One-meson sector in static models

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(Received 6 July 1981)

The explicit T matrix is given for a general static model in the one-meson subspace of the Tomonaga resolution of the meson field operator. A particular choice of the meson internal-mode function is made. The effective raising of the phase-shift threshold in the ground channel that is observed in pion-nucleon scattering is shown to occur in a natural way in the static model.

Consider the Hamiltonian for a meson field interacting with a single static source:

$$H=\int \omega(k)a^{\dagger}(k)\cdot a(k)dk \\ -\rho\cdot\int [v^*(k)a(k)+v(k)a^{\dagger}(k)]dk \ . \qquad (1)$$

Here $a(k)$ is the annihilation operator for a meson of momentum k and energy $\omega(k)=(k^2+m^2)^{1/2}$; the operator $a(k)$ and the source current operator ρ can be scalars or vectors in isospace or color space and/or configuration space. The case of isovector-vector or $SU_2\times SU_2$ is the usual static model for pions interacting with nucleons, which has been widely treated in the literature,[1] and which has recently received new attention in connection with theories of the interaction of pions with composite-quark nucleons.[2] The SU_3 case has also been considered recently.[3]

In this paper, a general treatment of the Hamiltonian is given in the one-meson sector, where the one-meson sector is defined in terms of the Tomonaga intermediate-coupling approximation.[4] A particular form for the Tomonaga internal mode is shown to have favorable properties. The effects of the orthogonality of the mesonic scattering states to the internal mode on the meson scattering matrix are calculated.

SEPARATION OF AN INTERNAL MODE

The first step in treating H is to follow Tomonaga[4] and separate $a(k)$ into an internal part, with normalized wave function $\varphi(k)$ and annihilation operator A, and a residual part $a_\perp(k)$:

$$a(k)=A\varphi(k)+a_\perp(k) \ . \qquad (2)$$

The $\perp$ subscript will be used to indicate orthogonality to $\varphi(k)$; any function $f(k)$ satisfies

$$f=(\varphi,f)\varphi+f_\perp \ , \\ (\varphi,f_\perp)=\int \varphi^*(k)f_\perp(k)dk=0 \ . \qquad (3)$$

With the separation of Eq. (2), H takes the form

$$\begin{aligned} H&=H_A+H_\perp+H_I+H_I^{\dagger} \ , \\ H_A&=WA^{\dagger}\cdot A-V\rho\cdot(A+A^{\dagger}) \ , \\ H_\perp&=\int \omega(k)a_\perp^{\dagger}(k)\cdot a_\perp(k)dk \\ &=\int \omega_\perp(p,q)a_\perp^{\dagger}(p)\cdot a_\perp(q)dpdq \ , \\ H_I&=-\rho\cdot\int v_\perp^*(k)a_\perp(k)dk \\ &\quad +A^{\dagger}\cdot\int \omega(k)\varphi^*(k)a_\perp(k)dk \ , \\ W&=\int \omega(k)\,|\varphi(k)|^2dk \ , \\ V&=\left|\int \varphi^*(k)v(k)dk\right| \ , \\ \omega_\perp(p,q)&=\omega(p)\delta(p-q) \\ &\quad -[\omega(p)+\omega(q)-W]\varphi(p)\varphi^*(q) \ . \end{aligned} \qquad (4)$$

FORM OF THE INTERNAL MODE

Usually the mode function $\varphi(k)$ is chosen[5] so as to minimize an approximation to $E_{Ag}(W,V)$, the lowest eigenvalue of H_A; this leads to the form

$$\varphi(k)=G_\lambda{}^{-1}\frac{v(k)}{\lambda+\omega(k)} \ , \\ G_\lambda{}^2=\int \frac{|v(k)|^2}{[\lambda+\omega(k)]^2}dk \ , \qquad (5)$$

where the single free parameter λ is determined by the minimization. Note that the form (5) is the most general form for which (in the following G is written for G_λ)

$$(\omega\varphi)_\perp \propto v_\perp \tag{6}$$

[in fact,

$$(\omega\varphi)_\perp = G^{-1} v_\perp] \tag{7}$$

and for which, therefore, H_I involves only the single function $v_\perp(k)$, or, equivalently, $\chi_\perp(k)$, where

$$\begin{aligned} \chi(k) &= G^{-1} v(k) = \omega(k)\varphi(k) , \\ \chi_\perp(k) &= [\omega(k) - W]\varphi(k) ; \end{aligned} \tag{8}$$

specifically, H_I is given by

$$H_I = (A^\dagger - G\rho)\cdot \int \chi_\perp^*(k) a_\perp(k) dk . \tag{9}$$

With the choice of φ given by Eq. (5), it also follows that

$$W = G^{-1} V - \lambda . \tag{10}$$

Let the ground state of H_A be denoted $|g_A\rangle$,

$$H_A |g_A\rangle = E_{Ag} |g_A\rangle . \tag{11}$$

In the Tomonaga procedure, $|g_A\rangle$ is the "no-meson" approximation to the ground state of H; the number of "mesons" is the number of $a_\perp^\dagger$ operators acting in a state (minus the number of $a_\perp$ operators). The source current operator is $A^\dagger - G\rho$. With H_A given in Eqs. (4),

$$[A, H_A] = WA - V\rho , \tag{12}$$

and it follows, using Eq. (10), that

$$\langle g_A | A^\dagger - G\rho | g_A \rangle = \frac{\lambda}{W} G \langle g_A | \rho | g_A \rangle . \tag{13}$$

The state $|g_A\rangle$ also has static field or field expectation value given by

$$\begin{aligned} \langle g_A | a(k) | g_A \rangle &= \varphi(k) \langle g_A | A | g_A \rangle \\ &= \frac{V}{W} \langle g_A | \rho | g_A \rangle \varphi(k) . \end{aligned} \tag{14}$$

COMPLETE SPECIFICATION OF THE INTERNAL MODE

The special choice $\lambda = 0$ for $\varphi(k)$ in Eq. (5) has several advantages. First, the static field of Eq. (14) has momentum dependence such that when it is transformed to configuration space the correct Yukawa dependence on r, namely e^{-mr}/r, is obtained when $v(k)$ is chosen to have the usual form. Second, the lowest-order effects of virtual $\perp$ mesons are minimized, since the expectation value of the source current is zero by Eq. (13). Finally, the unphysical values of λ determined by minimization of $E_{Ag}(\lambda)$ that were noted by Friedman, Lee, and Christian[6] are avoided. Numerical calculations have shown that (a) the change in $E_{Ag}(\lambda)$ between $\lambda = 0$ and $\lambda = \lambda_{\min}$ is quite small[7], (b) $\lambda \to 0$ as the variational subspace size is increased,[7] and (c) the single mode with $\lambda = 0$ gives a value of the ground-state energy that is comparable in accuracy with the value determined in a calculation using as many as six distinct mode functions.[8]

Thus, the first result: in the Tomonaga separation of Eq. (2), the function $\varphi(k)$ should be chosen to be

$$\begin{aligned} \varphi(k) &= \frac{v(k)}{G\omega(k)} , \\ G^2 &= \int \frac{|v(k)|^2}{\omega^2(k)} dk . \end{aligned} \tag{15}$$

With this choice for the function $\varphi(k)$, the Hamiltonian is

$$\begin{aligned} H &= H_A + H_\perp + H_I + H_I^\dagger , \\ H_A &= W[A^\dagger \cdot A - G\rho\cdot(A + A^\dagger)] , \\ H_\perp &= \int \omega_\perp(p,q) a_\perp^\dagger(p)\cdot a_\perp(q) dp\, dq , \\ H_I &= (A^\dagger - G\rho)\cdot \int \chi_\perp^*(k) a_\perp(k) dk , \end{aligned} \tag{16}$$

where $\omega_\perp(p,q)$ is the kinetic energy of the $\perp$ mesons,

$$\begin{aligned} \omega_\perp(p,q) = \omega(p)\delta(p-q) \\ - [\omega(p) + \omega(q) - W]\varphi(p)\varphi^*(q) , \end{aligned} \tag{17}$$

the function $\chi_\perp$ and the constant W are given by

$$\begin{aligned} \chi_\perp(k) &= [\omega(k) - W]\varphi(k) , \\ W &= G^{-2} \int \frac{|v(k)|^2}{\omega(k)} dk , \end{aligned} \tag{18}$$

and $\varphi(k)$ and G are given by Eqs. (15).

SOLVABILITY OF H IN THE ONE-MESON SUBSPACE

The significant feature of the Hamiltonian of Eqs. (16) [or also of the Hamiltonian of Eqs. (4) with the more general choice (5) for φ leading to

Eq. (9)] is that the interaction depends only on the single function $\chi_\perp(k)$, and that, therefore, the system can be solved explicitly within the one-meson subspace $\mathscr{M}_1$ consisting of eigenstates $|a\rangle$ of H_A,

$$H_A\,|a\rangle = E_{As}\,|a\rangle \ , \tag{19}$$

together with states of the form $a_\perp^\dagger(p)\,|a\rangle$. In the remainder of this paper the solution will be exhibited and compared with the case of a similar system without an internal mode. The threshold structure in $\mathscr{M}_1$ and the vanishing of the source current diagonal matrix elements of Eq. (13) can be used in a simple model to show that scattering in the channel with the quantum numbers of the ground state can be expected to be quite small well above the actual threshold for the scattering process. This picture provides a qualitative understanding of the observed pion-nucleon phase shift in the P_{11} partial wave.

SYMMETRY AND COUPLING

As a preliminary to considering the details of the solution in $\mathscr{M}_1$, some features of the classification of the eigenstates of H according to their transformation properties relative to the symmetries of H must be mentioned. In general, the states $|a\rangle$ of Eq. (19) are degenerate because H is invariant under isospin rotations, color rotations, space rotations, or their various combinations; the state $|a\rangle$ will be written $|\alpha\mu\rangle$,

$$H_A\,|\alpha\mu\rangle = E_{A\alpha}\,|\alpha\mu\rangle \ , \tag{20}$$

where the irreducible representation to which $|\alpha\mu\rangle$ belongs will be called $R(\alpha)$ with degeneracy $d_{R(\alpha)}$ and μ runs from 1 to $d_{R(\alpha)}$. The one-meson states are

$$\begin{aligned} &a_{\perp\lambda}^\dagger(p)\,|\alpha\mu\rangle \ , \\ &\langle \alpha\mu\,|\,\alpha_{\perp\lambda}(p)\alpha_{\perp\lambda'}^\dagger(q)\,|\,\beta\mu'\rangle = \delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{\alpha\beta}\delta_\perp(p,q) \ , \end{aligned} \tag{21}$$

where $\delta_\perp$ is the δ function for $\perp$ functions:

$$\delta_\perp(p,q) = \delta(p-q) - \varphi(p)\varphi^*(q) \ . \tag{22}$$

Because of the symmetry, it is useful to couple the representation R_0, to which $a_\perp^\dagger$ belongs, to $R(\alpha)$ to obtain the one-meson states in the representation R,

$$\begin{aligned} |\,R\alpha p\rangle_\nu &= \sum_{\lambda\mu} (R_0\lambda, R(\alpha)\mu\,|\,R\nu)\alpha_{\perp\lambda}^\dagger(p)\,|\,\alpha\mu\rangle \\ &\equiv \{\alpha_\perp^\dagger(p), |\,\alpha\rangle\}_\nu^R \ , \end{aligned} \tag{23}$$

where $(R_1\lambda, R_2\mu\,|\,R\nu)$ are the appropriate coupling coefficients determined by the symmetry group of H. With the usual orthonormality properties for the coupling coefficients, it follows that

$$\begin{aligned} &{}_\nu\langle R\alpha p\,|\,R'\beta q\rangle_{\nu'} = \delta_{RR'}\delta_{\alpha\beta}\delta_{\nu\nu'}\delta_\perp(p,q) \ , \\ &{}_\nu\langle R\alpha p\,|\,H\,|\,R'\beta q\rangle_{\nu'} \\ &\qquad = \delta_{RR'}\delta_{\alpha\beta}\delta_{\nu\nu'}[E_{A\alpha}\delta(p,q) + \omega_\perp(p,q)] \ , \\ &{}_\nu\langle\alpha\,|\,H\,|\,R'\beta q\rangle_{\nu'} \\ &\qquad = \delta_{R'R(\alpha)}\delta_{\nu\nu'}\chi_\perp^*(q)\,{}_\nu\langle\alpha\,|\,\{A^\dagger - G\rho, |\,\beta\rangle\}_{\nu'}^{R'} \ . \end{aligned} \tag{24}$$

Since H and the identity operator are diagonal in R and ν, the parameters R and ν need not be written explicitly; the system to be solved is

$$\begin{aligned} &\langle\alpha\,|\,\beta\rangle = \delta_{\alpha\beta} \ , \\ &\langle\alpha p\,|\,\beta q\rangle = \delta_{\alpha\beta}\delta_\perp(p,q) \ , \\ &\langle\alpha\,|\,\beta q\rangle = 0 \ , \\ &\langle\alpha\,|\,H\,|\,\beta\rangle = E_{A\alpha}\delta_{\alpha\beta} \ , \\ &\langle\alpha p\,|\,H\,|\,\beta q\rangle = [E_{A\alpha}\delta_\perp(p,q) + \omega_\perp(p,q)]\delta_{\alpha\beta} \ , \\ &\langle\alpha\,|\,H\,|\,\beta q\rangle = \langle\beta q\,|\,H\,|\,\alpha\rangle^* = \chi_\perp^*(q)C_{\alpha\beta} \ , \\ &C_{\alpha\beta} = \langle\alpha\,|\,\{A^\dagger - G\rho, |\,\beta\rangle\}\ . \end{aligned} \tag{25}$$

For a given representation R, the coupled states are $|\alpha\rangle$ for which $R(\alpha) = R$ and $|\beta q\rangle$ if R_0 and $R(\beta)$ can be coupled to give R.

COUPLED EQUATIONS

In the one-meson subspace, only states of the form

$$|\,s\rangle = \sum \theta_\alpha\,|\,\alpha\rangle + \sum_\beta \int dq\,\xi_{\perp\beta}(q)\,|\,\beta q\rangle \tag{26}$$

are considered. If $|s\rangle$ is a bound state, the parameter s can be taken to be its energy. For scattering states, s is taken to be γ, p_0^{in} or γ, p_0^{out}, where p_0 is the incident or outgoing momentum and $|\gamma\rangle$ is the eigenstate of H_A in the asymptotic state; that is,

$$\begin{aligned} &|\,\gamma, p_0^{\text{in}}\rangle \underset{t\to-\infty}{\longrightarrow} |\,\gamma p_0\rangle \ , \\ &|\,\gamma, p_0^{\text{out}}\rangle \underset{t\to+\infty}{\longrightarrow} |\,\gamma p_0\rangle \ , \end{aligned} \tag{27}$$

in the usual asymptotic sense. The energy of $|\gamma,p_0^{\rm in}\rangle$ or of $|\gamma,p_0^{\rm out}\rangle$ is $E_{A\gamma}+\omega(p_0)$. Let λ be the energy of the state $|s\rangle$; then

$$H_1|s\rangle=\lambda|s\rangle=\lambda(s)|s\rangle\ , \tag{28}$$

where H_1 is the restriction of H to the one-meson subspace; H_1 has the matrix elements of (21) with all others being zero, while H has many other nonzero matrix elements. The equations for θ and ξ then follow from Eqs. (25):

$$\begin{aligned}(\lambda-E_{A\alpha})\theta_\alpha&=\sum_\beta C_{\alpha\beta}\int \chi_1^*(q)\xi_{1\beta}(q)dq\ ,\\ [\lambda-E'_{A\alpha}-\omega(p)]\xi_{1\alpha}(p)&=\chi_1(p)\sum C^*_{\beta\alpha}\theta_\beta-\varphi(p)\int\omega(q)\varphi^*(q)\xi_{1\alpha}(q)dq\ ,\end{aligned} \tag{29}$$

where $E_{A\alpha}$ is used for states α that belong to the representation R of the scattering channel, while $E'_{A\alpha}$ indicates that the state α belongs to a representation R' that couples to R_0 to give the channel representation R.

SOLUTION OF COMPARISON SYSTEM

Equations (29) are similar to the set that arises in an extended Lee model,[9] namely, Eqs. (29) without the condition that ξ is orthogonal to $\varphi(k)$:

$$\begin{aligned}(\lambda-E_{A\alpha})\eta_\alpha&=\sum_\beta C_{\alpha\beta}\int x^*(q)\zeta_\beta(q)dq\ ,\\ [\lambda-E'_{A\alpha}-\omega(p)]\zeta_\alpha(p)&=x(p)\sum C^*_{\beta\alpha}\eta_\beta\ .\end{aligned} \tag{30}$$

It is the comparison of the solutions of Eqs. (29) with the solutions of Eqs. (30) that shows how the orthogonality to $\varphi(k)$ acts. Since Eqs. (30) are simpler to solve, their treatment is given first; it follows the standard procedure for separable potentials.

For the solution, it is first necessary to write the solution of the second of Eqs. (30). Suppose first that s is a bound state, so that λ is real and below any threshold $E_{A\alpha}+m$. Then $\zeta_\alpha(p)$ is

$$\zeta_\alpha(p)=\frac{x(p)}{\lambda-E'_{A\alpha}-\omega(p)}\sum_\beta C^\dagger_{\alpha\beta}\eta_\beta\ , \tag{31}$$

so that the first of Eqs. (30) becomes

$$\sum_\beta D_{\alpha\beta}(\lambda)\eta_\beta=0\ , \tag{32}$$

where the matrix $D_{\alpha\beta}(\lambda)$ is given by

$$D_{\alpha\beta}(\lambda)=(\lambda-E_{A\alpha})\delta_{\alpha\beta}-\sum_\mu C_{\alpha\mu}I_\mu(\lambda)C^\dagger_{\mu\beta} \tag{33}$$

with $I_\mu(\lambda)$ given by

$$I_\mu(\lambda)=\int\frac{|x(p)|^2}{\lambda-E'_{A\mu}-\omega(p)}dp\ . \tag{34}$$

In matrix notation, with E_A and $I(\lambda)$ diagonal matrices,

$$D(\lambda)=\lambda-E_A-CI(\lambda)C^\dagger\ . \tag{35}$$

Equation (32) has nontrivial solutions only if λ is such that

$$\det D(\lambda)=0\ , \tag{36}$$

and the values of λ that satisfy Eq. (36) are the bound-state energies. For such values of λ, the vector η_β of (32) can be determined and (31) gives the rest of the bound-state wave function.

If $|s\rangle$ is the scattering state $|\gamma,p_0\rangle$, then the solution for ζ_α has a δ-function part,

$$\zeta_\alpha(p)=\delta_{\alpha,\gamma}\delta(p-p_0)+\frac{x(p)}{\lambda-E'_{A\alpha}-\omega(p)}\sum_\beta C^\dagger_{\alpha\beta}\eta_\beta\ , \tag{37}$$

where λ is $E'_{A\gamma}+\omega(p_0)+i0$ if $|s\rangle$ is an "in" state and $E'_{A\gamma}+\omega(p)-i0$ if $|s\rangle$ is an "out" state. Substitution gives

$$\begin{aligned}\eta_\alpha^{\gamma p_0}&=\sum_\beta D^{-1}{}_{\alpha\beta}(\lambda)C_{\beta\gamma}x^*(p_0)\ ,\\ \zeta_\alpha^{\gamma p_0}(p)&=\delta_{\alpha,\gamma}\delta(p-p_0)+\frac{x(p)(C^\dagger D^{-1}(\lambda)C)_{\alpha\gamma}x^*(p_0)}{\lambda-E'_{A\alpha}-\omega(p)}\ .\end{aligned} \tag{38}$$

The orthonomality of the scattering and bound states can be demonstrated in the usual way,[1] and the T matrix turns out to be

$$\begin{aligned}T^{no\perp}_{\alpha p,\beta q}&=(C^\dagger D^{-1}(\lambda+i0)C)_{\alpha\beta}x(p)x^*(q)\ ,\\ E'_{A\alpha}+\omega(p)&=E'_{A\beta}+\omega(q)=\lambda\ ,\end{aligned} \tag{39}$$

as would be expected from the form of $\zeta_\alpha(p)$ in Eq. (38).

In order to relate this to a phase shift, consider the case that just one channel g is open, so that $I_\alpha(\lambda)$ is real (for λ real) unless α is g. Then

$$D_{\alpha\beta}(\lambda)=B_{\alpha\beta}(\lambda)-C_{\alpha g}I_g(\lambda)C^\dagger_{g\beta}\ ,$$
$$B_{\alpha\beta}(\lambda)=(\lambda-E_\alpha)\delta_{\alpha\beta}-\sum_{\mu\neq g}C_{\alpha\mu}I_\mu(\lambda)C^\dagger_{\mu\beta}\ , \quad (40)$$

where $B(\lambda)$ is Hermitian. Now D^{-1} can be worked out,

$$D^{-1}{}_{\alpha\beta}=B^{-1}{}_{\alpha\beta}+(B^{-1}C)_{\alpha g}\frac{I_g}{1-I_g(C^\dagger B^{-1}C)_{gg}}(C^\dagger B^{-1})_{g\beta}\ , \quad (41)$$

and, therefore, in this case of one open channel

$$T^{no\perp}_{gp,gp}=\frac{|x(p)|^2}{1/(C^\dagger B^{-1}C)_{gg}-I_g(\lambda+i0)}\ ,$$
$$\lambda=E'_{Ag}+\omega(p)\ . \quad (42)$$

The discontinuity in $I_g(\lambda)$ is

$$I_g(\lambda+i0)-I_g(\lambda-i0)=-2\pi i\,|x(p)|^2\frac{dp}{d\omega}\ , \quad (43)$$

so that the phase shift in the open channel is

$$\delta(\lambda)=-\text{phase}\left[\frac{1}{(C^\dagger B^{-1}C)_{gg}}-I_g(\lambda+i0)\right]\ . \quad (44)$$

This is the many-channel generalization of the well-known one-channel formula

$$\delta(\lambda)=-\text{phase}[\lambda-E_{A1}-|C|^2I_g(\lambda)]$$
$$=-\text{phase}D(\lambda) \quad (45)$$

that describes the scattering when C is 1 by 1 and B is just $\lambda-E_{A1}$.

SOLUTION OF ORIGINAL SYSTEM

In order to solve Eqs. (29) for the scattering case, note that the solution of the second of these equations has p dependence of the form

$$\xi_{1\alpha}(p)=\delta_{\alpha\gamma}\delta(p-p_0)+x\varphi(p)+y\frac{\varphi(p)}{\lambda-E'_{A\alpha}-\omega(p)}\ , \quad (46)$$

where the constants x and y must be determined so that

$$(\varphi,\xi_{1\alpha})=0=\delta_{\alpha,\gamma}\varphi^*(p_0)+x+yJ_\alpha(\lambda) \quad (47)$$

and the second of Eqs. (29) is satisfied, namely, taking coefficients of φ and $\omega\varphi$, respectively,

$$(\lambda-E'_{A\alpha})x+y=-W(C^\dagger\theta)_\alpha-\delta_{\alpha\gamma}\omega(p_0)\varphi^*(p_0)-Wx+y-(\lambda-E'_{A\alpha})J_\alpha(\lambda)y\ , \quad (48)$$

$$-x=(C^\dagger\theta)_\alpha\ ,$$

where

$$J_\alpha(\lambda)=\int\frac{|\varphi(p)|^2}{\lambda-E'_{A\alpha}-\omega(p)}dp\ . \quad (49)$$

Since λ is $E'_{A\gamma}+\omega(p_0)\pm i0$, the solution of these and of the first of Eqs. (29) is straightforward. The matrix $\Delta(\lambda)$ that is the analog of $D(\lambda)$ of the comparison system is given by

$$\Delta(\lambda)=\lambda-E_A-CK(\lambda)C^\dagger\ , \quad (50)$$

where the diagonal matrix $K(\lambda)$ that replaces $I(\lambda)$ is given by

$$K(\lambda)=\lambda-E'_A-W-J^{-1}(\lambda) \quad (51)$$

with J given by Eq. (49). Note that only $\varphi(p)$ appears, since $x_1(p)$ is related to $\varphi(p)$ by Eq. (18). Now the bound-state energies are the solution of

$$\det\Delta(\lambda)=0\ . \quad (52)$$

The solutions to the scattering equations, analogous to Eqs. (38), are

$$\theta^{\gamma p_0}_\alpha=[\Delta^{-1}(\lambda)C]_{\alpha\gamma}\varphi^*(p_0)\ ,$$
$$\xi^{\gamma p_0}_{1\alpha}(p)=\delta_{\alpha,\gamma}\delta(p-p_0)+\frac{\varphi(p)[J^{-1}C^\dagger\Delta^{-1}CJ^{-1}-J^{-1}]_{\alpha\gamma}\varphi^*(p_0)}{\lambda-E'_{A\alpha}-\omega(p)}-\varphi(p)[C^\dagger\Delta^{-1}CJ^{-1}]_{\alpha\gamma}\varphi^*(p_0)\ . \quad (53)$$

Again, orthonormality can be demonstrated. The T matrix is

$$T_{\alpha p,\beta q}=[J^{-1}C^{\dagger}\Delta^{-1}CJ^{-1}-J^{-1}]_{\alpha\beta}\varphi(p)\varphi^*(q) ,$$
$$E'_{A\alpha}+\omega(p)=E'_{A\beta}+\omega(q)=\lambda , \quad (54)$$

where Δ and J are to be evaluated at $\lambda+i0$. For the case of a single open channel,

$$\delta(\lambda)=-\text{phase}[(C^{\dagger}B_{\Delta}^{-1}C)_{gg}+J_g(\lambda+i0)] ,$$
$$(B_{\Delta})_{\alpha\beta}=\Delta_{\alpha\beta}-C_{\alpha g}J^{-1}{}_{g}C^{\dagger}_{g\beta} . \quad (55)$$

The T matrices of Eqs. (39) and (54) both satisfy the unitarity constraint

$$T^{\dagger}T=TT^{\dagger}=\frac{i}{2\pi}(T-T^{\dagger}) . \quad (56)$$

SIMPLE CHECKS

The first check of the one-meson-approximation T matrix is to compare it to the T matrix for systems where the answer is known. Two such systems are (a) vanishing coupling and (b) scalar or invariant field $a(k)$, where H of Eq. (1) can be solved by the well-known canonical transformation

$$\bar{a}(k)=a(k)-\frac{v(k)}{\omega(k)} . \quad (57)$$

In the set of equations (30), the only way to approach zero coupling is to let $C\to0$. Then (42) shows that $T\to0$ as it should. This does not happen in Eq. (54). For the set described by Eqs. (25) and (29), note that $\chi_{\perp}(q)$ and $\varphi(q)$ do not depend H_A and $C_{\alpha\beta}$. In fact, as $G\to0$, H_A goes to $WA^{\dagger}\cdot A=H_A(0)$, with ground state $|\Omega\rangle$ satisfying $A=H_A(0)$, with ground state $|\Omega\rangle$ satisfying $A|\Omega\rangle=0$ and excited states of the form $(A^{\dagger})^n|\Omega\rangle$. The first scattering process couples $A^{\dagger}|\Omega\rangle$ and $a_{\perp}^{\dagger}(p)|\Omega\rangle$, and from the definition of C in Eqs. (25), it is clear that $C=1$. The matrix E_A in Eq. (46) has the single entry W, while E'_A in Eq. (47) has the single entry 0. Thus, $\Delta(\lambda)=J^{-1}(\lambda)$ in this case and Eq. (54) shows that T is zero, as it should be.

In the case that $a(k)$ is a scalar field, so that H_A is H_{A0}

$$H_{A0}=W(A^{\dagger}A-G(A^{\dagger}+A))$$
$$=W((A-G)^{\dagger}(A-G)-G^2) , \quad (58)$$

the coupled states are again $(A-G)^{\dagger}|\Omega\rangle$, where the ground state $|\Omega\rangle$ satisfies $(A-G)|\Omega\rangle=0$, and $a_{\perp}^{\dagger}(p)|\Omega\rangle$. Again C is one and T vanishes as above. In the lowest channel, the well-known result that there is no scattering by a static source of scalar mesons is recovered. In higher channels, for example, $(A^{\dagger}-G)^2|\Omega\rangle$ coupled to $a_{\perp}^{\dagger}(A^{\dagger}-G)|\Omega\rangle$, the one-meson approximation is no longer exact; it gives a nonvanishing T matrix.

SCATTERED AND UNSCATTERED PACKETS

For the case of Eqs. (30), with T matrix given by Eq. (39), it is not surprising that wave packets orthogonal to $x(q)$ are not scattered. That is, let the wave packet $|y\alpha\rangle$ be defined by

$$|y\alpha\rangle=\int y(q)|\alpha q\rangle dq ; \quad (59)$$

then the off-shell T matrix for going from $|y\alpha\rangle$ to another state is proportional to

$$(x,y)=\int x^*(q)y(q)dq , \quad (60)$$

and if $(x,y)=0$, the state $|y,\alpha\rangle$ is not scattered. This is a general feature of scattering processes involving separable potentials or their equivalent mesonic interactions. On the other hand, the form of Eqs. (25) or (29) seems to indicate scattering of $\perp$ wave packets, but the solution (54) for the T matrix shows that $\perp$ wave packets are not scattered.

THRESHOLD BEHAVIOR

Now let the ground state of H_A belong to the representation R_g of the symmetry group of H, and let the various eigenstates of H_A belonging to R_g be $|0R_g\rangle$, $|1R_g\rangle$, $|2R_g\rangle$, etc. where $|0R_g\rangle$ is the ground state of H_A. Similarly let the representation of the first excited state of H_A not belonging to R_g be R_1, with states $|0R_1\rangle$, $|1R_1\rangle$, etc. For example, for pions and a nucleon R_g is ($T=\frac{1}{2}$, $S=\frac{1}{2}$, even parity) or "N" and R_1 is ($T=\frac{3}{2}$, $S=\frac{3}{2}$, even parity) or "Δ".

A very simple picture for the system involves neglecting all but the two eigenstates $|0R_g\rangle\equiv|g\rangle$ and $|0R_1\rangle\equiv|h\rangle$ of H_A. Consider first the scattering in states belonging to R_1. In the one-meson subspace $\mathscr{M}_1$, the coupled states are

$$|h\rangle ,$$
$$|gp\rangle=\{a_{\perp}^{\dagger}(p),|g\rangle\}^{R_1} , \quad (61)$$
$$|hp\rangle=\{a_{\perp}^{\dagger}(p),|h\rangle\}^{R_1} .$$

From Eqs. (25) and (13) with $\lambda=0$ it is seen that

$$\langle h \mid H \mid hp \rangle = 0 , \tag{62}$$

so that in this two-state approximation, the set of states (61) breaks up into one unscattered state $\mid hp \rangle$ and the coupled states $\mid h \rangle$ and $\mid gp \rangle$. The states $\mid gp \rangle$ have their threshold at $E_g + m$, so that the phase shift in the R_1 channel starts to be nonzero at $E_g + m$. The separation of $\mid hp \rangle$ simplifies the problem but does not otherwise alter its characteristics.

In the R_g states a similar splitting occurs. The state $\mid gp \rangle$ is unscattered and uncoupled from the states $\mid g \rangle$ and $\mid hp \rangle$. However, now the $\mid hp \rangle$ threshold is at $E_1 + m$, and the phase shift starts to be nonzero at $E_1 + m$. The physical threshold for the phase shift is at $E_g + m$, which is well below $E_1 + m$. The result is that in this first approximation, the phase shift remains zero above the physical threshold until the energy $E_1 + m$ of the second threshold is reached; at this latter energy the phase shift exhibits normal threshold behavior.[10]

Of course, when higher-energy channels are coupled, the previously uncoupled scattering states are coupled to the system. However, the preceding arguments show that this coupling goes through extra vertex factors $C_{\alpha\beta}$ and higher-energy virtual states; therefore, the R_g phase shift would be expected to be small in the interval between $E_g + m$ and $E_1 + m$.

The ground channel of the nucleon-pion system is the N channel and the corresponding phase shift is the P_{11} phase shift δ_g. The phase shift δ_g is in fact quite small until an energy above the energy $E_1 + m$, where E_1 is the energy of the resonance in the Δ channel. The above simple picture gives a qualitative explanation of this behavior.

Δ AS SOURCE IN H

In treatments based on the quark model[2] the Hamiltonian is different from that of Eq. (1); there are additional Δ terms, since the Δ is regarded as an excited state of the quark-gluon system. In the one-baryon sector H takes the form

$$H = \int \omega(k) a^\dagger(k) \cdot a(k) dk - \rho \cdot \int [v^*(k) a(k) + v(k) a^\dagger(k)] dk + \epsilon , \tag{63}$$

where ϵ is a diagonal 20×20 matrix that has zeros in the N entries and $E_\Delta - E_N$ in the Δ entries, and ρ is a 20×20 matrix whose leading 4×4 part is the ρ of Eq. (1) while the remainder describes $N\Delta\pi$ and $\Delta\Delta\pi$ vertices. Since the form of H of Eq. (63) is essentially the same as the form of H of Eq. (1), all of the preceding discussion applies to it as well. The only change is that H_A picks up the term ϵ. In particular, the effective elevation of the threshold in the N channel is unchanged.

GENERAL REMARKS

The T matrices of Eqs. (39) and (54), with corresponding elastic phase shifts of Eqs. (44) and (55), differ significantly. The explicit forms given may be useful in determining the existence of characteristic features of the scattering that can distinguish the presence of internal meson modes.

The important question of the range of validity of the one-meson approximation has not been discussed; I have nothing to contribute in this area beyond rather ordinary speculations.

SUMMARY

For general static models, the Hamiltonian has been shown to take the form given by Eqs. (15)–(18), when the preferred form for the internal mode is used. In the one-meson sector the explicit solution for the T matrix is given by Eq. (54). In a simple model that uses just two eigenstates of H_A, the threshold for scattering in the ground-state channel is raised above the physical threshold for scattering in this channel.

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