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Gauge fields of a non-Abelian classical source

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Solutions to the gauge-field equations in the presence of a classical, time-independent source are given in terms of elementary functions. Certain aspects of the linear stability of the ensuing field configurations are discussed. The physical electromagnetic fields of all our solutions are those of a magnetic monopole surrounded by an electric-dipole distribution.

I. INTRODUCTION

In the past few years, nonperturbative features of some quantum field theories have been discovered through the semiclassical approximation. In this scheme, quantum expansions in some coupling g are performed around classical field configurations which are, typically, nonanalytic about $g=0$. Thus, the resulting quantum-mechanical amplitudes are unattainable in any finite-order approximation that begins with the trivial $g=0$ term.[1]

Since the nonperturbative regime of all nontrivial field theories cannot be fully understood by existing techniques,[2] the study of classical field configurations is of primary interest.

In non-Abelian gauge theories, the most promising models of the physics of elementary particles, many aspects of the classical theory are known. The self-dual sector, for example, is almost completely solved. However, in the case of these theories, some areas still remain far from understood. In particular, the properties of field configurations produced by external sources are only partially known. Indeed, the concept of an external source itself is somewhat obscure.

In an effort to understand the physics of non-Abelian external sources, a large amount of work has been done recently in this area.[3−5]

In electrodynamics, the Abelian counterpart of Yang-Mills theory, the physics of external sources is fully understood. The general solution of the field equations is the static Coulomb potential plus an arbitrary radiation field. The situation with non-Abelian theories is much more complicated due, chiefly, to the inherently nonlinear nature of the theory. Here, it is known, there exist many solutions of the field equations which carry less energy than the Coulombic solution. In fact, it has been proved[4] that an extended, spherically symmetric source produces fields which are always less energetic than the corresponding Coulombic ones.

The sources considered in previous analyses have been purely electric.[6] In this work we consider sources with both electric and magnetic content. Our interest in this case is twofold. Firstly, this allows us to construct simple, explicit solutions to the field equations which, in turn, make it possible to explore analytically many of the properties of the ensuing field configurations, particularly their stability under linear deformations. Secondly, as it turns out, the properties of field configurations produced by electric sources emerge as well from our analysis.

We will be concerned here with extended, spherically symmetric sources.[7] When such sources are purely electric, it is known that the resulting field configurations fall into two broad classes termed type I and type II.[4] These two types are distinguished by the asymptotic behavior of the spatial components of the gauge potential. The perturbative (in powers of the source strength) and numerical analysis of these fields reveals the following basic properties.[4] Type-I fields resemble the Coulomb configuration in some ways: They exist for all values of the source strength Q, and their energy as a function of Q is, although always smaller, similar in structure to the Coulomb energy. Type-II fields have very different properties. There exists a minimal charge Q_c below which no type-II solutions exist. Moreover, for any charge above Q_c, there are always two solutions of different energy. A graph of energy versus Q thus shows a bifurcation at Q_c. These properties are explicitly verified by our solutions. However, we find new properties as well. In particular, we observe that bifurcation is not an exculsive property of type-II fields.

A general, abstract analysis of the stability of these configurations under linear deformation[5] shows that, for small charges, type-I fields are stable. Most interestingly, it was also shown that one of the branches of type-II configurations was necessarily unstable, whereas the other shares the stability properties of the critical solution. Thus, a deep connection between bifuraction and stability for type-II fields was seen. We argue that bifurcation and the onset of instability are always two aspects of the same phenomenon. This observation allows us to determine explicitly maximum stable charges as well as bifurcation charges from the solution of algebraic constraints.

This paper is organized as follows. In the following section we discuss some general properties of Yang-Mills fields in the presence of time-independent, spherically symmetric sources with both electric as well as magnetic content. Here we point out the existence of two new classes of configurations which are absent in the case of purely electric sources.

In Sec. III we give some representative examples of the solutions we have found and discuss their properties. Conclusions are given in the last section.

II. PRELIMINARIES

The field equations for a non-Abelian gauge field in the presence of a current denisty J_μ are given by

$$D_\mu F^{\mu\nu}=J^\nu \ , \tag{2.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu] \tag{2.2}$$

and

$$D_\mu \phi=\partial_\mu \phi+[A_\mu,\phi] \ . \tag{2.3}$$

In the above we have used a compact matrix notation for all objects. For the gauge group SU(2), $A_\mu \equiv A^a_\mu \sigma^a/2ig$ with σ^a the three Pauli matrices. The coupling constant g has been incorporated into the definition of J_μ, A_μ, and $F_{\mu\nu}$. Since the only effect of this rescaling is to multiply the action by an overall factor of g^{-2}, we will henceforth take g to be unity.

Under gauge transformations specified by

$$U=\exp\left[\frac{i}{2}\theta^a(x)\sigma^a\right]\in \mathrm{SU}(2) \ , \tag{2.4}$$

$$\begin{aligned} A_\mu &\to U^{-1}A_\mu U+U^{-1}\partial_\mu U \ , \\ F_{\mu\nu} &\to U^{-1}F_{\mu\nu}U \ . \end{aligned} \tag{2.5}$$

Equations (2.1) are therefore not gauge invariant; they are covariant provided J_μ transforms as a vector,

$$J_\mu \to U^{-1}J_\mu U \ . \tag{2.6}$$

Owing to its definition, the double covariant divergence of $F_{\mu\nu}$ vanishes. This imposes a constraint on J_μ,

$$D^\mu J_\mu=0 \ . \tag{2.7}$$

In striking contrast with electrodynamics, this integrability condition depends on the gauge potential. Thus, gauge invariance of the classical action is no longer automatic for a coupling of the form $\mathrm{Tr}(A^\mu J_\mu)$. For the time-independent solutions to (2.1), which we shall construct below, Eq. (2.7) is satisfied by a conserved current (in the ordinary sense) so that we meet the independent conditions

$$\partial^\mu J_\mu=\partial^i J_i=0 \ , \tag{2.8a}$$

$$[A^0,J_0]=0 \ , \tag{2.8b}$$

$$[A^i,J_i]=0 \ . \tag{2.8c}$$

In terms of the electric and magnetic components of $F^a_{\mu\nu}$

$$E^i_a=F^{i0}_a \ , \tag{2.9a}$$

$$B^i_a=-\tfrac{1}{2}\epsilon^{ijk}F^{jk}_a \ , \tag{2.9b}$$

the total energy of a given configuration takes the form

$$E=\int d^3x\left[\tfrac{1}{2}(E^i_aE^i_a+B^i_aB^i_a)+J^i_aA_{ia}\right] \ . \tag{2.10}$$

Solving Eqs. (2.1) for an arbitrary source is out of the question. However, the spherically symmetric sector of the theory is considerably simpler.

The most general, radially symmetric form for the current density and potentials can be written as[7]

$$A^0_a=a^0\hat{r}_a \ , \tag{2.11}$$

$$A^i_a=a_1\epsilon^{iaj}\hat{r}_j+a_2(\delta^{ia}-\hat{r}^i\hat{r}^a)+a_3\hat{r}^i\hat{r}^a \tag{2.12}$$

and

$$J_a^0 = j^0 \hat{r}_a \ , \tag{2.13}$$

$$J_a^i = j^1 \epsilon^{iaj} \hat{r}_j + j^2(\delta^{ia} - \hat{r}^i \hat{r}^a) + j^3 \hat{r}^i \hat{r}^a \ . \tag{2.14}$$

This ansatz reduces the field equations to a set of ordinary, coupled differential equations for the functions $a_\mu(r)$ and $j_\mu(r)$. Inspection of these equations reveals some redundancy in the ansatz, Eqs. (2.11)–(2.14). To resolve this redundancy and satisfy the constraints on J_μ, we specialize to the case $a_2 = a_3 = 0$ and $j_2 = j_3 = 0$. The remaining equations can be written in terms of a dimensionless variable $x \equiv r/r_0$ with r_0 an arbitrary scale. To do this, we rescale the current ansatz functions by defining $q(x) \equiv r_0{}^3 j_0(x)$ and $m(x) \equiv -r_0{}^3 j_1(x)$. With primes denoting x derivatives, Eqs. (2.1) are finally reduced to

$$-x^2 f'' + 2a^2 f = x^3 q \ , \tag{2.15}$$

$$-x^2 a'' + (a^2 - f^2 - 1)a = x^3 m \ , \tag{2.16}$$

where

$$a \equiv 1 - r a_1 \ , \quad f \equiv r a_0 \ .$$

These equations correspond to the non-Abelian counterparts of Gauss's and Ampere's laws, respectively. They are identical to those in Ref. 4 except for the appearance of a magnetic source in Eq. (2.16). It is this extra term that allows us to write down simple solutions.

In terms of $a(x)$ and $f(x)$, the components of $F_{\mu\nu}^a$ are given by

$$r_0{}^2 E_a^i = -\hat{r}^i \hat{r}^a \left[\frac{f}{x} \right]' - (\delta^{ia} - \hat{r}^i \hat{r}^a) \left[\frac{af}{x^2} \right] , \tag{2.17}$$

$$r_0{}^2 B_a^i = -\hat{r}^i \hat{r}^a \left[\frac{a^2 - 1}{x^2} \right] - (\delta^{ia} - \hat{r}^i \hat{r}^a) \left[\frac{a'}{x} \right] . \tag{2.18}$$

The total energy [Eq. (2.10)] now becomes $E = E_S + E_a$, where

$$E_S = \frac{4\pi}{r_0} \int_0^\infty dx \left[(a')^2 + \frac{1}{2} \left[\frac{a^2 - 1}{x} \right]^2 + \tfrac{1}{2} (f')^2 + \left[\frac{af}{x} \right]^2 \right] \tag{2.19}$$

and

$$E_A = \frac{8\pi}{r_0} \int_0^\infty dx \, x m (a - 1) \ . \tag{2.20}$$

Partial integration together with the equations of motion reduce E_S to a simpler form involving q and m explicitly:

$$E_S = -\frac{4\pi}{r_0} \int_0^\infty dx \left[x^3 q \left[\frac{f}{x} \right]' + 2x^2 m a' \right] . \tag{2.21}$$

We define the total non-Abelian electric and magnetic charges of a solution as the volume integrals of q and m. These are given by

$$Q = 8\pi r_0{}^3 \int_0^\infty dx \frac{a^2 f}{x} \tag{2.22}$$

and

$$M = 4\pi r_0{}^3 \left[a(\infty) - a(0) + \int_0^\infty dx \frac{(a^2 - f^2 - 1)}{x^2} \right] . \tag{2.23}$$

The above expressions do not, of course, correspond to physical, measurable quantities. The physical electromagnetic fields and charges are obtained by a gauge-invariant projection[8] of the expressions given by Eqs. (2.17) and (2.18). Before constructing these fields we discuss the requirements which must be met by smooth solutions of Eqs. (2.14) and (2.18) so that the fields carry finite total energy.

From Eq. (2.19), finite energy demands that

$$a^2(0) = a^2(\infty) = 1 \ , \tag{2.24}$$

$$f(0) = f(\infty) = 0 \ . \tag{2.25}$$

For a purely electric source, the ansatz does not fully fix the gauge; there still remains a residual U(1) subgroup defined by rotations around the direction in group space specified by J_0^a. This symmetry reflects itself in the fact that the $m = 0$ theory is invariant under $a \to -a$. Hence, in that case, Eqs. (2.24) and (2.25) define only two distinct classes of configurations given by, say, $a(0) = a(\infty) = +1$ and $a(0) = -a(\infty) = +1$. These have been called type-I and type-II fields, respectively.[4] The solutions approach their asymptotic values as

$$\delta a = \delta f = O(x^2) \ , \quad x \to 0 \tag{2.26a}$$

$$\delta a = \delta f = O(1/x) \ , \quad x \to \infty \ . \tag{2.26b}$$

The above analysis is modified when $J_a^i \neq 0$, since, in this case, J_μ^a specifies *two* orthogonal directions in group space, thereby fixing the gauge completely. The two new families of solutions, type Im and type IIm, follow from the previous solutions by transforming $a \rightarrow -a$; the degeneracy seen for $m=0$ is removed by the asymmetric term in the total energy E_A, given by Eq. (2.20). These solutions, which are particular to the case $m \neq 0$, will not be discussed further. Rather, in the next section we will concentrate on the properties of those configurations which have a direct $m=0$ counterpart. As we shall see, the properties of the latter solutions are in essence identical to those of the purely electric objects and so will be referred to as type-I and type-II fields.

As remarked above, J_0^a defines the direction of a U(1) subgroup. This fact allows one to define the physical electromagnetic fields produced by the source: Calling $\hat{\rho}^a$ the direction of J_0^a, the Abelian fields are given by[8]

$$\mathscr{F}_{\mu\nu}=\hat{\rho}^a F_{\mu\nu}^a-\epsilon_{abc}\hat{\rho}^a D_\mu \hat{\rho}^b D_\nu \hat{\rho}^c \; . \tag{2.27}$$

For the radial ansatz, using Eqs. (2.17) and (2.18), one finds

$$\mathscr{E}_i=\mathscr{F}_{0i}=\hat{r}_i \frac{d}{dr}\left[\frac{f}{r}\right] \tag{2.28}$$

and

$$\mathscr{B}_i=-\tfrac{1}{2}\epsilon_{ijk}\mathscr{F}^{jk}=\frac{\hat{r}_i}{r^2} \; , \tag{2.29}$$

these correspond to the fields of a magnetic monopole of unit charge located at the origin surrounded by an electric-dipole cloud[9] [recall Eq. (2.26b)]. Thus the Abelian magnetic and electric charges of all the solutions are one and zero, respectively. The fact that $\mathscr{B}_i$ is independent of the ansatz functions is not as surprising as it might at first seem. It is well known[10] that, in a coupled Yang-Mills-Higgs system, the topology of the gauge field is completely determined by that of the Higgs field, whose role is played here by J_0^a. In fact, for a general configuration, the divergence of the dual of $\mathscr{F}_{\mu\nu}$ defines a magnetic current. This is easily seen to be given by

$$M_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\epsilon_{abc}\partial^\nu \hat{\rho}^a \partial^\alpha \hat{\rho}^b \partial^\beta \hat{\rho}^c \; , \tag{2.30}$$

which is independent of A_μ^a.

We pass now to the discussion of some representative examples of the solutions we have found.

III. SOLUTIONS

Solving Eqs. (2.15) and (2.16) for an arbitrary, prescribed source is, in general, extremely difficult. Moreover, even if this is done, there is no guarantee that the results thus obtained would be useful in understanding the general properties of such configurations: A simple source does not necessarily produce simple fields. However, a different approach to this problem is possible. It was noted by numerical analysis,[11] and borne out by our explicit solutions, that most of the important properties of a given solution are, to a large extent, independent of the detailed shape of the charge distribution. More precisely, it is usually possible to distinguish general properties of these configurations from those which are specific to the particular source considered.

Thus, for a wide range of sources with the general properties of being concentrated in a small region of space with a fast decay at large distances (faster than x^{-4}), the properties of the ensuing field configurations can be established by studying a particular source distribution. A convenient source can be constructed using the numerical solutions for $m=0$ (Ref. 4) as a guide. For purely electric sources the system of equations (2.15) and (2.16) is too restrictive. Examples of analytic solutions for these sources are generally too complicated to be useful.[12] However, the freedom afforded by a nonvanishing magnetic source in Eq. (2.16) allows one to construct simple solutions for simple sources.[13]

In light of the above discussion, in this section we present the details of two examples corresponding to a type-I and a type-II solution. These examples validate numerical and perturbative analyses[4,5] and permit the explicit evaluation of critical and bifurcation charges. The main purpose of this exercise is to demonstrate the usefulness of our approach; it is by no means an exhaustive study of possible solutions to the field equations.[14]

The ansatz functions for the two examples considered here can be written in the form

$$f(x)=\mu\phi(x) \tag{3.1a}$$

and

$$a_{\mathrm{I,II}}(x)=\psi_{\mathrm{I,II}}(x)+\lambda\phi(x) \; , \tag{3.1b}$$

where μ and λ are real parameters and

$$\phi(x)=\frac{x^2}{1+x^4} \; , \tag{3.2}$$

$$\psi_{\mathrm{I}}(x)=1 \tag{3.3a}$$

for type-I solutions and

$$\psi_{\mathrm{II}}(x)=\frac{1-x^4}{1+x^4} \tag{3.3b}$$

for type-II solutions.

We have plotted f and $a_{\rm I}$ for several values of (μ,λ) in Fig. 1. The corresponding curves for $a_{\rm II}$ are shown in Fig. 2. These functions resemble the numerical results of Ref. 4 and constitute the input in our analysis.

The charge distributions corresponding to Eqs. (3.1) are simple rational functions which, for the choice given in Eqs. (3.2) and (3.3), satisfy the requirements discussed above. These functions are shown in Figs. 3 and 4 for several values of μ and λ.

A useful feature of our parametrization is that, for both classes of solutions, for fixed λ, Q is linear in μ whereas M is quadratic. The explicit results are readily obtained from Eqs. (2.22) and (2.23). They are given by

$$Q_{\rm I}=\pi r_0{}^3\mu\left[\frac{\pi}{4}\lambda^2+4\lambda+2\pi\right], \qquad (3.4a)$$

$$M_{\rm I}=\frac{\pi r_0{}^3}{8}[\lambda(\pi\lambda^2+24\lambda+16\pi) -\mu^2(\pi\lambda+8)] \qquad (3.4b)$$

for type-I solutions, and

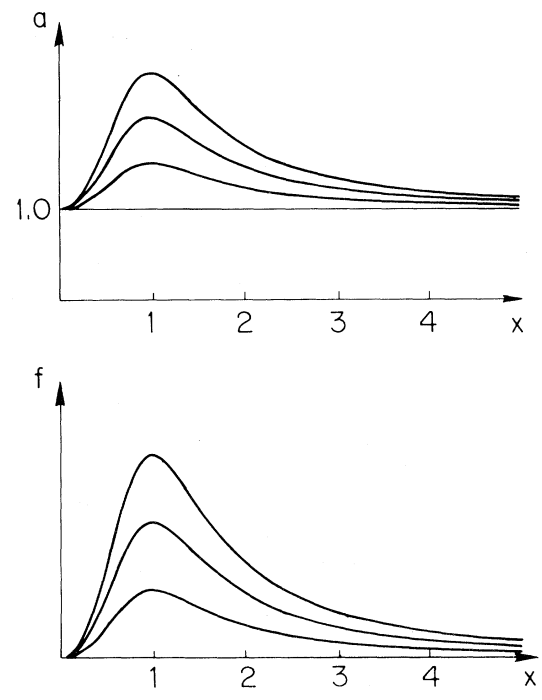

FIG. 1. The functions $a_{\rm I}(x)$ and $f(x)$ for a type-I solution for several values of μ and λ.

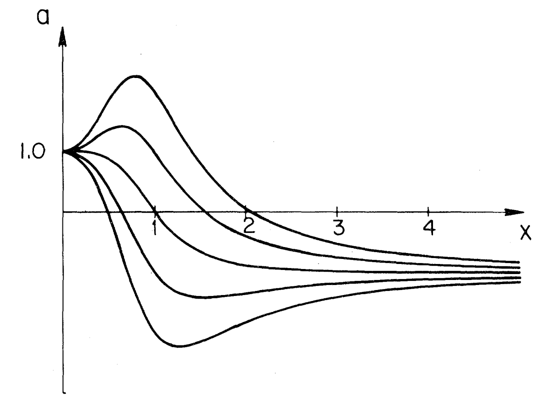

FIG. 2. The function $a_{\rm II}(x)$ for a type-II solution for several values of λ.

$$Q_{\rm II}\frac{\pi^2 r_0{}^3}{4}\mu(\lambda^2+4), \qquad (3.5a)$$

$$M_{\rm II}=4\pi r_0{}^3\left[\frac{\pi\lambda}{32}(\lambda^2-\mu^2+4)-2\right] \qquad (3.5b)$$

for type-II solutions.

Apart from the total charges given above, the most useful object for the analysis that follows is the total energy. From Eqs. (2.19) [or (2.21)] and (2.20) a straightforward computation leads to the results

$$E_{\rm I}=E_0[\lambda^2(25\lambda^2+192\lambda+672) +2\mu^2(56-5\lambda^2)], \qquad (3.6a)$$

$$E_{\rm II}=E_0\{\lambda^2[25\lambda^2-8\lambda+10(36-\mu^2)] +40\lambda(\mu^2-12)+8(242-7\mu^2)\}, \qquad (3.6b)$$

where

$$E_0=\frac{\pi^2}{128\sqrt{2}r_0}.$$

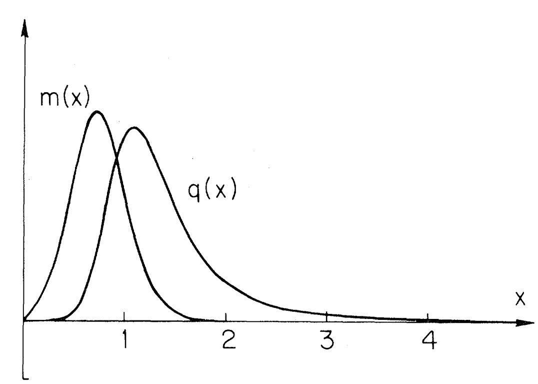

FIG. 3. The charge densities for a type-I solution.

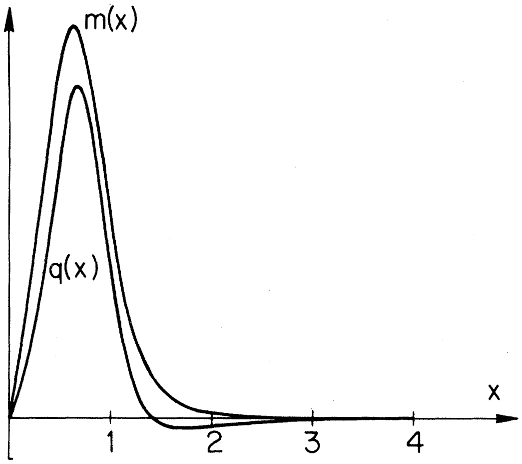

FIG. 4. The charge densities for a type-II solution.

The energies can be written as functions of the total charges by fixing the parameters to satisfy some specific relation $\mu=\mu(\lambda)$. Then, Eqs. (3.4) to (3.6) can be thought of as the parametric equations defining $E=E(Q,M)$. The choice of parametrization is important. As we discuss at length below, bifurcation as well as destabilization occur when, as functions of μ, E and Q (or E and M) have a common extremum. This condition imposes strong constraints on $\mu(\lambda)$. Given such a parametrization, the known or conjectured properties of the solutions[4,5] emerge quite naturally. We find that our solutions for $m\neq 0$ differ in nature from the previously known $m=0$ fields by factors whose origin can be identified. That is, it is possible to extract the properties of the $m=0$ configurations from our analysis. Thus, a type-I class as well as a type-II class with bifurcation can be found easily. Examples for the present case are shown in Figs. 5(a) and 5(b).

Analysis of the small fluctuations around a time-independent solution to the Yang-Mills equations in the presence of a radially symmetric source can be carried out by studying the spectrum of a Schrödinger-type Hermitian operator L:

$$L\chi=\omega^2\chi\ , \tag{3.7}$$

where χ is a vector of small fluctuations and ω^2 is the square of their frequency. (In this analysis, fluctuations are *assumed* to be monochromatic. The following results depend on this assumption.) The explicit form of L has been given for the $m=0$ case,[5] its extension to the present case is trivial. For the solutions given in Eqs. (3.1) to (3.3), one can show that $L\chi$ is a continuous function of μ and λ. Therefore, if a stable oscillation ($\omega^2>0$) becomes unstable ($\omega^2<0$) as μ and λ are varied along some curve in the $\mu-\lambda$ plane, $\mu(\lambda)$, a point on this curve must have $\omega^2=0$. Although we cannot, in general determine ω^2, if we know its sign for some point of the curve, we can determine the critical point where this sign changes. At such critical points the energy and charge (Q or M) must have a common extremum.

If $\partial E/\partial Q$ (or $\partial E/\partial M$) is continuous across this point ω^2 still changes sign, but there is no bifurcation; this can occur only if the common extremum is an inflection point. If the extremum is either a minumum or a maximum, the point of destabilization will also correspond to a bifurcation in $E(Q)$ [or $E(M)$].

Choosing λ as the parameter, the critical points

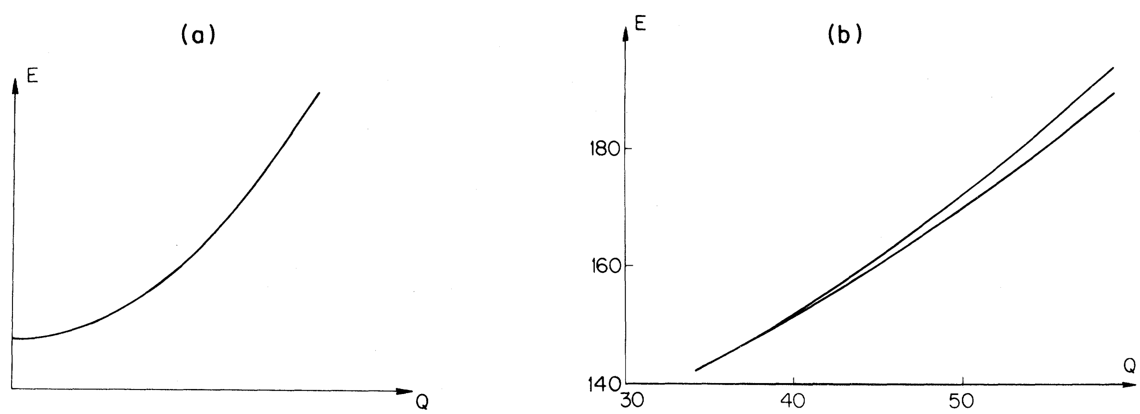

FIG. 5. (a) Energy versus total electric charge for a type-I solution. (b) Energy versus total electric charge for a bifurcating type-II solution.

are determined from the solution of

$$\frac{\partial E}{\partial \lambda}+\frac{d\mu}{d\lambda}\frac{\partial E}{\partial \mu}=0\ , \tag{3.8a}$$

$$\frac{\partial Q}{\partial \lambda}+\frac{d\mu}{d\lambda}\frac{\partial Q}{\partial \mu}=0\ . \tag{3.8b}$$

This system will have a nontrivial solution if, and only if, its determinant vanishes. Therefore, the critical point along the curve $\mu(\lambda)$ is such that the "Poisson bracket" of E and Q (or E and M) with respect to (λ,μ) vanishes:

$$\{E,Q\}_{\lambda\mu}=\frac{\partial E}{\partial \lambda}\frac{\partial Q}{\partial \mu}-\frac{\partial E}{\partial \mu}\frac{\partial Q}{\partial \lambda}=0\ . \tag{3.9}$$

The critical slope, $\mu'(\lambda)$ is given by Eqs. (3.8) for the solutions of Eq. (3.9).

We now illustrate the application of this formalism in the above examples (we look for critical electric charges only; the procedure to find critical magnetic charges is entirely analogous).

The curves in the μ-λ plane for which $\omega^2=0$ for the type-I solution are shown in Fig. 6(a) with critical slopes shown in Fig. 6(b). They correspond to the functions

$${\mu_{\rm crit}}^2=\frac{\lambda(25\lambda^2+144\lambda+336)\left[\frac{\pi}{4}\lambda^2+4\lambda+2\pi\right]}{\frac{3\pi}{4}\lambda^3+16\lambda^2+38\pi\lambda+224}\ , \tag{3.10}$$

$$\mu'_{\rm crit}=\frac{\mu(2\pi\lambda+16)}{\pi\lambda^2+16\lambda+8\pi}\ , \tag{3.11}$$

evaluated at the critical point. Since this solution is stable for small charges,[5] a parametrization $\mu(\lambda)$ which starts with $Q=0$ and crosses the curves (3.10) with slope (3.11) will have $\omega^2=0$ at $Q(\lambda_c)$ and, by continuity, $\omega^2<0$ for $Q>Q(\lambda_c)$. For $\partial E/\partial Q$ to be continuous across this point, $\mu(\lambda)$ must also satisfy

$$\mu''=\mu\frac{\frac{1}{8}\pi^2\lambda^2+2\pi\lambda+16-\pi^2}{(\frac{1}{4}\pi\lambda^2+4\lambda+2\pi)^2} \tag{3.12}$$

at λ_c.

The corresponding results for the type-II solution are shown in Figs. 7(a) and 7(b). These curves are given by

$${\mu_{\rm crit}}^2=-\frac{(25\lambda^3-6\lambda^2+180\lambda-120)(\lambda^2+4)}{5\lambda^3-30\lambda^2-76\lambda+40} \tag{3.13}$$

with

$$\mu'_{\rm crit}=\frac{-2\mu\lambda}{\lambda^2+4} \tag{3.14}$$

evaluated at the critical point. The upper branch is unstable for all Q, whereas the lower one is stable for $Q_b<Q<Q_c$, where Q_b is the bifurcation point obtained from the above equaions and Q_c is a second solution with $\partial^2 Q/\partial\lambda^2=0$ at the critical point. We have checked this analysis for specific choices of the parametrization.

There are, of course, solutions for which $\mu(\lambda)$ does not satisfy the requirements given above for ω^2 to change sign. Since it is physically unlikely that such configurations are stable for arbitrary Q or M, we conjecture that they are always unstable.

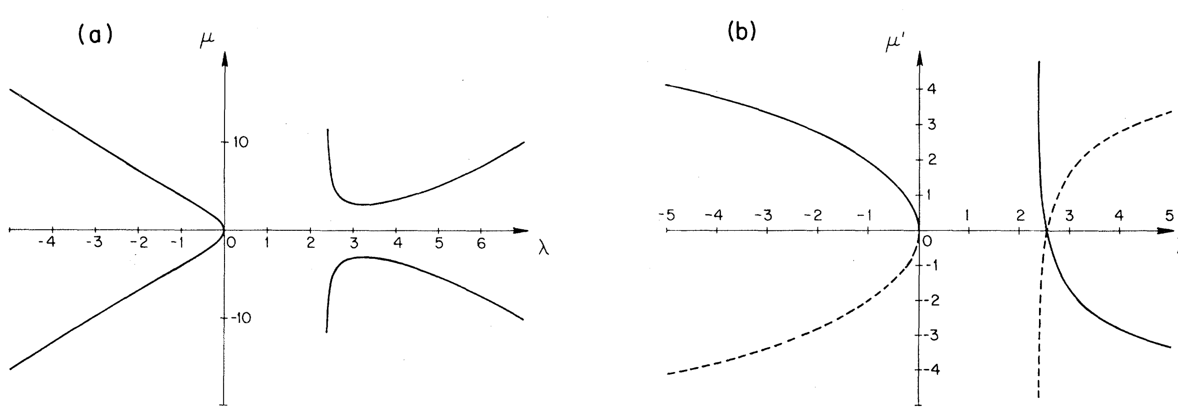

FIG. 6. (a) Points in the μ-λ plane for which $\omega^2=0$ in a type-I solution. (b) Critical slope for bifurcation or destabilization of type-I solutions. The dashed line corresponds to $\mu_{\rm crit}<0$.

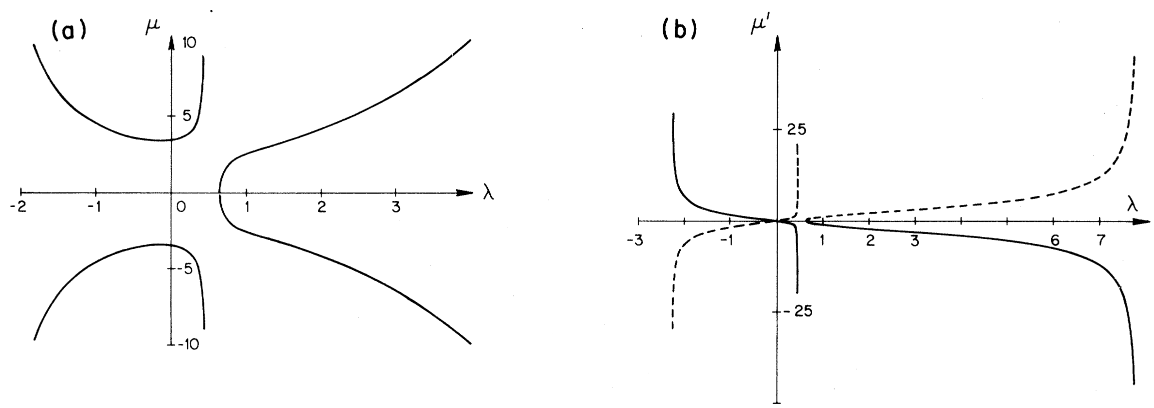

FIG. 7. (a) Points in the μ-λ plane for which $\omega^2=0$ in a type-II solution. (b) Critical slope for bifurcation or destabilization of type-II solutions. The dashed line corresponds to $\mu_{\text{crit}}<0$.

IV. CONCLUSIONS

We have presented an analysis of the nature of gauge fields associated with extended, radially symmetric classical sources. The crucial ingredient which allows the calculability of all objects of interest is the observation that global properties of the configurations depend only weakly on the detailed shape of the charge distribution.

Our analysis of linear stability is based on the assumption that monochromatic fluctuations are the relevant ones. Other than this, our results only depend on the continuity of the fluctuation equations, which could, in fact, include general, radially asymmetric oscillations.

When magnetic sources are included, we find four distinct classes of solutions; the twofold degeneracy observed in the case of purely electric sources is removed by the specification of two orthogonal vectors in group space inherent in the definition of the source.

The interesting problem of point sources is not discussed. Some recent work in this area[15] should complement our results and clarify some remaining problems. The physics of non-Abelian sources remains obscure, but we hope that the availability of explicit solutions may serve as a basis for further understanding of the classical theory.

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[1]Many good references exist on the semiclassical approximation in field theory. See, for example, S. Coleman, in *New Phenomena in Subnuclear Physics,* proceedings of the 14th Course of the International School of Subnuclear Physics, Erice, 1975, edited by A. Zichichi (Plenum, New York, 1977) and *The Whys of Subnuclear Physics,* proceedings of the International School of Subnuclear Physics, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1979); R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

[2]Recently some powerful techniques have been introduced which, in a lattice regulated theory, permit the analysis of nonperturbative aspects of field theory. These techniques are described in M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rev. Lett. 42, 1390 (1979); Phys. Rev. D 20, 1915 (1979); K. Wilson, in Cargese Lectures, 1979 (unpublished).

[3]A partial list of references follows. J. Mandula, Phys. Rev. D 14, 3497 (1976); M. Magg, Phys. Lett. 77B, 199 (1978); P. Sikivie and N. Weiss, Phys. Rev. Lett. 40, 1411 (1978), Phys. Rev. D 18, 3809 (1978); 20, 487 (1979); K. Cahill, Phys. Rev. Lett. 41, 599 (1978); M. Kalb, Phys. Rev. D 18, 2909 (1978); H. Ardoź, Acta Phys. Polon. B12, 429 (1981); N. Weiss, Phys. Rev. D 21, 1603 (1980); D. Horvat and K. S. Viswanathan, *ibid.* 23, 937 (1981); J. Kiskis, *ibid.* 21, 421 (1980).

[4]R. Jackiw, L. Jacobs, and C. Rebbi, Phys. Rev. D 20, 474 (1979).

[5]R. Jackiw and P. Rossi, Phys. Rev. D 21, 426 (1980).

[6]When this manuscript was being readied for publication we were informed of related work by L. Mathelitsch, H. Mitter, and F. Widder, following paper, Phys. Rev. D 25, 1123 (1982). We thank R. Jackiw for bringing this reference to our attention.

[7]In a non-Abelian gauge theory, radial symmetry can be realized in two ways: Either all objects point in a fixed direction in group space and depend only on the magnitude of the radius, or objects depend on angles but do so in a way in which radial asymmetry can always be compensated by a local gauge rotation.

[8]G. 't Hooft, Nucl. Phys. B79, 276 (1974); see also J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975); L. Jacobs, Ph.D thesis, MIT, 1976 (unpublished).

[9]Unlike the Julia-Zee dyon [Phys. Rev. D 11, 2227 (1975)], where the exponential decay of the function corresponding to our $f(x)$ leads to a nonvanishing electric charge.

[10]J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975); G. Woo, Phys. Rev. D 16, 1015 (1977).

[11]R. Jackiw, L. Jacobs, and C. Rebbi (unpublished) and Ref. 4.

[12]See, for example, C. H. Oh, R. Teh, and W. K. Koo, Phys. Rev. D (to be published).

[13]Of course, important problems like determining all the solutions corresponding to a given source cannot be investigated in this context.

[14]A systematic analysis of many examples, including bifurcating type-I solutions, nonbifurcating type-II solutions, and the case $q=0$, $m\neq 0$ are given in J. Wudka, B.S. thesis, UNAM, 1981 (unpublished).

[15]J. Mandula (private communication). An interesting development, currently being studied by R. Giles (private communication), uses the lattice regulator to avoid the problems associated with the singular nature of point sources.