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Some exact solutions of Einstein-Dirac-Maxwell fields and massive neutrino

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Some exact solutions of Einstein-Dirac-Maxwell equations are presented for a zero-mass neutrino. It is found that, in the presence of an electromagnetic field, even the time-dependent Dirac field has ghost solutions. But the solution becomes "ghost-free" in the presence of (charged) matter. It is shown that the time-independent Dirac field has a ghost-free solution if, in the context of the current speculations, the neutrinos are considered to possess some mass.

I. INTRODUCTION

Using Cartan's formalism Davis and Ray[1] (DR) obtained an exact solution of the Einstein-Dirac equations for a zero-mass neutrino in a static plane-symmetric space-time where the Dirac field is time independent. The interesting feature of their solution is that the neutrino energy-momentum tensor vanishes, whereas the neutrino field and current density do not. Hence they refer to this neutrino as a "ghost neutrino."

Later, Pechenick and Cohen[2] (PC) presented a more general solution of Einstein-Dirac equations with the same metric as in Ref. 1, but with a time-dependent Dirac field. They obtained in their solution a neutrino field with a nonzero neutrino energy-momentum tensor and current density. Thus, one may think that their solution may represent physical neutrinos.

Sections II A and II B of this paper are devoted to the electromagnetic versions of the above two solutions in cylindrically symmetric metrics. In both cases our solution represents "ghost neutrinos" only. In Sec. II C we have therefore proceeded to obtain an interior solution of Einstein-Dirac-Maxwell equations in a charged fluid. The solution is now "ghost-free."

In the context of the current speculations[3,4] that the neutrinos may possess some mass, we obtain in Sec. III a ghost-free solution for a time-independent Dirac field. One may then say that general relativity demands that the neutrino should possess some mass.

In this paper we have used cylindrically symmetric Weyl, Einstein-Rosen, and Marder metrics. It has been shown in the Appendix that broadly similar results may also be obtained with the plane-symmetric metric used by DR and PC.

II. FIELD EQUATIONS AND THEIR SOLUTIONS

The field equations of Einstein-Dirac-Maxwell fields are

$$R_{jk}-\tfrac{1}{2}g_{jk}R=-8\pi GE_{jk}+8\pi GT_{jk}\,, \tag{1}$$

$$\gamma^j\psi_{;j}+m\psi=0, \tag{2}$$

$$F^{jk}{}_{,k}=0, \tag{3}$$

$$F_{jk,l}+F_{kl,j}+F_{lj,k}=0, \tag{4}$$

with

$$T_{jk}=\tfrac{1}{4}(\psi^\dagger\gamma_j\psi_{;k}-\psi^\dagger_{;k}\gamma_j\psi+\psi^\dagger\gamma_k\psi_{;j}-\psi^\dagger_{;j}\gamma_k\psi), \tag{5}$$

$$E_{jk}=-F^l{}_kF_{jl}+\tfrac{1}{4}g_{jk}F^{lm}F_{lm}. \tag{6}$$

We use units in which $\hbar=c=1$. We adopt the conventions of Jauch and Rohrlich[5] for Dirac γ matrices and the notations of Brill and Wheeler[6] with regard to $\psi^\dagger$, ψ^*, and $\nabla_\mu\psi$.

In the solutions of this section we shall put $m=0$ (zero-mass neutrino).

A. Time-independent Dirac field

In this subsection we shall restrict our discussion to the static case only. Hence, the Dirac field ψ is a function of r only.

Let us consider the line element (Weyl form).

$$ds^2=e^{2u}dt^2-e^{2k-2u}(dr^2+dz^2)-r^2e^{-2u}d\phi^2, \tag{7}$$

where u and k are functions of r alone. Equation (2) can be expressed as

$$\psi_{,1}+\tfrac{1}{2}\left(\frac{1}{r}+k_{,1}-u_{,1}\right)\psi=0, \tag{8}$$

where a comma indicates ordinary differentiation. Solving Eq. (8) we get

$$\psi=\frac{1}{\sqrt{r}}\,e^{(u-k)/2}\psi_0, \tag{9}$$

ψ_0 being an arbitrary constant spinor. From Eq. (5) we get

$$T_{20}=\tfrac{1}{4}e^{u-k}(2u_{,1}-k_{,1})\psi^\dagger\gamma^1\gamma^2\gamma^0\psi \tag{10}$$

and

$$T_{30}=\tfrac{1}{4}e^{u-k}\left(2u_{,1}-\frac{1}{r}\right)\psi^\dagger\gamma^1\gamma^3\gamma^0\psi. \tag{11}$$

All other components of the energy-momentum tensor vanish. The only nonvanishing components of the electromagnetic field tensor are F_{02} and F_{31}. From Eqs. (3) and (4) we get

$$F_{02}=C_1/e^k \text{ and } F_{31}=C_2/e^k \tag{12}$$

where C_1 and C_2 are constants of integration.

The field equation (1) for the line element (7) can now be written as

$$u_{,11}+\frac{u_{,1}}{r}=C^2e^{-2u}, \tag{13}$$

$$k_{,11}-\frac{k_{,1}}{r}-u_{,11}-\frac{u_{,1}}{r}+2u_{,1}{}^2=-C^2e^{-2u}, \tag{14}$$

$$k_{,11}+\frac{k_{,1}}{r}-u_{,11}-\frac{u_{,1}}{r}=C^2e^{-2u}, \tag{15}$$

where

$$C^2=4\pi G(C_1{}^2+C_2{}^2).$$

The solution of Eqs. (13) and (14) are given by

$$e^{2u}=\left(r^{1-a}+\frac{C^2}{2a^2}r^{1+a}\right)^2, \tag{16}$$

$$e^{2k}=\left(r^{1-a}+\frac{C^2}{2a^2}r^{1+a}\right)^4, \tag{17}$$

$$F_{02}=C_1\left(r^{1-a}+\frac{C^2}{2a^2}r^{1+a}\right)^{-2}, \tag{18}$$

$$F_{31}=C_2\left(r^{1-a}+\frac{C^2}{2a^2}r^{1+a}\right)^{-2}, \tag{19}$$

where a is a constant.

Proceeding exactly in the same way as in Ref. 1, one gets from Eqs. (10) and (11)

$$\psi_\nu=\begin{bmatrix}1\\ \pm1\\ i\\ \pm i\end{bmatrix}\psi_c, \tag{20}$$

where ψ_c is a scalar.

Then from Eq. (9), one gets the neutrino solution

$$\psi_\nu=\left(r^{2-a}+\frac{C^2}{2a^2}r^{2+a}\right)^{-1/2}\psi_{\nu_0}, \tag{21}$$

where

$$\psi_{\nu_0}=\alpha_0\begin{bmatrix}1\\ \pm1\\ i\\ \pm i\end{bmatrix}, \tag{22}$$

α_0 being an arbitrary constant.

The current density $S^j=i\psi^\dagger\gamma^j\psi$ when evaluated will take the form

$$S^j=4|\alpha_0|^2\frac{[r^{-a}+(C^2/2a^2)r^{a-1}]^{-1}}{r^2}(\delta_0^j\pm\delta_1^j). \tag{23}$$

Equation (21) together with Eqs. (16)–(19) will give the complete solution.

B. Time-dependent Dirac field

In this subsection we shall consider a field function which depends on r and t. Separating the variables, ψ can be written in the form $\psi_0(r)e^{-i\omega t}$, where ψ_0 is a spinor function of r, and ω is a positive real number.

We shall consider the Einstein-Rosen metric

$$ds^2=e^{2(\alpha-\beta)}(dt^2-dr^2)-r^2e^{-2\beta}d\phi^2-e^{2\beta}dz^2. \tag{24}$$

where α and β are functions of r alone.

Using the notations and results of Ref. 2, we get from Eq. (2)

$$\frac{d\psi_0}{dr}=\left[i\omega\gamma^1\gamma^0-\tfrac{1}{2}\left(\frac{1}{r}+\alpha_{,1}-\beta_{,1}\right)\right]\psi_0. \tag{25}$$

From the solution of Eq. (25) we get

$$\psi=\frac{1}{\sqrt{r}}e^{(\beta-\alpha)/2}e^{i\omega(\gamma^1\gamma^0 r-t)}\psi_c, \tag{26}$$

where ψ_c is an arbitrary constant spinor.

The nonvanishing components of the energy-momentum tensor from Eq. (5) are

$$T_{00}=T_{11}=-\frac{e^{2\beta-2\alpha}}{r}\omega\psi_c^*\psi_c, \tag{27}$$

$$T_{01}=T_{10}=\tfrac{1}{4}e^{\beta-\alpha}\psi^\dagger(4i\omega\gamma^1)\psi, \tag{28}$$

$$T_{02}=T_{20}=\tfrac{1}{4}e^{\beta-\alpha}[\psi^\dagger(2i\omega\gamma^2)\psi+(\alpha_{,1}-2\beta_{,1})\psi^\dagger\gamma^1\gamma^2\gamma^0\psi], \tag{29}$$

$$T_{03}=T_{30}=\tfrac{1}{4}e^{\beta-\alpha}\left[\psi^\dagger(2i\omega\gamma^3)\psi+\left(\alpha_{,1}-\frac{1}{r}\right)\psi^\dagger\gamma^1\gamma^3\gamma^0\psi\right]. \tag{30}$$

The nonvanishing components of the electromagnetic field tensor are given by (12).

The field equations (1) for the line element (24) will reduce to

$$e^{2\beta-2\alpha}\left(\alpha_{,11}-\beta_{,11}-\frac{\beta_{,1}}{r}+\frac{\alpha_{,1}}{r}\right)=C^2e^{-2\alpha}-8\pi GT_{00}, \tag{31}$$

$$e^{2\beta-2\alpha}\left(\alpha_{,11}-\beta_{,11}+2\beta_{,1}{}^2-\frac{\beta_{,1}}{r}-\frac{\alpha_{,1}}{r}\right)=-C^2e^{-2\alpha}+8\pi GT_{11}, \tag{32}$$

$$e^{2\beta-2\alpha}\left(\beta_{,11}+\frac{\beta_{,1}}{r}\right)=C^2e^{-2\alpha}. \tag{33}$$

Adding Eqs. (31) and (32) we get

$$\alpha_{,11}-\beta_{,11}+\beta_{,1}{}^2-\frac{\beta_{,1}}{r}=0. \quad (34)$$

Equation (34) is identical to the result obtained by adding Eqs. (14) and (15). Again, Eq. (33) is identical to Eq. (13). Hence, the solutions for α and β will be similar to those for k and u, respectively.

It then implies that

$$T_{00}=T_{11}=0. \quad (35)$$

Since all other components of T_{jk} are zero from the field equations, we have

$$T_{jk}=0 \text{ for all } j \text{ and } k. \quad (36)$$

Applying Eqs. (26) and (36) to Eqs. (27)–(30), one gets

$$\psi_c=\alpha_0\begin{pmatrix}1\\ \pm 1\\ i\\ \pm i\end{pmatrix}=\psi_{\nu_0}. \quad (37)$$

The wave function is

$$\psi=\frac{[r^{-a}+(C^2/2a^2)r^a]^{-1/2}}{r}(\cos\omega r+i\gamma^1\gamma^0\sin\omega r)e^{-i\omega t}\psi_{\nu_0}. \quad (38)$$

The current density will be given by Eq. (23).

The ghost solution in the case of Sec. II A does not cause surprise, but one may not expect such a solution in the case of Sec. II B. Let us therefore see if a ghost-free solution can be obtained for the Dirac time-dependent field with the electromagnetic field in the presence of matter.

C. An interior solution

We shall consider here an interior solution of a charged fluid at rest. The field equations (1) will take the form

$$R_{jk}-\tfrac{1}{2}g_{jk}R=-8\pi G(E_{jk}+M_{jk})+8\pi GT_{jk}, \quad (39)$$

where

$$M_{jk}=\rho u_{,j}u_{,k}-p(g_{jk}-u_{,j}u_{,k}), \quad (40)$$

and T_{jk} and E_{jk} are given by Eqs. (5) and (6). The quantities involved in Eq. (40) have their usual meanings.

Let us consider here the metric (due to Marder[7])

$$ds^2=e^{2(\alpha-\beta)}(dt^2-dr^2)-r^2e^{-2\beta}d\phi^2-e^{2(\beta+\nu)}dz^2, \quad (41)$$

where α, β, and ν are functions of r alone.

The nonvanishing components of T_{jk} are

$$T_{01}=T_{10}=\tfrac{1}{4}e^{\beta-\alpha}\psi^\dagger(4i\omega\gamma^1)\psi, \quad (42)$$

$$T_{02}=T_{20}=\tfrac{1}{4}e^{\beta-\alpha}[\psi^\dagger(2i\omega\gamma^2)\psi+(\alpha_{,1}-2\beta_{,1}-\nu_{,1})\psi^\dagger\gamma^1\gamma^2\gamma^0\psi], \quad (43)$$

$$T_{03}=T_{30}=\tfrac{1}{4}e^{\beta-\alpha}\left[\psi^\dagger(2i\omega\gamma^3)\psi+\left(\alpha_{,1}-\frac{1}{r}\right)\psi^\dagger\gamma^1\gamma^3\gamma^0\psi\right], \quad (44)$$

$$T_{00}=T_{11}=\tfrac{1}{4}e^{\beta-\alpha}\psi^\dagger(-4i\omega\gamma^0)\psi. \quad (45)$$

Again for the line element (41),

$$R_{01}=R_{02}=R_{03}=0.$$

Thus,

$$T_{01}=T_{02}=T_{03}=0. \quad (46)$$

Proceeding as before, we get

$$\psi=\frac{1}{\sqrt{r}}e^{(\beta-\alpha-\nu)/2}e^{i\omega(\gamma^1\gamma^0r-t)}\psi_c. \quad (47)$$

Equations (46) and (47) together with the Eqs. (42)–(45) give

$$\psi_c=\begin{pmatrix}s\\ \pm s\\ q\\ \pm q\end{pmatrix}e^{i\phi}, \quad (48)$$

where s, q, and ϕ are arbitrary real numbers. Also

$$T_{00}=T_{11}=\frac{M}{r}e^{2\beta-2\alpha-\nu} \quad (49)$$

and

$$S^0=-\frac{M}{\omega r}e^{\beta-\alpha-\nu}, \quad (50)$$

where

$$M=-\omega\psi_c^*\psi_c. \quad (51)$$

All other components of S^j are zero.

The field equations (39) for the line element (41) reduce to

$$e^{2\beta-2\alpha}\left(\nu_{,11}-\alpha_{,1}\nu_{,1}+2\beta_{,1}\nu_{,1}+\beta_{,1}{}^2+\nu_{,1}{}^2-\frac{\alpha_{,1}}{r}+\frac{\nu_{,1}}{r}\right)=-8\pi G\rho-4\pi GC^2e^{-2\alpha-2\nu}-8\pi GT_{00}, \quad (52)$$

$$e^{2\beta-2\alpha}\left(-\beta_{,1}{}^2-2\beta_{,1}\nu_{,1}+\alpha_{,1}\nu_{,1}+\frac{\alpha_{,1}}{r}+\frac{\nu_{,1}}{r}\right)=8\pi Gp+4\pi GC^2e^{-2\alpha-2\nu}+8\pi GT_{11}, \quad (53)$$

$$e^{2\beta-2\alpha}\left(\alpha_{,11}-2\beta_{,11}-\frac{2\beta_{,1}}{r}+\beta_{,1}{}^2\right)=8\pi Gp-4\pi GC^2e^{-2\alpha-2\nu}, \quad (54)$$

$$e^{2\beta-2\alpha}(\alpha_{,11}+\nu_{,11}+2\beta_{,1}\nu_{,1}+\beta_{,1}^2+\nu_{,1}^2) = 8\pi G p + 4\pi G C^2 e^{-2\alpha-2\nu}, \quad (55)$$

where

$$C^2 = C_1^2 + C_2^2 .$$

A solution of Eqs. (52)–(55) is as follows:

$$e^{2\alpha} = \frac{r^2}{\cos^2 ar} \exp\left(\frac{16\pi GM}{a^2}\cos ar\right), \quad (56)$$

$$e^{2\beta} = \frac{r^2}{\cos^2 ar}, \quad (57)$$

$$e^{2\nu} = \frac{r^2}{\cos^2 ar}, \quad (58)$$

$$p = \left(-M\cos ar - \frac{C^2}{2}\right)\exp\left(-\frac{16\pi GM}{a^2}\cos ar\right), \quad (59)$$

$$\rho = \left(-M\cos ar + \frac{C^2}{2}\right)\exp\left(-\frac{16\pi GM}{a^2}\cos ar\right), \quad (60)$$

where M is given by Eq. (51) and $a^2 = 8\pi GC^2$. From Eqs. (47) and (50) we get

$$\psi \sim \frac{1}{(\cos ar)^{1/2}} \exp\left(-\frac{4\pi GM}{a^2}\cos ar\right), \quad (61)$$

$$T_{00} = T_{11} \sim \frac{1}{\cos ar}\exp\left(-\frac{16\pi GM}{a^2}\cos ar\right), \quad (62)$$

$$S^0 \sim \frac{1}{\cos ar}\exp\left(-\frac{8\pi GM}{a^2}\cos ar\right). \quad (63)$$

The above solution represents a fluid cylinder of finite radius r_0 given from Eq. (59) by

$$r_0 = \frac{1}{a}\cos^{-1}\left(\frac{C^2}{2\omega\psi_c^*\psi_c}\right).$$

The exterior solution will be the same as that of the preceding subsection (i.e., Sec. II B).

We have obtained the above solution with a limited motive, namely, to see if a ghost-free solution of Einstein-Dirac-Maxwell equations can be had in the presence of matter. The solution is ghost-free as well as singularity-free.

III. MASSIVE NEUTRINO?

We find that the time-dependent Dirac field which has a ghost-free solution in an otherwise empty space as in Ref. 2 has a "ghost" solution in an electromagnetic field. Again it has a ghost-free solution in an electromagnetic field in a charged fluid. It is therefore obvious that the neutrino theory may suffer from some basic weakness. All solutions should be ghost-free if this weakness is identified and removed. In the context of the current speculations[3,4] that the neutrinos possess some mass, we think that we should attempt to find a solution for the massive neutrino. We do this here.

Equation (2), when solved for the line element (41), will give

$$\psi = \frac{1}{\sqrt{r}}\exp\left(-m\gamma^1\int e^{\alpha-\beta}dr\right)e^{(\beta-\alpha-\nu)/2}, \quad (64)$$

ψ_c being arbitrary constant spinor. The only nonvanishing components of T_{jk} are T_{11}, T_{02}, and T_{03}. The Einstein-Dirac field equations for the line element (41) can be written as

$$\alpha_{,1}\nu_{,1} + \frac{\alpha_{,1}}{r} + \frac{\nu_{,1}}{r} - \beta_{,1}^2 - 2\beta_{,1}\nu_{,1} = \frac{8\pi Gm\lambda}{r}e^{\beta-\alpha-\nu}, \quad (65)$$

$$\nu_{,11} - \alpha_{,1}\nu_{,1} + 2\beta_{,1}\nu_{,1} + \beta_{,1}^2 + \nu_{,1}^2 - \frac{\alpha_{,1}}{r} + \frac{\nu_{,1}}{r} = 0, \quad (66)$$

$$\alpha_{,11} - 2\beta_{,11} + \beta_{,1}^2 - \frac{2\beta_{,1}}{r} = 0, \quad (67)$$

$$\alpha_{,11} + \nu_{,11} + 2\beta_{,1}\nu_{,1} + \beta_{,1}^2 + \nu_{,1}^2 = 0, \quad (68)$$

with

$$T_{11} = \frac{\lambda m}{r}e^{\beta-\alpha-\nu}, \quad (69)$$

where $\lambda = \psi_c^\dagger\psi_c$ and T_{20} and T_{30} are given by

$$T_{20} = \tfrac{1}{4}e^{\beta-\alpha}(\alpha_{,1} - 2\beta_{,1} - \nu_{,1})\psi^\dagger\gamma^1\gamma^2\gamma^0\psi, \quad (70)$$

$$T_{30} = \tfrac{1}{4}e^{\beta-\alpha}\left(\alpha_{,1} - \frac{1}{r}\right)\psi^\dagger\gamma^1\gamma^3\gamma^0\psi. \quad (71)$$

The solution of Eqs. (65)–(68) is

$$\alpha = \ln r + 3\ln(r+A) - \ln[(r+A)^3 + a^3], \quad (72)$$

$$\beta = \ln r + \ln(r+A) - \ln[(r+A)^3 + a^3], \quad (73)$$

$$\nu = \ln(r+A) - \ln r + \ln[(r+A)^3 + a^3], \quad (74)$$

where a and A are constants of integration and

$$8\pi Gm\lambda = 12. \quad (75)$$

Since $R_{20} = R_{30} = 0$, we have

$$T_{20} = T_{30} = 0. \quad (76)$$

Equations (64) and (76) together with (70) and (71) give

$$\psi_c = \begin{bmatrix} s \\ \pm s \\ q \\ \pm q \end{bmatrix} e^{i\phi}, \quad (77)$$

where s, q, and ϕ are arbitrary real numbers. From Eqs. (64), (69), and (72)–(74), we have

$$\psi=(r+A)^{-3/2}[(r+A)^3+a^3]^{-1/2} \times\exp\left[-\frac{m}{3}(r+A)^3\gamma^1\right]\psi_c\,, \quad (78)$$

$$T_{11}=\lambda\, m(r+A)^{-3}[(r+A)^3+a^3]^{-1} \quad (79)$$

and

$$S^j=4|\alpha_0|^2(r+A)^{-3}[(r+A)^3+a^3]^{-1}(\delta_0^j+\delta_1^j)\,. \quad (80)$$

The metric for the massive neutrino will be

$$ds^2=(r+A)^4(dt^2-dr^2) -\frac{[(r+A)^3+a^3]^2}{(r+A)^2}\,d\phi^2-(r+A)^4dz^2\,. \quad (81)$$

To investigate the possibility whether the space-time is singular at any point, let us calculate the components of the Riemann tensor. The nonzero components of the Riemann tensor are

$$\begin{aligned} R^0{}_{101}&=R^1{}_{212}=\frac{2}{(r+A)^6}\,,\\ R^0{}_{202}&=-\frac{4}{(r+A)^6}\,,\\ R^1{}_{313}&=-\frac{4}{(r+A)^6}+\frac{6(r+A)^{-3}}{(r+A)^3+a^3}\,,\\ R^0{}_{303}&=R^2{}_{323}=\frac{2}{(r+A)^6}-\frac{6(r+A)^{-3}}{(r+A)^3+a^3}\,. \end{aligned} \quad (82)$$

The Kretchmann scalar K is calculated to be

$$K=24(r+A)^{-12}[(r+A)^3+a^3]^{-2}[5(r+A)^6+4a^6]\,. \quad (83)$$

Equation (83) shows that K is finite at $r=0$ if A is positive and tends to zero as $r\to\infty$. It should be noted from Eqs. (82) that all the nonzero components of Riemann tensor approach zero as $r\to\infty$. Hence, the space-time is asymptotically flat. Also S^j, ψ, and T_{11} all tend to zero as $r\to\infty$.

The energy-momentum tensor component T_{00} will have a nonzero value if the Dirac field is time dependent. However, from the results of this section, one may infer that general relativity is consistent with the existence of the massive neutrino. In other words, the discovery of the massive neutrino may be an important test of the theory of general relativity.

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APPENDIX

Here we shall briefly derive with the plane-symmetric metric all the results corresponding to those already obtained in this paper. The metric is

$$ds^2=e^{2u}(dx^2-dt^2)+e^{2v}(dy^2+dz^2)\,, \quad (A1)$$

where u and v are functions of x alone.

1. Time-independent Dirac field

The time-independent Dirac function ψ for a massless neutrino is given by

$$\psi=e^{-(v+u/2)}\psi_0\,. \quad (A2)$$

The nonvanishing components of the energy-momentum tensor are

$$T_{20}=\tfrac{1}{4}e^{-u}(v_{,1}-u_{,1})\psi^\dagger\gamma^1\gamma^2\gamma^0\psi\,, \quad (A3)$$

$$T_{30}=\tfrac{1}{4}e^{-u}(v_{,1}-u_{,1})\psi^\dagger\gamma^1\gamma^3\gamma^0\psi\,. \quad (A4)$$

The nonvanishing components of the electromagnetic field tensors are

$$F_{20}=C_1e^{-2v}\text{ and }F_{23}=C_2e^{-2v} \quad (A5)$$

where C_1 and C_2 are constants of integration.

The field equations can be written as

$$e^{-2u}(u_{,11}+2u_{,1}v_{,1})=\frac{k}{2}(C_1^2+C_2^2)e^{-4v}\,, \quad (A6)$$

$$e^{-2u}(u_{11}+2v_{,11}-2u_{,1}v_{,1}+2v_{,1}^2)=\frac{k}{2}(C_1^2+C_2^2)e^{-4v}\,, \quad (A7)$$

$$e^{-2u}(v_{,11}+2v_{,1}^2)=-\frac{k}{2}(C_1^2+C_2^2)e^{-4v}\,, \quad (A8)$$

where $k=8\pi G$ A solution of Eqs. (A6)–(A8) is given by[8]

$$e^{2u}=C_3A(3Ax+B)^{-2/3}\,, \quad (A9)$$

$$e^{2v}=(3Ax+B)^{2/3}\,, \quad (A10)$$

$$F_{01}=C_1(3Ax+B)^{-2/3}\,, \quad (A11)$$

$$F_{23}=C_2(3Ax+B)^{-2/3}\,, \quad (A12)$$

where C_3, B are constants and $A=(k/2)(C_1^2+C_2^2)$. Then from Eq. (A2) one gets the neutrino solution.

$$\psi_\nu=(C_3A)^{-1/4}(3Ax+B)^{-1/6}\psi_{\nu_0}\,, \quad (A13)$$

and

$$S^j=4|\alpha_0|^2(C_3A)^{-1/2}(3Ax+B)^{-1/3}(\delta_0^j\pm\delta_1^j)\,. \quad (A14)$$

2. Time-dependent Dirac field

As in Sec. II B we have the following field equations:

$$e^{-2u}(u_{,11}+2u_{,1}v_{,1})=\frac{k}{2}(C_1^2+C_2^2)e^{-4v}+k\,T_{00}\,, \quad (A15)$$

$$e^{-2u}(u_{,11}+2v_{,11}-2u_{,1}v_{,1}+2v_{,1}^2) = \frac{k}{2}(C_1^2+C_2^2)e^{-4v}-kT_{11}, \quad (A16)$$

$$e^{-2u}(v_{,11}+2v_{,1}^2) = -\frac{k}{2}(C_1^2+C_2^2)e^{-4v}. \quad (A17)$$

Adding Eqs. (A15) and (A16) we get

$$e^{-2u}(u_{,11}+v_{,11}+v_{,1}^2) = \frac{k}{2}(C_1^2+C_2^2)e^{-4v}, \quad (A18)$$

which is identical to the equation obtained by adding (A6) and (A7). Equation (A8) is identical to (A17). Hence, the solution of Eqs. (A15)–(A17) is given by (A9)–(A12). It implies that

$$T_{00} = T_{11} = 0. \quad (A19)$$

The field function ψ becomes

$$\psi = e^{-(v+u/2)}\exp(i\gamma^1\gamma^0\omega x)e^{-i\omega t}\psi_{\nu_0}, \quad (A20)$$

where u and v are given by (A9) and (A10).

3. An interior solution

Proceeding as in Sec. II C for the line element (A1) we get

$$\psi = e^{-(v+u/2)}e^{-i\omega t}r \times e^{i\phi}\left[\begin{pmatrix}1\\ \pm 1\\ \mp i\\ -1\end{pmatrix}e^{i\omega x}e^{i\theta}+\begin{pmatrix}1\\ \pm 1\\ \pm i\\ i\end{pmatrix}e^{-i\omega x}e^{-i\theta}\right], \quad (A21)$$

$$S^0 = 8r^2e^{-(2v+u)}, \quad (A22)$$

where r, θ, and ϕ are real constants.

The field equations reduce to

$$e^{-2u}(2v_{,11}-2u_{,1}v_{,1}+3v_{,1}^2) = k\rho - \frac{k}{2}(E^2-H^2)-kT_{00}, \quad (A23)$$

$$e^{-2u}(2u_{,1}v_{,1}+v_{,1}^2) = kp + \frac{k}{2}(E^2-H^2)+kT_{11}, \quad (A24)$$

$$e^{-2u}(u_{,11}+v_{,11}+v_{,1}^2) = kp - \frac{k}{2}(E^2-H^2), \quad (A25)$$

where

$$F_{01}F^{01} = E^2 \text{ and } F_{23}F^{23} = H^2.$$

The Maxwell equation will take the form

$$\frac{dF^{01}}{dx}+2F^{01}(u_{,1}+v_{,1}) = 4\pi\sigma, \quad (A26)$$

where σ is the charge density.

Since we have four equations (A23)–(A26) to determine six quantities, let us take u and v as free variables and choose them as

$$u = \tfrac{1}{2}ax^2, \quad (A27)$$

$$v = \tfrac{1}{2}(bx^2+c), \quad (A28)$$

where a, b, and c are arbitrary constants. Using Eqs. (A27) and (A28), we get

$$T_{00} = T_{11} = 8\omega r^2e^{-c}e^{-(a+b)x^2}, \quad (A29)$$

$$S^0 = 8r^2e^{-c}e^{-(a/2+b)x^2}, \quad (A30)$$

$$k(E^2-H^2) = -(a+b-2abx^2)e^{-ax^2} - 8k\omega r^2e^{-c}e^{-(a+b)x^2}, \quad (A31)$$

$$kp = \tfrac{1}{2}(a+b)(1+2bx^2)e^{-ax^2} - 4k\omega r^2e^{-c}e^{-(a+b)x^2}, \quad (A32)$$

$$k\rho = \tfrac{1}{2}(3a-b)(1+2bx^2)e^{-ax^2} - 4k\omega r^2e^{-c}e^{-(a+b)x^2}, \quad (A33)$$

$$\frac{k\sigma}{2} = \frac{d}{dx}(F^{01})+2x(a+b)F^{01}. \quad (A34)$$

4. Massive neutrino

Proceeding as in Sec. III we write the field equations as

$$2u_{,1}v_{,1}+v_{,1} = -8\pi\psi_0^\dagger\psi_0 me^{u-2v}, \quad (A35)$$

$$u_{,11}+v_{,11}+v_{,1}^2 = 0, \quad (A36)$$

$$2v_{,11}-2u_{,1}v_{,1}+3v_{,1}^2 = 0, \quad (A37)$$

with

$$T_{11} = \psi_0^\dagger\psi_0 me^{-u-2v}. \quad (A38)$$

The solutions of Eqs. (A36)–(A37) are

$$u = v = 2\ln\left(A+\frac{\lambda}{2}\sqrt{m}\,x\right), \quad (A39)$$

where A is a constant of integration and

$$\lambda^2 = -\tfrac{8}{3}\pi\psi_0^\dagger\psi_0.$$

Thus we get

$$\psi = \left(A+\frac{\lambda}{2}\sqrt{m}\,x\right)^{-3}e^{\xi}\psi_{\nu_0}, \quad (A40)$$

$$S^j = 4|\alpha_0|^2\left(A+\frac{\lambda}{2}\sqrt{m}\,x\right)^{-6}e^{-2\xi}(\delta_0^j\pm\delta_1^j), \quad (A41)$$

and

$$T_{11} = \psi_0^\dagger\psi_0 m\left(A+\frac{\lambda}{2}\sqrt{m}\,x\right)^{-6}, \quad (A42)$$

where

$$\xi = m\gamma^1x(A^2+\tfrac{1}{2}A\lambda\sqrt{m}\,x+\tfrac{1}{12}\lambda^2mx^2). \quad (A43)$$

The metric for the massive neutrino is conformally flat. Calculating the Riemann tensor and the Kretchmann scalar, one can conclude that the space-time cannot be extended beyond $x = -2A/\lambda\sqrt{m}$, while the space-time is asymptotically flat on the other side of the origin. Thus a sort of bound state occurs as in Ref. 2.

[1]T. M. Davis and J. R. Ray, Phys. Rev. D 9, 331 (1974).
[2]K. R. Pechenick and J. M. Cohen, Phys. Rev. D 19, 1635 (1979).
[3]D. V. Nanopoulos, Lectures at the International School of Nuclear Physics, Erice, Sicily, 1980 (Report No. Th. 2871-CERN, 1980) (unpublished), and the references quoted therein.
[4]D. N. Schram, talk at a seminar in the Tata Institute of Fundamental Research, Bombay (India), 1980 (unpublished).
[5]J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer, New York, 1976), Appendix A_2.
[6]D. Brill and J. Wheeler, Rev. Mod. Phys. 29, 465 (1957).
[7]L. Marder, Proc. R. Soc. London A244, 524 (1958).
[8]S. Patnaik, Proc. Camb. Phil. Soc. 67, 127 (1970).