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Schrödinger versus Dirac equation for a massless quark

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Starting from the Dirac equation for a massless particle in a scalar one-body potential, we investigate the limits within which the single-particle energy levels and the corresponding rms radii and magnetic moments may be reproduced by a Schrödinger equation with constant effective mass.

It is well known that an impressive body of hadronic spectroscopic data may be reproduced by the nonrelativistic quark model.[1] The rich spectrum of the nucleon and the other baryons are described well by the nonrelativistic model of Isgur and Karl[2] and even the spectra of the lighter mesons, generated by solving the Schrödinger equation for the $q\bar{q}$ system in an effective potential,[3] are in good agreement with experiment. In these naive quark models, one generally fixes the quark masses $m_u(=m_d)$ around 330 MeV to reproduce the nucleon magnetic moments, and furthermore the effective potential is chosen to roughly yield the proton charge radius. On the whole, one also obtains semiquantitative agreement with the observed transition strengths in various channels. Despite these successes of the naive quark model, it is disturbing to note that the motion of a quark of mass 330 MeV in the chosen effective potential is quite relativistic, and its kinetic energy is comparable to its mass, often exceeding it in excited states.[4,5] One is therefore faced with the old question: How valid is the nonrelativistic quark model, and in what light should its success be viewed?

In a relativistic theory, the near conservation of chiral symmetry requires the masses of m_u and m_d to be close to zero.[6,7] In order to see the connection between the nonrelativistic quark model and a relativistic one, we start from the one-body Dirac equation for simplicity. In this "relativistic shell model," a nucleon may be regarded as three noninteracting zero-mass quarks moving in a confinement potential. The one-gluon-exchange attraction and the hyperfine interaction may, somewhat questionably, be included perturbatively. Such a model, for fitting the baryonic data, was investigated by Ferreira *et al*,[8] and is reminiscent of the precursor to the bag model.[9]

Our objective in this paper is not in fitting the baryonic data. We use the one-body model only to generate relativistically determined quantities such as the ground-state magnetic moment, the radius, and the excitation spectrum of a zero-mass quark in a linear confinement potential. We then find these "data" may be fairly well reproduced by solving the Schrödinger equation with a constant effective mass, in a confinement potential of the same form but of weaker strength. The kinetic energy of the quark is again of the same order as the effective mass. This is very encouraging in the following sense: By solving the Schrödinger equation one is reproducing the "data" generated by the Dirac equation of a zero-mass particle. That the motion in the Schrödinger equation is relativistic is not surprising: One is actually solving a very relativistic problem through the Schrödinger equation. Of course, in this one-body problem there was no practical need to take this crooked route, because solving the Dirac equation is equally easy. Our calculation is only to demonstrate that for this simple case, the Schrödinger equation with a constant effective mass is able, within limits, to reproduce the relativistic results. It is conceivable that such an equivalence may persist even in the Bethe-Salpeter equations for two- and three-body systems.[10] Solving the corresponding Schrödinger equations with an effective potential to simulate the relativistic data, as is done in the quark model, may then make some sense.

Consider a Dirac particle of mass m_D moving in a scalar and potential V and a vector potential (fourth component) W. The stationary Dirac equation is

$$[c\vec{\alpha}\cdot\vec{p}+\beta(m_Dc^2+V)+W]\psi=E\psi\ . \quad (1)$$

The four-component spinor is written as $\psi=\binom{\phi}{\chi}$, and a coupled set of two equations results. Eliminating the small component χ gives a single second-order differential equation:

$$\left[\vec{\sigma}\cdot\vec{\mathrm{p}}\frac{c^2}{E+m_Dc^2+U_-}\vec{\sigma}\cdot\vec{\mathrm{p}}+U_+\right]\phi$$
$$=(E-m_Dc^2)\phi\ , \quad (2)$$

where σ is the usual Pauli matrix, and

$$U_\pm=V\pm W\ . \quad (3)$$

If we now use the identity

$$(\vec{\sigma}\cdot\vec{\nabla})f(\vec{\mathrm{r}})(\vec{\sigma}\cdot\vec{\nabla})$$
$$=\vec{\nabla}\cdot f(\vec{\mathrm{r}})\vec{\nabla}-i(\vec{\nabla}f)\cdot(\vec{\sigma}\times\vec{\nabla})\ ,$$

the Dirac equation may be written as

$$\left[-\vec{\nabla}\cdot\frac{\hbar^2}{2M}\vec{\nabla}-i\left[\vec{\nabla}\frac{\hbar^2}{2M}\right]\cdot\vec{\nabla}\times\vec{\sigma}\right]\phi$$
$$=(E-m_Dc^2-U_+)\phi\ , \quad (4)$$

where M is both E and $\vec{\mathrm{r}}$ dependent:

$$2Mc^2=m_Dc^2+E+U_-(\vec{\mathrm{r}})\ . \quad (5)$$

It is amusing to note that Eq. (4) is identical in form to the nonrelativistic Hartree-Fock equation with a Skyrme-type effective interaction[11] which is extensively used in nuclear physics, with M there being only $\vec{\mathrm{r}}$ dependent. The form (4) of the Dirac equation exhibits the well known energy dependence of the effective mass, and the apparent hopelessness of trying to generate the same spectrum through a Schrödinger equation with constant mass. The energy and coordinate dependence of M comes about through the elimination of the lower component spinor χ. It not only affects the energy spectrum, but the lower component contributes significantly to the magnetic moment and the radius of the orbit, making the latter larger. However, one may fit approximately the Dirac energy spectrum and accommodate the increased radius by solving the Schrödinger equation in a shallower potential. The constant effective mass may be chosen to yield the ground-state magnetic moment.

The magnetic moment of a Dirac particle with charge e_q may be calculated easily using the electric current density $\vec{j}=e_qc\psi^\dagger\,\vec{\alpha}\psi$, and is given by

$$\mu=\frac{1}{2c}\int(\vec{\mathrm{r}}\times\vec{j})_z d^3r\ . \quad (6)$$

In standard notation, writing $\phi=[G(r)/r]\,Y_{jl_A}^{m_z}$, and $\chi=i[F(r)/r]\,Y_{jl_B}^{m_z}$, it can be shown[12,13] that for a scalar potential Eq. (6) reduces (taking $m_z=j$) to

$$\mu=\frac{e_q\hbar c}{2E}\frac{2\kappa j}{(2\kappa+1)}\left[1+\frac{2}{2\kappa-1}\int_0^\infty F^2dr\right] \quad (7)$$

with

$$\kappa=\pm(j+\tfrac{1}{2})\ ,$$
$$l_A=(j\mp\tfrac{1}{2})\ ,\quad l_B=(j\pm\tfrac{1}{2})\ . \quad (8)$$

In the above expressions we have assumed the normalization

$$\int_0^\infty(G^2+F^2)dr=1\ .$$

On solving Eq. (4) in a given partial wave, G is obtained, and the lower component F is then given by

$$F=\frac{\hbar c}{2Mc^2}\left[G'+\frac{\kappa}{r}G\right]\ . \quad (9)$$

Using the above relations, the energy spectrum, magnetic moment, and the mean square radius

$$\langle r^2\rangle=\int_0^\infty(G^2+F^2)r^2dr$$

may easily be computed.

In our numerical computation, we set the vector potential $W=0$, and take the scalar potential

$$V=\lambda r\ . \quad (10)$$

We solve the Dirac equation for a zero-mass ($m_D=0$) particle in the above potential. The strength λ of the Dirac potential is adjusted to fit the rms radius of the ground state to 0.9 fm, a reasonable value in keeping with the charge radius of the proton. We then find

$$\lambda=400\,\mathrm{MeV\,fm}^{-1}\quad\text{with}\ m_D=0\ , \quad (11)$$

and the resulting Dirac spectrum is shown in Table I. The ground-state magnetic moment, using Eq. (7), is found to be $\mu_{1s}=1.875e_q/e$ in units of the proton magneton $e\hbar/2M_pc$. We then proceed to solve the Schrödinger equation for a particle of mass m_{eff} in the corresponding potential

$$V_{\mathrm{eff}}=\lambda' r\ . \quad (12)$$

The mass m_{eff} is chosen such that the Dirac magnetic moment μ_{1s} is reproduced by the effective nonrelativistic expression $e_q\hbar/2m_{\mathrm{eff}}c$, which fixes $m_{\mathrm{eff}}c^2=M_pc^2/\mu_{1s}\approx 500$ MeV. *This effective mass is then kept unchanged in calculating all the excited-state properties.* The strenght λ' of V_{eff} is adjusted to yield the same rms radius of 0.90 fm for the ground state, and results in a shallower po-

TABLE I. Comparison of the Dirac and Schrödinger spectra for a scalar linear confining potential (see text for the parameters). The energies are in MeV, the radii in fm, and the magnetic moments in units of the proton magneton. The numbers in parentheses in the second column are the weighted means of the spin-orbit doublets.

State	Dirac equation Energy		$\langle r^2\rangle^{1/2}$	μ	Schrödinger equation Energy	$\langle r^2\rangle^{1/2}$	μ
$1s_{1/2}$	455		0.90	1.88	455	0.90	1.88
$2s_{1/2}$	731		1.32	1.14	703	1.57	1.25
$3s_{1/2}$	925		1.65	0.90	906	2.11	0.99
$4s_{1/2}$	1083		1.92	0.76	1085	2.60	0.83
$1p_{3/2}$	603	(617)	1.14	2.94	600	1.24	2.90
$1p_{1/2}$	645		1.10	0.66			0.49
$1d_{5/2}$	722	(737)	1.34	3.74	726	1.55	3.65
$1d_{3/2}$	760		1.30	1.65			1.46
$1f_{7/2}$	824	(839)	1.51	4.41	839	1.82	4.24
$1f_{5/2}$	859		1.48	2.50			2.27
$1g_{9/2}$	915	(929)	1.67	5.00	945	2.08	4.74
$1g_{7/2}$	947		1.63	3.23			2.95

tential. This strength λ' is also kept fixed in calculating the excited states. We thus find, for the Schrödinger case,

$$\lambda'=270\,\mathrm{MeV}\,fm^{-1}\,, \quad m_{\mathrm{eff}}c^2=500\,\mathrm{MeV}\,. \tag{13}$$

The Schrödinger spectrum obtained in this way is also shown in Table I for comparison. Whereas the Dirac energy eigenvalues are absolute, the Schrödinger energies are all shifted by a constant amount to adjust the $1s_{1/2}$ eigenvalues to be identical. (The absolute Schrödinger eigenvalue for the ground state was $m_{\mathrm{eff}}c^2+E_{1s}=831$ MeV.) Since we do not add any spin-orbit potential in the Schrödinger potential, the weighted mean of the doublets for the Dirac case is also shown in Table I. Note that the absolute energies in the Dirac spectrum have changed by more than a factor of 2, yet the Schrödinger equation with fixed parameters (13) is able to reproduce the energies fairly. The agreement will get poorer for more highly excited states. The weakness of the equivalence shows up more in the radii. Whereas the ground-state radius was constrained to be the same, the radii of the excited states in the Schrödinger case tend to increase much more rapidly. Similarly, it is not possible to get agreement in the excited-state magnetic moments. In the naive Schrödinger picture, the magnetic moment (using $g_s=2$) is simply given by

$$\mu=\frac{e_q\hbar c}{2m_{\mathrm{eff}}c^2}\frac{2\kappa j}{(2\kappa+1)}\,, \tag{14}$$

where $m_{\mathrm{eff}}c^2=500$ MeV is fixed. This formula yields, for example, the same μ for all the nodal excited states with fixed κj, whereas the Dirac magnetic moments decrease rapidly due to the increasing excitation energies [see Eq. (7)]. One can, however, simulate this effect by modifying Eq. (14) to

$$\mu=\frac{e_q\hbar c}{2(m_{\mathrm{eff}}c^2+\Delta E)}\frac{2\kappa j}{(2\kappa+1)}\,, \tag{15}$$

where $\Delta E=(E_{\kappa j}-E_{1s})$ is the excitation energy of a given state with respect to the ground state, as calculated through the Schrödinger equation. The magnetic moments calculated by this prescription for the potential specified by Eqs. (12) and (13) are shown in the last column of Table I to be compared with the corresponding Dirac moments given in column 4.

We have also performed similar numerical calculations with other power-law scalar potentials, and with half-and-half scalar-vector type of potentials as used in Ref. 8. The agreement in the spectra improves if the power is less than one, and deteriorates with steeper potentials, but the general trend is the same. For example, with a harmonic potential $V=300r^2$ MeV in the Dirac case, again with $m_D=0$, we got $\langle r^2\rangle^{1/2}=0.90$ fm and $\mu=2.1e_q/e$. With the above procedure, we find the Schrödinger mass $m_{\mathrm{eff}}\approx 440$ MeV, and $\hbar\omega=162$ MeV for the harmonic potential. The spectra comes out fairly in the energy range 386 MeV ($1s_{1/2}$) to 1037 MeV ($3s_{1/2}$), in which lie all Dirac states up to $l=4$. The maximum error of

47 MeV between the two spectra is again for the $2s_{1/2}$ state.

We were motivated in this investigation by the success of the Schrödinger equation in simple quark-model calculations.[14] For the lighter hadrons, the term "nonrelativistic" is misleading, because the kinetic energy of a quark is an appreciable fraction of its effective rest mass. A zero-mass fermion moving in a field has obviously no nonrelativistic limit. Within the one-body context, however, an effective mass and an effective potential in the Schrödinger dynamics may be chosen to reproduce the ground-state moments and the excited-state moments. However, the Dirac magnetic moments may be reasonably well reproduced by taking into account the excitation energy as obtained from the Schrödinger spectrum.

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[1]See, e.g., R. H. Dalitz, in *Fundamentals of Quark Models*, edited by I. M. Barbour and A. T. Davies (SUSSP Publications, Edinburgh, 1977), p. 151.

[2]N. Isgur and G. Karl, Phys. Rev. D 11, 4187 (1978); 19, 2653 (1979); 20, 1191 (1979).

[3]R. K. Bhaduri, L. E. Cohler, and Y. Nogami, Nuovo Cimento (to be published).

[4]R. K. Bhaduri, L. E. Cohler, and Y. Nogami, Phys. Rev. Lett. 44, 1369 (1980).

[5]H. J. Lipkin, in *Baryon 80*, proceedings of the IVth International Conference on Baryon Resonances, Toronto, edited by N. Isgur (University of Toronto Press, Toronto, 1981), p. 461.

[6]H. Pagels and S. Stokar, Phys. Rev. D 22, 2876 (1980).

[7]A. Katz, Phys. Rev. D 23, 800 (1981).

[8]P. Leal Ferreira, I. A. Helayel, and N. Zagury, Nuovo Cimento 55A, 215 (1980).

[9]P. N. Bogolioubov, Ann. Inst. Henri Poincaré 8, 163 (1967).

[10]A. N. Mitra, Z. Phys. C 8, 25 (1981); A. N. Mitra and I. Santhanam, *ibid.* 8, 33 (1981).

[11]D. Vautherin, Phys. Rev. C 7, 296 (1973).

[12]A. M. Perelomov and V. S. Popov, Yad, Fiz. 14, 661 (1971) [Sov. J. Nucl. Phys. 14, 370 (1972)].

[13]P. F. Smith and J. D. Lewin, Phys. Lett. 94B, 484 (1980).

[14]I. Cohen and H. J. Lipkin, Phys. Lett. 93B, 56 (1980).