi-\tfrac{1}{6})-\frac{5\lambda}{3(16\pi^2)^2}(\xi-\tfrac{7}{30}) \tag{3.25}$$

and

$$D(\xi)=\frac{\lambda}{(16\pi^2)^2}(\xi-\tfrac{1}{6}) . \tag{3.26}$$

Again it must be emphasized that Eq. (3.24) does not in general represent the entire two-loop contribution to $\langle\phi^2\rangle_0$, but rather gives that part which is determined solely by the coefficient of the pole term; that is, by d_1. The form of the two-loop corrections given in Eq. (3.24) may, however, be used to assess the limits of validity of the one-loop approximation, as will be discussed in Sec. V.

IV. PARTICULAR MODELS

A. Static Einstein universe

Consider the static Einstein spacetime whose line element is given by

$$ds^2=dt^2-a^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\phi^2)] , \tag{4.1}$$

and whose topology is $R^1\times S^3$. The scalar curvature for this model is a constant, $R=6a^{-2}$.

The classical vacuum state $\phi_c=0$ is easily seen to be stable only for $\xi\geq 0$ since the scalar curvature acts like a mass term here. For $\xi<0$ there is symmetry breaking at the tree level with the field developing a nonzero curvature-dependent vacuum expectation value of

$$\phi_c=\pm\left[-\frac{3\xi}{\lambda}R\right]^{1/2} . \tag{4.2}$$

This is just the usual result for the wrong-sign mass model, and the solutions in Eq. (4.2) are locally stable if $\xi<0$.

In order to examine the effects of one-loop corrections on stability $\langle\phi^2\rangle_0$ is required. This was calculated in Ref. 9 for a massive field. Equation (44) of that paper, with the correction of an erroneous numerical factor and a slight change of notation, reads

$$\langle\phi^2\rangle_0=(16\pi^2)^{-1}\left[\rho^2a^{-2}\ln(\rho^2\mu^{-2}a^{-2})-m^2\ln\left[\frac{m^2}{\mu^2}\right]\right]-(2\pi^2a^2)^{-1}[f_{-1/2}(\rho)+\rho^2f_{1/2}(\rho)] , \tag{4.3}$$

where

$$\rho^2=a^2[m^2+(\xi-\tfrac{1}{6})R] , \tag{4.4}$$

m is the mass of the field, and $f_{\pm 1/2}$ are functions whose form is irrelevant to the present discussion. If $m=0$, then $\rho^2=(6\xi-1)$ is a constant independent of a. In this case, the second term in Eq. (4.3) is proportional to R and can be removed by a finite ξ renormalization. This results in

$$\langle\phi^2\rangle_0=(16\pi^2)^{-1}(\xi-\tfrac{1}{6})R\ln(R\mu^{-2}) \tag{4.5}$$

which is recognized to be of the form in Eq. (3.20) predicted by the general arguments of Sec. III.

The stability of $\phi_c=0$ is determined by whether or not the solutions to Eq. (2.11) with $\langle\phi^2\rangle_0$ given in Eq. (4.5) remain bounded for all times. We have $\Box=\partial^2/\partial t^2-(1/a^2)\nabla_3{}^2$ where $\nabla_3{}^2$ is the Laplacian on S^3 whose eigenvalues are given by $n(n+2)$ where $n=0,1,2,\ldots$. Assuming a time dependence for Φ of the form $e^{-i\omega t}$, we find that the eigenfrequencies are given by

$$\omega^2=\frac{n(n+2)}{a^2}+\xi R+\frac{\lambda}{16\pi^2}(\xi-\tfrac{1}{6})R\ln(R\mu^{-2}) . \tag{4.6}$$

The $n=0$ mode will therefore be unstable (i.e., the modes will grow exponentially in time) if

$$\xi R+\frac{\lambda}{16\pi^2}(\xi-\tfrac{1}{6})R\ln(R\mu^{-2})<0 . \tag{4.7}$$

Assuming that $\xi<\frac{1}{6}$, $\phi_c=0$ is found to be unstable if the radius satisfies $a<a_c$ where a_c is the critical radius defined by

$$a_c=\sqrt{6}\mu^{-1}\exp\left[\frac{8\pi^2\xi}{(\xi-\frac{1}{6})\lambda}\right] . \tag{4.8}$$

Note that this critical radius is invariant under the renormalization-group transformation in Eq. (3.18). If $a>a_c$, then the classical ground state $\phi_c=0$ is stable also at the one-loop level. If $\xi>\frac{1}{6}$, then $\phi_c=0$ will be unstable if the radius satisfies $a>a_c$, and stable otherwise.

We can find the ground state in the case of an instability by solving

$$\Box\phi_c+\xi R\phi_c+\frac{\lambda}{3}\phi_c{}^3+\frac{\lambda}{16\pi^2}(\xi-\tfrac{1}{6})R[\ln(R\mu^{-2})]\phi_c=0 . \tag{4.9}$$

Assuming ϕ_c to be a constant we find either $\phi_c=0$ or

$$\phi_c{}^2=-\frac{3\xi}{\lambda}R-\frac{3}{16\pi^2}(\xi-\tfrac{1}{6})R\ln(R\mu^{-2}) . \tag{4.10}$$

In order to examine the stability of these solutions we must solve for the eigenfrequencies of Eq. (2.11). The lowest ($n=0$) eigenfrequency is given by

$$\omega^2=-2\left[\xi R+\frac{\lambda}{16\pi^2}(\xi-\tfrac{1}{6})R\ln(R\mu^{-2})\right] \tag{4.11}$$

which is positive when $\phi_c=0$ is unstable. The states given in Eq. (4.10) are therefore locally stable for $a<a_c$ when $\xi<\frac{1}{6}$, and for $a>a_c$ when $\xi>\frac{1}{6}$.

If we imagine a closed Robertson-Walker universe whose scale factor grows sufficiently slowly so that it may be regarded as a sequence of static Einstein universes of increasing radii, then as the radius passees through the critical value of a_c there will be a phase transition.

Another static case where $\langle\phi^2\rangle_0$ takes the form of Eq. (3.5) is de Sitter spacetime.[27] Because the scalar curvature is constant the results will be essentially equivalent to those found here for the static Einstein universe.

B. Robertson-walker universe with a power-law expansion

In this model we consider a spatially flat, topologically trivial Robertson-Walker model whose line element is

$$ds^2=dt^2-a^2(t)(dx_1{}^2+dx_2{}^2+dx_3{}^2) , \tag{4.13}$$

where the scale factor is given by the power law

$$a(t)=\sigma t^c \tag{4.14}$$

for σ and c constants. It is assumed that $c\neq 1$; however, the case $c=1$ has been examined separately, and there were found to be no essential differences with the $c\neq 1$ case treated here. The scalar curvature is

$$R(t)=6c(2c-1)t^{-2} \tag{4.15}$$

which approaches zero at late times.

The classical wave equation for the free theory can be solved in terms of Hankel functions. [See Eq. (4.18).] Bunch and Davies[28] have calculated $\langle\phi^2\rangle_{\rm reg}$, for the case $\xi=0$, in the out vacuum state using point separation. The result, their Eq. (3.10), may be written as

$$G(x'',x')=\langle\phi^2\rangle_{\rm reg}=-(16\pi^2\epsilon^2\Sigma)^{-1}+\frac{1}{48\pi^2}(R_{\alpha\beta}t^\alpha t^\beta+\tfrac{1}{3}R)-\frac{R}{96\pi^2}[2\gamma+\ln|\tfrac{1}{6}R\mu^{-2}\Sigma|+\ln(\epsilon^2\mu^2)]+O(\epsilon^2\ln\epsilon) , \tag{4.16}$$

where t^α is the separation vector, $\Sigma=t^\alpha t_\alpha=\pm 1$, ϵ is one-half of the proper distance between x' and x'', γ is Euler's constant, and we have introduced the arbitary unit of mass μ to keep the arguments of the logarithms dimensionless. If the directions of t^α are averaged over,[29] then $t^\alpha t^\beta\to g^{\alpha\beta}$, and $R_{\alpha\beta}t^\alpha t^\beta\to R$. The terms in Eq. (4.16) which are singular when the regularization is removed by taking the $\epsilon\to 0$ limit may be absorbed into mass and ξ renormalizations; the result after renormalization is

$$\langle\phi^2\rangle_0=-(96\pi^2)^{-1}R\ln|R\mu^{-2}| . \tag{4.17}$$

This is seen to agree with the result in Eq. (3.20) for $\xi=0$. In this model, as in the Einstein universe, the entire expression for $\langle\phi^2\rangle_0$ is the contribution discussed in Sec. III. For $\xi\neq 0$, the result for $\langle\phi^2\rangle_0$ is given by Eq. (3.20); this follows either from the arguments of Sec. III, or by explicit calculation.

Let us first consider the stability of this model at the classical level. The solutions to Eq. (2.11) with $\lambda=0$ are of the form

$$\Phi=\eta^{(1-3c)/2(1-c)}[b_1H_\nu^{(1)}(k\eta)+b_2H_\nu^{(2)}(k\eta)]e^{i\vec{k}\cdot\vec{x}} , \tag{4.18}$$

where b_1 and b_2 are constants,

$$\eta=\sigma^{-1}(1-c)^{-1}t^{1-c}$$

and

$$\nu^2=(1-6\xi)\frac{c(2c-1)}{(1-c)^2}+\tfrac{1}{4} . \tag{4.19}$$

Hence ν is either pure imaginary or real and nonnegative. These solutions are always oscillatory.

Consequently the theory is stable at the classical level for any choice of ξ. Although it is possible to choose ξ so that ξR in Eq. (2.11) has the same sign as a tachyonic mass term, the time dependence of R prevents the theory from being unstable.

In order to examine the stability of the state $\phi_c=0$ when one-loop effects are included we must examine the solutions to Eq. (2.11). These will be of the form $\Phi=e^{i\vec{k}\cdot\vec{x}}F_k(t)$. The stability may be determined by considering the $\vec{k}=0$ mode, as this is the first to begin to grow at the onset of an instability. The solution is expressible in terms of Hankel functions whose argument may be either real or imaginary depending upon whether t is greater than or less than a critical value t_c given by

$$t_c=\sqrt{6c}\,|2c-1|^{1/2}\mu^{-1}\exp\left\{\frac{8\pi^2}{(\xi-\frac{1}{6})\lambda}\left[\xi-\frac{1}{6}-\frac{(1-c)^2}{24c(2c-1)}\right]\right\}. \tag{4.20}$$

Let

$$B=-\frac{3\lambda}{4\pi^2}(\xi-\tfrac{1}{6})\frac{c(2c-1)}{(1-c)^3}\ . \tag{4.21}$$

If $B<0$ and $t<t_c$, or if $B>0$ and $t>t_c$, then

$$F_0(t)=t^{(1-3c)/2}z^{1/2}\left[c_1H^{(1)}_{1/3}\left[\frac{2}{3|B|}z^{3/2}\right]+c_2H^{(2)}_{1/2}\left[\frac{2}{3|B|}z^{3/2}\right]\right] \tag{4.22}$$

where

$$z=|(1-c)B\ln(\mu t)+A-\tfrac{1}{4}|\ , \tag{4.23}$$

$$A=(6\xi-1)\frac{c(2c-1)}{(1-c)^2}\left[1+\frac{\lambda}{16\pi^2}\ln(6c\,|2c-1|)\right], \tag{4.24}$$

and c_1 and c_c are constants. If $B<0$ and $t>t_c$, or if $B>0$ and $t<t_c$, the solution is

$$F_0(t)=t^{(1-3c)/2}z^{1/2}\left[c_1H^{(1)}_{1/3}\left[\frac{2i}{3|B|}z^{3/2}\right]+c_2H^{(2)}_{1/3}\left[\frac{2i}{3|B|}z^{3/2}\right]\right]\ . \tag{4.25}$$

The critical time given in Eq. (4.20) is that value for which the argument of the Hankel functions vanishes, (i.e., for which $z=0$). It is invariant under the rescaling $\mu\to\mu'$ and $\xi\to\xi'$ in Eq. (3.18) in the same way that the critical radius was in Sec. IV A.

When $|\ln\mu t|$ is large, the mode given in Eq. (4.25) behaves like

$$F_0(t)\propto t^{(1-3c)/2}|\ln\mu t|^{-1/4}\exp(\tfrac{2}{3}|B|^{1/2}|1-c|^{3/2}|\ln\mu t|^{3/2}) \tag{4.26}$$

and therefore grows faster than any power of t. We identify such modes as corresponding to an instability of the state $\phi_c=0$ since the perturbation grows with time faster than the Robertson-Walker scale factor (or indeed, faster than any power of the scale factor). The behavior in Eq. (4.26) may also be obtained from a WKB analysis of the wave equation (2.11).

Therefore, we have found that as the time increases beyond a critical value, if $B<0$, then the tree-level vacuum state $\phi_c=0$ becomes unstable and the discrete symmetry $\phi\to-\phi$ is broken. If $B>0$, then for $t<t_c$, the vacuum state is not the tree-level one, but as t increases beyond t_c this new state becomes unstable and the discrete symmetry gets restored.

In order to characterize the stable ground state of the theory in a regime in which the $\phi\to-\phi$ symmetry is broken, the nonlinear differential equation (2.5) must be solved. We have not been able to do this exactly, however, it is possible to find the asymptotic form of this solution which is valid as $t\to\infty$ in the case $B<0$. Assume that the time derivatives of ϕ_c in Eq. (2.5) may be neglected. In this case the solution becomes

$$\phi_c{}^2 = -3\lambda^{-1}\left[\xi R + \frac{\lambda}{16\pi^2}(\xi - \tfrac{1}{6})R \ln|R\mu^{-2}|\right] \tag{4.27}$$

which is positive. As $t \to \infty$ this is a self-consistent solution. [The time after which this asymptotic form becomes a good approximation to an exact solution may be estimated by computing $\dot{\phi}_c$ and $\ddot{\phi}_c$ from Eq. (4.27).] With this form for ϕ_c, Eq. (2.11) becomes

$$\Box\Phi + \tfrac{2}{3}\phi_c{}^2\Phi = 0 . \tag{4.28}$$

Because $\phi_c{}^2 > 0$ in the regime where the symmetry is broken, the solutions of Eq. (4.28) will be oscillatory, verifying that this nonzero ϕ_c is associated with a stable vacuum state. Note that as $t \to \infty$, $\phi_c \to 0$ so that the broken and unbroken phases become indistinguishable at very late times.

V. DISCUSSION

It has been shown in the previous section that the one-loop radiative corrections in $\lambda\phi^4$ theory in curved spacetimes such as the Einstein universe or an expanding spatially flat Robertson-Walker universe can cause a phase transition between the symmetric $\phi_c = 0$ vacuum and a nonsymmetric $\phi_c \neq 0$ vacuum. We have, however, not addressed the question of the limits of validity of this one-loop approximation. From Eqs. (3.24), (3.25), and (3.26), we see that the two-loop contribution to $\langle\phi^2\rangle_0$ is small compared to the one-loop contribution provided that

$$|\lambda \ln|R\mu^{-2}|| << 16\pi^2 . \tag{5.1}$$

Let us consider what constraints this requirement imposes upon the results obtained in the previous section. In the Einstein universe, the critical radius is given by Eq. (4.8) and satisfies

$$\xi R_c + \frac{\lambda}{16\pi^2}(\xi - \tfrac{1}{6})R_c \ln(R_c\mu^{-2}) = 0 , \tag{5.2}$$

where $R_c = 6a_c{}^{-2}$ is the scalar curvature at the critical radius. In order that Eq. (5.1) be satisfied near $a = a_c$ we must have

$$|\xi| << \tfrac{1}{6} . \tag{5.3}$$

Similarly, in the expanding-universe model of Sec. IV B,

$$\frac{\lambda}{16\pi^2}\ln|R_c\mu^{-2}| = \frac{(1-c)^2}{24c(2c-1)(\xi-\frac{1}{6})} - 1 . \tag{5.4}$$

Equation (5.1) implies that

$$\xi - \tfrac{1}{6} \approx \frac{(1-c)^2}{24c(2c-1)} . \tag{5.5}$$

For example, if $c = \frac{1}{3}$, then $\xi \approx 0$.

Equations (5.3) and (5.5) give the conditions under which the one-loop approximation is valid near the phase transition, and hence that Eqs. (4.8) and (4.20) are correct. For other choices of ξ, the one-loop approximation will fail, and two- and higher-loop processes will be important near the transition. The situation is rather analogous to that found by Coleman and Weinberg[14] for massless scalar electrodynamics in flat spacetime. In general, the one-loop approximation is not adequate to characterize the stable ground state of the theory, but for certain choices of coupling constants higher-loop processes can be neglected. Another analogous situation is finite-temperature field theory[30] where higher-loop effects are important near the phase transition, but the high-temperature limit can be described by the one-loop approximation.

It may appear rather strange that Eqs. (5.3) and (5.5), which involve ξ but not μ, are apparently not invariant under renormalization-group transformations. However, recall that changes in ξ under a rescaling are of order λ [see Eq. (3.18)]. The implication of Eqs. (5.3) and (5.5) is that λ must be sufficiently small so that ξ and ξ' both approximately satisfy these conditions. Thus, as might be expected, the validity of the one-loop approximation requires smallness of the coupling constant.

In the Einstein universe, Eq. (5.3) results in the $\phi_c = 0$ phase being stable for $a > a_c$ and the $\phi_c \neq 0$ phase being stable for $a < a_c$. Thus an adiabatically expanding closed universe undergoes a phase transition which restores the $\phi \to -\phi$ symmetry. In the case of a spatially flat Robertson-Walker universe with a power-law expansion, Eq. (5.5) requires that $B < 0$, so that the symmetric phase is stable for $t < t_c$ and the nonsymmetric phase is stable for $t > t_c$.

The expression for a_c and t_c involve the mass parameter μ. If we suppose that ξ and λ are given, then μ has a definite but unknown value which the theory cannot predict, and which must be determined empirically. The physically significance of the phase transition produced by radiative effects depends crucially upon the value of μ. If it is of the order of the Planck mass, then t_c will be of the order of the Planck time. On the other hand, it seems equally plausible that μ might be of order 1 GeV, in which case $t_c \approx 10^{-23}$ sec. It is often argued that effects due to spacetime curvature ought to be important only at the Planck time; however, in the present case, there seems to be no way to exclude the possibility of such effects being large at much later times.

We have assumed that the quantum state of the field is the vacuum. (For the Einstein universe it is the vacuum defined by the timelike Killing vector; for the power-law expansion it is the out vacuum.) If the system is in some other state then additional terms will appear in Eq. (2.11). In particular, for a thermal state, finite-temperature corrections must be taken into account.

ACKNOWLEDGMENTS

We would like to thank R. Jackiw for a helpful discussion. L.H.F. would like to thank C. J. Isham for hospitality at Imperial College while this work was undertaken. D.J.T. would like to thank the Natural Sciences and Engineering Research Council of Canada for a NATO postdoctoral fellowship. This research was supported in part by the National Science Foundation (PHY-81-06978).

[1] L. H. Ford and T. Yoshimura, Phys. Lett. 70A, 89 (1979).
[2] D. J. Toms, Phys. Rev. D 21, 928 (1980).
[3] D. J. Toms, Phys. Rev. D 21, 2805 (1980).
[4] D. J. Toms, Ann. Phys. (N.Y.) 129, 334 (1980).
[5] G. Denardo and E. Spallucci, Nucl. Phys. B169, 514 (1980).
[6] G. Denardo and E. Spallucci, Nuovo Cimento A58, 243 (1980); A64, 27 (1981).
[7] G. Kennedy, Phys. Rev. D 23, 2884 (1981).
[8] L. H. Ford, Phys. Rev. D 21, 933 (1980).
[9] L. H. Ford, Phys. Rev. D 22, 3003 (1980).
[10] D. J. Toms, Phys. Rev. D (to be published).
[11] G. M. Shore, Ann. Phys. (N.Y.) 128, 376 (1980).
[12] G. W. Gibbons, J. Phys. A 11, 1341 (1978).
[13] J. Goldstone, Nuovo Cimento 19, 154 (1961).
[14] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
[15] A. A. Grib and V. M. Mostepanenko, Pis'ma Zh. Eksp. Teor. Fiz. 25, 302 (1977) [JETP Lett. 25, 277 (1977)].
[16] L. F. Abbott, Nucl. Phys. B185, 233 (1981).
[17] D. J. Toms, in *Quantum Structure of Space and Time*, proceedings of the Nuffield Workshop on Quantum Gravity held at Imperial College, London, 1981, edited by M. J. Duff and C. J. Isham (Cambridge University Press, to be published).
[18] T. S. Bunch and L. Parker, Phys. Rev. D 20, 2499 (1979).
[19] N. D. Birrell and P. C. W. Davies, Phys. Rev. D 22, 322 (1980).
[20] L. S. Brown and J. C. Collins, Ann. Phys. (N.Y.) 130, 215 (1980).
[21] T. S. Bunch, J. Phys. A 14, L218 (1981).
[22] N. D. Birrell, J. Phys. A 13, 569 (1980).
[23] T. S. Bunch, P. Panangaden, and L. Parker, J. Phys. A 13, 901 (1980).
[24] T. S. Bunch and P. Panangaden, J. Phys. A 13, 919 (1980).
[25] N. D. Birrell and L. H. Ford, Phys. Rev. D 22, 330 (1980).
[26] T. S. Bunch, Ann. Phys. (N.Y.) 131, 118 (1981).
[27] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A360, 117 (1978).
[28] T. S. Bunch and P. C. W. Davies, J. Phys. A 11, 1315 (1978).
[29] S. L. Adler, J. Lieberman and Y. J. Ng, Ann. Phys. (N.Y.) 106, 279 (1977).
[30] L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).