PHYSICAL REVIEW D VOLUME 25, NUMBER 5 1 MARCH 1982

Sum-rule inequalities for pion polarizabilities

I. Guiaşu
Central Institute of Physics, Bucharest, Romania

E. E. Radescu*
International Centre for Theoretical Physics, Trieste, Italy and Joint Institute for Nuclear Research, Dubna, U.S.S.R.

(Received 11 January 1982)

The peculiar influence of the annihilation-channel affects on the electric (α) and magnetic (β) polarizabilities of the (light) hadrons is shown in connection with a bound on $\alpha-\beta$ for pions established in terms of differential cross sections for backward $\gamma\pi\to\gamma\pi$ scattering and forward $\pi\pi\to\gamma\gamma$ annihilation.

The question of determining experimentally the pion polarizabilities (e.g., by trying to extract the low-energy pion Compton scattering from the radiative scattering of high-energy pions on nuclear Coulomb fields[1] or from radiative single-pion photoproduction on protons[2]) is presently under active investigation. Information on pion polarizabilities could also be obtained from the study of the reaction $e^+e^-\to e^+e^-\pi\pi$. There are many theoretical predictions derived within various assumptions and approaches (quark models, current algebras, chiral Lagrangians, dispersion sum rules, etc.), which give results spread over a quite large spectrum (see, for instance, the review in Ref. 3 and the table in Ref. 1). In this context we report here a bound on the difference between the electric (α) and magnetic (β) polarizabilities of the pion with the intention of noting certain possible unusual features of the (light-) hadron polarizabilities and also with the aim of bringing some quantative theoretical clarifications. In a previous note[4] it has been pointed out on the basis of a bound on the sum $\alpha+\beta$ that the $\pi\pi$ annihilation into two photons seems to control somehow the ability of the hadronic cloud surrounding the pion to become polarized in the presence of electric and magnetic fields. However, the sum $\alpha+\beta$ is expected (on quite general grounds) to be rather small[5] with respect to the standard unit of 10^{-3} fm^3 and an exact bound on the more relevant object $\alpha-\beta$ might better reveal such interconnections between physical quantities describing the annihilation and the direct ($\gamma\pi\to\gamma\pi$) channels.

We start by considering the t-channel helicity amplitude $G_{++}\equiv F_{0,0;1,1}$ (in the notations and normalizations of Ref. 4) at $\theta=180°$ [θ is the center-of-mass (c.m.) scattering angle for the s channel $\gamma\pi\to\gamma\pi$] as a function of t [$(t)^{1/2}$ is the total c.m. energy in the t channel $\gamma\gamma\to\pi\pi$]. The usual variables s,t,u [$s+t+u=2\mu^2$, μ is the pion mass, and $(s)^{1/2}$ is the total c.m. energy in the s channel] satisfy then the relations $t=-(s-\mu^2)^2/s$, $u=\mu^4/s$. Assuming, for instance, Mandelstam analyticity (we work to the lowest order in electromagnetism), one can view $f_{++}(t)$ as a real analytic function in the complex t plane cut along the real axis from $4\mu^2$ to ∞ (t-channel unitarity cut) and from $-\infty$ to $-9\mu^2/4$ (s-u-channel cut). At $\theta=180°$ $|f_{++}(t)|$ on the s-channel cut is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\gamma\pi\to\gamma\pi}(s,\theta=180°)=\frac{1}{64\pi^2 s}|f_{++}|^2 \tag{1}$$

while on the t-channel cut $|f_{++}(t)|$ is related to the cross section at $\psi=0°$ or $180°$ (ψ is the c.m. scattering angle in the t channel):

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\gamma\gamma\to\pi\pi}(t,\psi=0°)=\frac{1}{64\pi^2 t}\frac{(t-4\mu^2)^{1/2}}{2(t)^{1/2}}|f_{++}|^2 \ . \tag{2}$$

Instead of Eq. (2) one may use as well

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\pi\pi\to\gamma\gamma}(t,\psi=0°)=\frac{2}{64\pi^2 t}\frac{(t)^{1/2}}{(t-4\mu^2)^{1/2}}|f_{++}|^2 \ . \tag{2$'$}$$

At $t=0$ (the location of the s-u-channel Born poles in the charged-pion case which we treat first) one has (see the identifications from Ref. 4)

$$f_{++}^{(\pi^\pm)}(t=0)=-2e^2,\quad \frac{e^2}{4\pi}\simeq\frac{1}{137}\ , \tag{3}$$

$$\left.\frac{df_{++}^{(\pi^\pm)}}{dt}\right|_{t=0}=-2\mu\pi(\alpha-\beta)^{(\pi^\pm)}\ . \tag{4}$$

Then the maximum-modulus theorem leads through simple procedures to the constraints

$$G \geqslant 2e^2 \ , \tag{5}$$

$$\tfrac{144}{25}\mu^2 \left| -2\mu\pi(\alpha-\beta)^{(\pi^\pm)} + 2e^2 \frac{G'}{G} \right| \leqslant G - \frac{4e^4}{G} \ , \tag{6}$$

where

$$G = \exp\left\{ \frac{3\mu^2}{2\pi} \int_{4\mu^2}^{\infty} \frac{dt}{t} \frac{\ln\left[128\pi^2 t \left(\frac{t}{t-4\mu^2} \right)^{1/2} \left(\frac{d\sigma}{d\Omega'} \right)_{\gamma\gamma\to\pi^+\pi^-} (t;\psi=0°) \right]}{(t-4\mu^2)^{1/2}(t+9\mu^2/4)^{1/2}} + \frac{3\mu^2}{2\pi} \int_{4\mu^2}^{\infty} ds \frac{\ln\left[64\pi^2 s \left(\frac{d\sigma}{d\Omega} \right)_{\gamma\pi^\pm\to\gamma\pi^\pm} (s;\theta=180°) \right]}{(s-\mu^2)(s-4\mu^2)^{1/2}(s-\mu^2/4)^{1/2}} \right\} \tag{7}$$

and

$$\frac{G'}{G} = \frac{3}{2\pi} \int_{4\mu^2}^{\infty} \frac{dt}{t} \frac{(\frac{7}{72}+\mu^2/t)}{(t-4\mu^2)^{1/2}(t+9\mu^2/4)^{1/2}} \ln\left[128\pi^2 t \left(\frac{t}{t-4\mu^2} \right)^{1/2} \left(\frac{d\sigma}{d\Omega'} \right)_{\gamma\gamma\to\pi^+\pi^-} (t;\psi=0°) \right] + \frac{3}{2\pi} \int_{4\mu^2}^{\infty} ds \frac{[\frac{7}{72} - \mu^2 s/(s-\mu^2)^2]}{(s-\mu^2)(s-4\mu^2)^{1/2}(s-\mu^2/4)^{1/2}} \ln\left[64\pi^2 s \left(\frac{d\sigma}{d\Omega} \right)_{\gamma\pi^\pm\to\gamma\pi^\pm} (s;\theta=180°) \right] . \tag{8}$$

The technical steps of the derivation are as follows:

(a) Construct the function

$$G(t) = \exp\left\{ \frac{(4\mu^2-t)^{1/2}(t+9\mu^2/4)^{1/2}}{\pi} \left(\int_{4\mu^2}^{\infty} - \int_{-\infty}^{-9\mu^2/4} \right) dt' \frac{\ln|f_{++}^{(\pi^\pm)}(t')|}{(t'-t)[(t'-4\mu^2)(t'+9\mu^2/4)]^{1/2}} \right\} ,$$

which is analytic in the cut complex t plane, has no interior zeros, and has the property that $|G(t')| = |f_{++}(t')|$ on both cuts.

(b) Apply the maximum-modulus theorem for the analytic function $\phi(t) \equiv f_{++}^{(\pi^\pm)}(t)/G(t)$ which is of modulus 1 on both cuts. This leads to the inequality (5); $G \equiv G(0)$.

(c) Use the conformal mapping

$$\left(\frac{t-4\mu^2}{t+9\mu^2/4} \right)^{1/2} = i \left(\frac{4\mu^2}{9\mu^2/4} \right)^{1/2} \frac{1-z}{1+z} \ ,$$

which applies the cut t plane onto the unit disk in the complex z plane such that $t=0$ goes into $z=0$.

(d) Use the known bound (see, for instance, Ref. 6)

$$\left| \left[\frac{d\phi(t(z))}{dz} \right]_{z=0} \right| \leqslant 1 - |\phi(0)|^2 \ .$$

This lead to the inequality (6)

$$\frac{G'}{G} \equiv \frac{(dG/dt)_{t=0}}{G(0)} \ .$$

The inequalities (5) and (6) are optimal in the sense that they become just equalities if $f_{++}^{(\pi^\pm)}(t)$ has no interior zeros (real or complex) in the cut t plane.

Analogous consideration lead in the neutral-pion case (when no Born-pole contributions have to be taken care of) to the inequality

$$|2\pi\mu^3(\alpha-\beta)^{(\pi^0)}| \leqslant \exp\left\{ \frac{3\mu^2}{2\pi} \int_{4\mu^2}^{\infty} \frac{dt}{t} \frac{\ln\left[\frac{128\pi^2\mu^4}{(t)^{1/2}(t-4\mu^2)^{1/2}} \left(\frac{d\sigma}{d\Omega'} \right)_{\gamma\gamma\to\pi^0\pi^0} (t;\psi=0°) \right]}{(t-4\mu^2)^{1/2}(t+9\mu^2/4)^{1/2}} + \frac{3\mu^2}{2\pi} \int_{4\mu^2}^{\infty} ds \frac{\ln\left[\frac{64\pi^2\mu^4 s^3}{(s-\mu^2)^4} \left(\frac{d\sigma}{d\Omega} \right)_{\gamma\pi^0\to\gamma\pi^0} (s;\theta=180°) \right]}{(s-\mu^2)(s-4\mu^2)^{1/2}(s-\mu^2/4)^{1/2}} \right\} , \tag{9}$$

which is again optimal [i.e., saturated if $(f_{++}^{(\pi^0)}/t)$ has no interior zeros].

If on the s-channel cut instead of the modulus of f_{++} one takes as known the imaginary part

$$\mathrm{Im}\, f_{++}(s,\theta=180°)=(s-\mu^2)\sum_{l=1}^{\infty}(-1)^{l-1}[\sigma_{El}(s)-\sigma_{Ml}(s)] \tag{10}$$

(specifies in terms of total cross sections for photoabsorption of El and Ml photons) one can derive through somewhat more sophisticated techniques (see, for instance, Ref. 7) another sum-rule inequality for $\alpha-\beta$. Below we display the result only in the π^0 case. This time one uses the conformal mapping $t(z)=-4\mu^2 z/(1-z)^2$ which brings the t-channel cut into the circle $|z|=1$ and the s-channel cut into the segment $[x_1=\frac{1}{9},1]$ of the new real axis $x=\mathrm{Re}z$. The bound reads[8]

$$[\phi(0)]^2-2[\phi(0)]I_1+I_2\leq 1 \ , \tag{11}$$

where

$$\phi(z)=\frac{1}{S(z)}\left[\frac{\mu^2 f_{++}(t(z))}{t(z)}\right],\quad \phi(0)=-\frac{2\mu^3\pi(\alpha-\beta)^{\pi^0}}{S(0)} \ , \tag{12}$$

$$S(z)=S(t(z)) \ ,$$

$$S(t)=\exp\left\{\frac{(4\mu^2-t)^{1/2}}{2\pi}\int_{4\mu^2}^{\infty}\frac{dt'\ln\left[\frac{128\pi^2\mu^4}{(t')^{1/2}(t'-4\mu^2)^{1/2}}\left(\frac{d\sigma}{d\Omega'}\right)_{\gamma\gamma\to\pi^0\pi^0}t';\psi=0°)\right]}{(t'-t)(t'-4\mu^2)^{1/2}}\right\} \ , \tag{13}$$

$$I_1=\frac{1}{\pi}\int_{x_1}^{1}\frac{dx}{x}\mathrm{Im}\phi(x),\quad I_2=\frac{1}{\pi^2}\int_{x_1}^{1}\int_{x_1}^{1}dx\,dy\,\frac{\mathrm{Im}\phi(x)\,\mathrm{Im}\phi(y)}{(1-xy)xy} \ .$$

If the amplitude f_{++} is taken in Born approximation the bounds would imply $\alpha-\beta=0$; together with the bound from Ref. 4 which in Born approximation leads to $\alpha+\beta=0$, this gives $\alpha=\beta=0$ in the absence of hadronic structure, as it should.

The above results may be used as checks for model calculations. They show in a way unobscured by approximations that annihilation-channel affects may play a quite peculiar role for (light-) hadron polarizabilities, unlike the case of atoms or molecules when both the electric and magnetic polarizabilities are obtained just by summing over contributions from the excited states of the system. The bad high-energy asymptotics (at least as far as the Regge behavior is concerned) of the Compton amplitude related to $\alpha-\beta$ prevents one from writing down a subtraction-free sum rule involving only s-channel contributions (as for $\alpha+\beta$) and that is why the contact with the usual quantum-mechanical calculations is being lost. The derivation of the above bounds relies on very weak assumptions concerning the asymptotics (due to the logarithms appearing a polynomial boundness is in fact sufficient) and so the conclusions are valid irrespective of subtractions.

Quark-model calculations of hadron polarizabilities should, in the light of the above discussion, be regarded with some caution unless the models are so devised as to account also for the dynamics in the annihilation channel, since otherwise the results may refer only to the static, nonrelativistic correspondents of α and β which risk to have little in common with the structure constants actually measured in photon-hadron reactions.

Next we shall shortly present the results of some simple numerical evaluations of the above bounds based, in the absence of direct experimental information, on reasonable Breit-Wigner (BW) models for the cross sections which are involved. The integrals over the $\gamma\gamma\to\pi\pi$ cross section in Eqs. (7) and (12) and inequality (9) are computed by joining to the Born approximation ($f_{++}^{(\pi^{\pm})}=-2e^2$, $f_{++}^{(\pi^0)}=0$) an $\epsilon(0^+)$-exchange-model contribution in BW form[9] with $M_\epsilon\simeq 660$ MeV, $\Gamma_{\mathrm{total}}\simeq\Gamma_{\epsilon\to\pi\pi}\simeq 640$ MeV and taking $\Gamma_{\epsilon\to\gamma\gamma}\simeq 1.3$ keV. The integrals over the backward Compton cross section in Eq. (7) and inequality (9) are analogously (but more reliably) computed including, apart from the Born approximation (in the $\pi^{\pm}$ case) the ρ, A_1, and A_2 resonances and (in the π^0 case) the ρ and ω resonances.[10] In the charged-pion case [Eq. (7)] beyond the resonance regions on both cuts (to ∞) we have taken the cross sections in Born approximation; in the neutral-pion case [Eq. (12) and inequality (9)] the integrals have been extended only up to around $\sqrt{s}$, $\sqrt{t}\simeq 1.5$ GeV.[11] Numerical integration has lead to the following results [for $(\alpha-\beta)$ we use everywhere units of 10^{-4} fm^3]: $G\simeq 0.189$, $\mu^2 G^1/G\simeq 0.008$, and hence $-2\lesssim(\alpha-\beta)^{(\pi\pm)}\lesssim 15$ from inequality (6); $|(\alpha-\beta)^{(\pi^0)}|\lesssim 17$ from inequality (9). The bound (11) has been evaluated by calculating $S(t)$ in Eq. (13) analogously with the first term in the right-hand side of inequality (9) but expressing, for computational simplicity, $\mathrm{Im} f_{++}^{(\pi^0)}$ in Eq. (10) in narrow-width approximation (only $M1$ transitions with ρ^0 and ω states are retained). $I_{1,2}$ in Eqs. (13) can then be easily found integrating over δ functions and we have obtained $S(0) \simeq 235 \times 10^{-5}$, $S(0) I_1 \simeq -25 \times 10^{-5}$, $S^2(0) I_2 \simeq 0.8 \times 10^{-7}$ so that[12] $-9 \lesssim (\alpha - \beta)^{(\pi^0)} \lesssim 11.5$. More refined evaluations of such sum-rule inequalities with a better consideration of the threshold and asymptotic regions in the integrations (upon which the bounds are quite sensitive) are in progress and will be reported elsewhere together with full details of the saturation.

Note added. The contribution of the f(1270 MeV) meson to the present bounds is irrelevant since the coupling $f \rightarrow \gamma\gamma$ is very small in the helicity channel specified by the amplitude f_{++} we are dealing with. Indeed, the Crystal Ball Group at SPEAR has found from the angular distribution of the $f \rightarrow \pi^0\pi^0$ decay that the production of the f meson in $\gamma\gamma$ scattering is strongly dominated by photon pairs with opposite helicity, thus confirming previous theoretical predictions [see, for instance, D. L. Burke, Report No. SLAC-PUB-2745, 1981 (unpublished) and the literature cited therein]. We thank Dr. S. B. Gerasimov and Dr. A. B. Govorkov for helpful comments.

One of the authors (E.E.R.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. Useful discussions with D. R L. V. Filkov and Professor V. A. Mescheryakov are gratefully acknowledged.

---

*On leave of absence from Central Institute of Physics, P.O. Box 5206, Bucharest, Romania.

[1] A. S. Galperin, G. V. Mitselmacher, A. G. Olshevsky, and V. N. Pervushin, JINR Dubna Report No. P2-80-35 (unpublished).

[2] I. M. Akhmedov and L. V. Filkov, Lebedev Physical Institute, Moscow, Report No. 214, 1978 (unpublished).

[3] V. A. Petrunkin, Fiz. Elem. Chastits At. Yadra 12, 692 (1981) [Sov. J. Part. Nuc. 12, 278 (1981)].

[4] E. E. Radescu, Phys. Rev. D 18, 2329 (1978).

[5] For instance, one finds (see in Ref. 3) $(\alpha + \beta)^{(\pi^\pm)} \simeq 0.4 \times 10^{-4}$ fm^3 and $(\alpha + \beta)^{(\pi^0)} \simeq 10^{-4}$ fm^3 by saturating sum rule involving the total cross section for photoabsorption on pions.

[6] E. E. Radescu, Phys. Rev. D 5, 135 (1972).

[7] I. Caprini, I. Guiaşu, and E. F. Radescu, Phys. Rev. D (to be published).

[8] Inequality (11) is only the simplest from a family and not necessarily the best; the best one which, for simplicity, is not considered here can be obtained (see again Ref. 7) by subsequent extremization with respect to a remaining weight function [taken as 1 in inequality (11)] running over a certain class of analytic functions.

[9] D. M. Akhmedov and L. V. Filkov, Yad. Fiz. 33, 1083 (1981) [Sov. J. Nucl. Phys. 33, 573 (1981)].

[10] The necessary parameters in the Breit-Wigner formulas, e.g., masses and the strong and radiative width are taken from: Particle Data Group, Rev. Mod. Phys. 48, S1 (1976); Phys. Lett. B 75, 1 (1978); T. Ferbel, in *Proceedings of the First Workshop on Ultrarelativistic Nuclear Collisions, Berkeley, California, 1979* (LBL, Berkeley, 1979), p. 1; Univ. of Rochester Report No. UR-727, 1979 (unpublished); E. N. May *et al.*, Phys. Rev. D 16, 1983 (1977). In the BW formulas the corresponding angular momentum factors in the half-widths have been included.

[11] It is worthwhile noting that the inequalities derived here weaken but remain valid if under the logarithms one insert instead of the modulus of the actual amplitude a majorizing quantity. So this crude treatment of the asymptotic behavior (which supposes $f_{++} \rightarrow$ constant at ∞) should not endanger but somewhat weaken the results if f_{++} actually decreases at ∞ as one may expect if s-channel helicity conservation at high energies holds in pion Compton scattering (see Ref. 3 and the literature cited therein in this context).

[12] It is seen that unlike inequality (9) the bound (11) or possible variants of it may be helpful in clarifying the interesting question of the sign of $\alpha^{(\pi^0)}$.