Scaling behavior of interacting quantum fields in curved spacetime

Bruce L. Nelson and Prakash Panangaden
Department of Physics, University of Utah, Salt Lake City, Utah 84112
(Received 13 October 1981)

We propose a version of the 't Hooft-Weinberg renormalization-group equation which is valid in general curved spacetimes. The key idea is to regard the Green's functions as functions of the metric and to scale the metric rather than the coordinates or momenta. We apply these equations to the infrared limit of the effective conformal coupling in $\lambda\phi^4$ theory.

I. INTRODUCTION

Recently, there has been much interest in the renormalization properties of interacting quantum field theories in curved spacetimes.[1-10] Much of this work has focused on developing techniques for extracting the singular parts of Feynman integrals in general curved spacetimes[4,7] and applying them to examine the renormalizability of field-theoretic models such as ϕ^3 field theory,[3] ϕ^4 field theory,[4-10] and quantum electrodynamics.[8]

In this paper we set up a formalism for studying the asymptotic properties of coupling constants by means of renormalization-group (RG) equations which are valid in curved spacetime. Work along these lines has already been undertaken by various authors.[11,12] However, the earlier work has been concerned with the gravitational action[11] on conformally flat spacetimes.[12] The RG equation we use is based on the idea of regarding the Green's functions as functions of the metric and scaling the metric instead of scaling the coordinates or momenta.

The methods we use closely correspond to the approach of 't Hooft,[13] Weinberg,[14] and Macfarlane and Collins.[15] Thus the RG equations we derive involve an auxiliary scaling variable and are homogeneous unlike the Callan-Symanzik equation.

We apply the RG equation to examine the long-distance (low-energy) behavior of the effective coupling constant governing the coupling of the quantum field to the scalar curvature. Earlier work on the renormalizability of ϕ^4 field theory in curved spacetime[4-7] showed that such a coupling must necessarily arise through the process of renormalization. This is a feature not seen in flat spacetime so it is interesting to examine the asymptotic form of this coupling constant. Curiously, we find that it tends to the value $\frac{1}{6}$ which (for a massless field) would correspond to a conformally invariant field theory. This result has been obtained earlier by Gass[12] for conformally flat spacetimes. Our result shows this to be valid in a general curved spacetimes as well.

We expect the main applications of our work to be to non-Abelian asymptotically free gauge theories in curved spacetimes. Recently, these theories have become popular as possible models to explain the cosmological baryon asymmetry. Our methods can be applied to these theories to establish that asymptotic freedom is not vitiated by high curvature. Earlier work on QED[8] has shown that the renormalization constants are the same as in flat spacetime and that gauge invariance prohibits couplings of the fields to the curvature. Thus the asymptotic behavior of the effective electric charge is the same as in flat spacetime. For non-Abelian gauge theories, however, the question remains open.

II. RENORMALIZATION-GROUP EQUATION IN CURVED SPACETIME

In this section we will derive a renormalization-group equation valid in a general curved spacetime and discuss the solutions of this equation. For simplicity we consider a scalar field theory with a quartic self-interaction together with a coupling to the scalar curvature.

The renormalization-group equation has been discussed in the flat-space limit by Fujikawa[16] and in conformally flat spacetimes by Gass.[12] In the former case one obtains renormalization-group equations for the one-particle-irreducible (1PI) Green's functions expressed in terms of the position variables. In the later case one uses a conformal transform to flat spacetime. One can then Fourier transform the Green's functions and obtain an RG equation expressed in terms of the scaling of the "momentum" variables.

We shall begin by considering the position-space version of the 't Hooft-Weinberg RG equation (rather than the Callan-Symanzik RG equation). The derivation closely follows the discussion of Collins and Macfarlane.[15] The Lagrangian for the theory we consider is

$$L = \tfrac{1}{2}\sqrt{g}\,[(\partial_\mu\phi_B)(\partial_\nu\phi_B)g^{\mu\nu} + (m_B{}^2+\xi_B R)\phi_B{}^2+\tfrac{1}{12}\lambda_B\phi_B{}^4] \quad (2.1)$$

where $g_{\mu\nu}$ is the spacetime metric tensor and ϕ_B is the bare (unrenormalized) field and m_B, ξ_B, and λ_B stand for the bare mass, the bare conformal coupling constant, and the bare self-interaction coupling constant. The scalar curvature of the spacetime is denoted by R. The renormalized fields and parameters will be denoted by unsubscripted variables.

We use dimensional regularization to render infinite quantities formally finite. The spacetime dimension is n and the singularities of Green's functions will be displayed as poles at $n=4$. The renormalized Green's functions are obtained by making the parameters m_B and λ_B depend on n and by multiplying the Green's function by a suitable power of the wave-function renormalization constant. To keep λ_B dimensionless in n dimensions we must introduce a mass scale parameter μ.

The bare parameters are related to the renormalized parameters by[13]

$$m_B(n)=m+\sum_{\nu=1}\frac{b_\nu(\lambda,m,\mu)}{(n-4)^\nu}=Z_m m\ , \quad (2.2)$$

$$\lambda_B(n)=\mu^{4-n}\left[\lambda+\sum_{\nu=1}\frac{a_\nu(\lambda,m,\mu)}{(n-4)^\nu}\right]=Z_\lambda\lambda\ , \quad (2.3)$$

$$\xi_B(n)=\xi+\sum_{\nu=1}\frac{d_\nu(\lambda,m,\mu)}{(n-4)^\nu}=Z_\xi\xi\ . \quad (2.4)$$

The renormalized field ϕ is related to ϕ_B by

$$\phi_B=Z_1{}^{1/2}\phi\ , \quad (2.5)$$

where

$$Z_1=1+\sum_{\nu=1}\frac{c_\nu(\lambda,m,\mu)}{(1-4)^\nu}\ . \quad (2.6)$$

The poles in Eqs. (2.2), (2.3), (2.4), and (2.6) are defined to be precisely those needed to subtract the poles in the corresponding Feynman integrals.

In the usual treatment of the RG equation[15] we are concerned with the scaling behavior of a renormalized, amputated, and connected l-point Green's function $\Gamma^{(l)}(p_1,\ldots,p_l;\lambda,m,\mu)$ as the external momenta are scaled. In position space[16] we are interested in the scaling behavior of the corresponding l-point Green's function $\Gamma^{(l)}(x_1,\ldots,x^l;\lambda,m,\mu)$ as the external points $x_1,\ldots,x^l$ are brought close together. This can be done by considering the Green's function for the scaled arguments $\Gamma^{(l)}(x_1/\kappa,\ldots,x_l/\kappa;\lambda,m,\mu)$. This is meaningful because Minkowski space is a vector space and expressions such as $(1/\kappa)x_1$ are legitimate. In a general curve spacetime this cannot be done.

Let $\Gamma_u^{(l)}(x_1,\ldots,x^l;\lambda_B,m_B,\mu)$ denote the unrenormalized, n-dimensional, l-point Green's function calculated using Feynman perturbation theory with the bare Lagrangian. Then the n-dimensional renormalized Green's function is

$$\tilde\Gamma^{(l)}(x_1,\ldots,x_l;\lambda(n),m(n),\mu,n)=Z^{l/2}(\lambda_B\mu^{n-4},n)\Gamma_u^{(l)}(x_1,\ldots,x^l;\lambda_B(n),m_B(n),n)\ , \quad (2.7)$$

where

$$\lambda(n)=\lambda(\lambda_B(n)\mu^{n-4}) \quad (2.8a)$$

and

$$m(n)=m_B(n)Z_m{}^{-1}(\lambda_B(n)\mu^{n-4})\ . \quad (2.8b)$$

The physical renormalized Green's function at $n=4$ is

$$\Gamma^{(l)}(x_1,\ldots,x_l;\lambda,m,\mu)=\lim_{n\to4}\tilde\Gamma^{(l)}\ . \quad (2.9)$$

We now differentiate (2.7) with respect to μ and multiply by μ to obtain

$$\left[\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda}+Z_m m_\mu\frac{\partial Z_m{}^{-1}}{\partial\mu}\frac{\partial}{\partial m}\right]\Gamma^{(l)}$$
$$=\mu\left[\frac{l}{2}\right]Z^{l/2-1}\frac{\partial Z}{\partial\mu}\Gamma_u^{(l)}$$
$$=\mu\left[\frac{l}{2}\right]Z^{l/2}\frac{\partial \ln Z}{\partial\mu}\Gamma_u^{(l)}$$
$$=\mu\left[\frac{l}{2}\right]\frac{\partial \ln Z}{\partial\mu}\tilde{\Gamma}^{(l)} . \quad (2.10)$$

We now define the following quantities:

$$\beta(\lambda)\equiv \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}\ln\lambda(\lambda_B(n)\mu^{n-4},n) , \quad (2.11a)$$

$$\gamma_m(\lambda)\equiv \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}\ln Z_m(\lambda_B(n)\mu^{n-4},n) , \quad (2.11b)$$

$$\gamma(\lambda)\equiv \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}\ln Z(\lambda_B(n)\mu^{n-4},n) . \quad (2.11c)$$

Taking $n\to 4$ in Eq. (2.10) and using (2.11) we have

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}-\gamma_m m\frac{\partial}{\partial m}-\frac{l}{2}\gamma\right]\Gamma^{(l)}=0 . \quad (2.12)$$

To obtain an equation that gives the short-distance behavior of these Green's functions we must use a naive scaling equation to eliminate $\partial/\partial\mu$ in favor of derivatives with respect to x.

The scaling equation can be obtained by the following simple considerations. Let the physical dimensions of $\Gamma^{(l)}$ be mass D_l. Then since x has the dimension mass^{-1} we have

$$\Gamma^{(l)}\left[\frac{x_1}{\alpha},\ldots,\frac{x_l}{\alpha};\lambda,\alpha m,\alpha\mu\right]=\alpha^{D_l}\Gamma^{(l)}(x_1,\ldots,x_l;\lambda,m,\mu) , \quad (2.13)$$

where α is a scaling variable. We now differentiate (2.13) with respect to α and set $\alpha=1$ to get

$$\left[-\sum_{i=1}^{l}x_{(i)}^{\mu}\partial_{(i)\mu}+m\frac{\partial}{\partial m}-D_l\right]\Gamma^{(l)}=-\mu\frac{\partial\Gamma^{(l)}}{\partial\mu} . \quad (2.14)$$

From Eqs. (2.12) and (2.14) we obtain

$$\left[\sum_{i=1}^{l}x_{(i)}^{\mu}\partial_{(i)\mu}-(1+\gamma_m)m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial\lambda}+D_l-\frac{l}{2}\gamma\right]\Gamma^{(l)}=0 . \quad (2.15)$$

This is the position-space RG equation valid in flat spacetimes. The $x^\mu\partial_\mu$ term is the generator of dilations and thus (2.15) shows the behavior of Green's functions under changes of the length scale. This form cannot be used in curved spacetime because we cannot give meaning to an expression such as $x^\mu\partial_\mu$.

In conformally flat spacetimes we can use a conformal transformation to flat spacetime and derive RG equations (in either position or momentum space) for the conformally transformed Green's functions. In a recent paper Gass[12] indicated how this can be done. For a general curved spacetime, however, the natural way to proceed is to examine the change of the Green's functions as the metric is scaled. Intuitively, this can be thought of as follows. One can imagine that the position arguments of a Green's function to all lie in a Riemann normal coordinate neighborhood. One wishes to investigate the behavior of the Green's function as the points are all brought close together. This can be achieved by moving the points along the geodesics connecting them to the origin of the coordinate system or alternatively by scaling the geodesic distance function, i.e., the metric. For the purposes of this paper we will examine metric rescalings that are spacetime constants. A later paper will examine the effect of rescalings that are spacetime-dependent functions.[17]

The derivation of the renormalization-group equation (2.12) can be carried through almost unchanged except that now we have nonzero curvature so the $\xi R\phi^2$ term in the Lagrangian cannot be ignored. Therefore, we have a Green's function that is also a function of the bare conformal coupling constant $\xi_B(n)$. Thus Eq. (2.12) is modified to read

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}-\gamma_m m\frac{\partial}{\partial m}-\gamma_\xi\xi\frac{\partial}{\partial\xi}-\frac{l}{2}\gamma\right]\Gamma^{(l)}=0 , \quad (2.16)$$

where

$$\gamma_\xi\equiv \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}\ln Z_\xi . \quad (2.17)$$

The scaling equation, however, must be treated differently. We fix the fiducial background metric to be $g_{\alpha\beta}$. We shall then consider a one-parameter family of metrics $\kappa^{-1}g_{\alpha\beta}$ where κ is a (spacetime-constant) scaling variable. The Green's function is now a function of κ as well as of λ, ξ, and m and is written $\Gamma^{(l)}(x_1,\ldots,x_l;\lambda,\xi,m,\mu,\kappa)$. The quantity κ has effectively the dimensions of mass so if $\Gamma^{(l)}$ has dimensions mass D_l we have

$$\Gamma^{(l)}(x_1,\ldots,x_l,\lambda,\xi,\alpha m,\alpha\mu,\alpha\kappa) = \alpha^{D_l}\Gamma^{(l)}(x_1,\ldots,x_l;\lambda,\xi,m,\mu,\kappa)\ . \quad (2.18)$$

The differential version of this scaling formula is

$$\left[\mu\frac{\partial}{\partial\mu}+m\frac{\partial}{\partial m}+\kappa\frac{\partial}{\partial\kappa}-D_l\right]\Gamma^{(l)}=0\ . \quad (2.19)$$

This gives rise to the familiar RG equation

$$\left[-\kappa\frac{\partial}{\partial\kappa}-m(1+\gamma_m)\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial\lambda} -\xi\gamma_\xi\frac{\partial}{\partial\xi}+D_l-\frac{l}{2}\gamma\right]\Gamma^{(l)}=0\ , \quad (2.20)$$

where κ is now interpreted as the metric scaling variable rather than as the momentum scaling variable.

Under changes in κ the contravariant metric $g^{\alpha\beta}$ scales as κ^2 whereas the curvature tensors $R^{\alpha}{}_{\beta\gamma\delta}$ and $R_{\alpha\beta}$ do not scale. The scalar curvature $R\equiv R_{\alpha\beta}g^{\alpha\beta}$ will, however, scale as κ^2. Thus as $\kappa\to\infty$, the analog of the short-distance or high-momentum limit, we have $R\to\infty$, thus we are also looking at the high-curvature limit.

This is an important point since there is much interest in examining the mechanisms for baryon generation in grand unified theories in the early universe.[18] These calculations are made on the assumption that the theories one uses asymptotically free at high energies, and hence, one can obtain reliable estimates from perturbative calculations. There is, however, the possibility that the high curvature of the early universe may result in effects that vitiate asymptotic freedom. The existence of an RG equation such as (2.20) strongly suggest that metric scaling together with the associated curvature scaling lead to RG equations similar to the original RG equations, and hence, high curvature would not necessarily spoil asymptotic freedom.

The solution of Eq. (2.20) is virtually identical to the solution given by Weinberg[14] viz.

$$\Gamma^{(l)}\left[x_1,\ldots,x_l;\lambda,m,\xi,\mu,\frac{g_{\alpha\beta}}{\kappa^2}\right] = \kappa^{D_l}\Gamma^{(l)}(x_1,\ldots,x_l;\lambda(\kappa),m(\kappa),\xi(\kappa),\mu,g_{\alpha\beta}) \times\exp\left[-\frac{l}{2}\int_1^{\kappa}\gamma(\lambda(\kappa'))\frac{d\kappa'}{\kappa'}\right] \quad (2.21)$$

with

$$\kappa(\partial\lambda/\partial\kappa)=\beta(\lambda(\kappa)),\ \ \lambda(1)=\lambda_R \quad (2.22a)$$

$$\kappa(\partial m/\partial\kappa)=-[1+\gamma_m]m(\kappa),\ \ m(1)=m_R \quad (2.22b)$$

$$\kappa(\partial\xi/\partial\kappa)=-\gamma_\xi\xi(\kappa),\ \ \xi(1)=\xi_R\ . \quad (2.22c)$$

To calculate the scaling behavior of the parameters λ, ξ, and m we need to know the functions β, γ_m, and γ_ξ. These can be obtained in terms of the renormalization constants a_ν, b_ν, and d_ν, respectively, defined in Eqs. (2.2)–(2.4). The constants a_ν, b_ν, and d_ν are defined to be precisely those needed to cancel the poles appearing in Feynman diagrams. The mass scale μ occurs only in the form $\mu^{4-n}\lambda$ so using the expansion

$$\mu^{4-n}=1+(4-n)\ln\mu+O((4-n)^2) \quad (2.23)$$

we see that the Feynman integrals will contain powers of $\ln\mu$. On the other hand, m_R and ξ_R only occur as polynomials. Thus the pole terms will contain only positive powers of m_R and ξ_R for all n. However, we know that the a_ν and d_ν are dimensionless and b_ν, m_R, and μ have mass dimension 1. Thus a_ν, b_ν, and d_ν do not depend on μ, and a_ν and d_ν do not depend on μ, and a_ν and d_ν do not depend on m_R.

The arguments of the preceding paragraph are well known.[15] The only new feature is that we have applied them to the d_ν coefficients. The fact that ξ_R appears in the Lagrangian in almost the same fashion as does m_R allows us to assert that the pole terms only contain polynomials in ξ_R. The fact that the d_ν are dimensionless in n dimensions follows from the fact that R (the scalar curvature) has mass dimension 2 for all n, and hence ξ is dimensionless for all n; we do not need any term such as $\xi\mu^{4-n}$ in our Lagrangian.

We are now in a position to express β, γ_m, and γ_ξ in terms of the a_ν, b_ν, and d_ν. For β and γ_m the equations are derived by Collins and Macfarlane[15] and we merely quote the results:

$$\beta(\lambda_R)=\left[1-\lambda_R\frac{d}{d\lambda_R}\right]a_1 \quad (2.24)$$

and

$$\gamma_m = \frac{\lambda_R}{m_R} \frac{d(b_1)}{d\lambda_R} . \tag{2.25}$$

For γ_ξ we proceed as follows. From Eq. (2.17) we have

$$\gamma_\xi = \lim_{n\to 4} \mu \frac{\partial}{\partial \mu} \ln Z_\xi . \tag{2.17}$$

Since $\xi(n)$ does not depend on μ we have

$$\mu \frac{\partial Z_\xi}{\partial \mu} = \left[\mu \frac{\partial \lambda_R}{\partial \mu} \right] \sum_{v=1}^{\infty} \frac{d'_v}{(n-4)\xi_R} , \tag{2.26}$$

where $d'_v \equiv (d/d\lambda_R)(d_v)$. It is well known that $\mu(\partial\lambda_R/\partial\mu)$ can be expressed as

$$\mu \frac{\partial \lambda_R}{\partial \mu} = \beta(\lambda_R) + \lambda_R(n-4) . \tag{2.27}$$

Since $\mu\partial \ln Z_\xi/\partial\mu$ appears in the RG equation for a renormalized Green's function it must be an analytic function of n at $n=4$; hence, we can express it as a power series

$$\mu \frac{\partial \ln Z_\xi}{\partial \mu} = Z_0 + Z_1(n-4) + \dots , \tag{2.28}$$

where the coefficients Z_0, Z_1 are unknown. Using Eqs. (2.28) and (2.27) in Eq. (2.26), we have

$$\begin{aligned} Z_0 + Z_1(n-4) + \dots + \frac{Z_1 d_1}{\xi_R} + \dots \\ = Z_\xi \mu \frac{\partial \ln Z_\xi}{\partial \mu} \\ = \beta \sum_{v=1}^{\infty} \frac{d'_v}{(n-4)\xi_R} + \lambda_R \frac{d_1}{\xi_R} + \dots . \end{aligned} \tag{2.29}$$

Since the right-hand side does not contain any positive power of $(n-4)$ we conclude that only Z_0 in Eq. (2.28) is nonzero. Equating the constant terms on each side we obtain

$$Z_0 = \frac{\lambda_R d_1}{\xi_R}$$

and thus

$$\gamma_m = \frac{\lambda_R}{\xi_R} \frac{d}{d\lambda_R}(d_1) . \tag{2.30}$$

The quantities a_v, b_v, and d_v have been calculated to second order in λ_R for $\lambda\phi^4$ field theory in a general curved spacetime by Bunch and Panangaden[6,7] and by Birrell.[4] Thus we can use their results to obtain β, γ_m, and γ_ξ in a general curved spacetime.

In the next section we will use these formulas to obtain the infrared behavior of λ, ξ, and m in a general curved spacetime.

III. SCALING BEHAVIOR FOR THE CONFORMAL COUPLING CONSTANT IN $\lambda\phi^4$ THEORY

In this section we discuss the long-distance behavior of the effective coupling constant $\lambda(\kappa)$, effective mass $m(\kappa)$, and effective coupling to the curvature $\xi(\kappa)$. The coefficients of the renormalization-group equations are given by the coefficients of the $(n-4)^{-1}$ singularities of the renormalization constants Z_λ, Z_m, and Z_ξ. Because of the structure of the β function, if λ_R is sufficiently small, $\lambda(\kappa)$ will go to zero as κ goes to zero (the theory is infrared free). Studying the small-κ behavior of $m(\kappa)$ and $\xi(\kappa)$ will thus be equivalent to studying their small-λ behavior. This fact lends some credence to the study of the renormalization-group equations in terms of perturbation theory.

As in the flat-space case γ_m depends only on λ (not on m or ξ) so the equation for $m(\kappa)$ can be easily integrated. We find that $m(\kappa)\to\infty$ as $\kappa\to 0$ [unless $m_R=0$, in which case $m(\kappa)$ is zero for all κ]. The equation for $\xi(\kappa)$ is much more interesting, however, since γ_m depends on both λ and ξ. Despite the apparent singular nature of the equation for $\xi(\kappa)$, we are able to extract the small-κ dependence and show that as $\kappa\to 0$, $\xi(\kappa)$ approaches $\frac{1}{6}$ regardless of ξ_R [unless $\xi_R=0$, in which case $\xi(\kappa)=0$ for all κ]. We are thus able to show that if $m_R=0$, the $\kappa\to 0$ limit of the Green's function is a scaling factor times the appropriate Green's function of the conformally invariant theory. This result holds regardless of what value ξ_R takes (as long as it is not zero).

The scaling of λ with κ is given by the equation

$$\kappa \frac{d\lambda}{d\kappa} = \beta(\lambda) \tag{2.22a}$$

with $\lambda(\kappa=1)=\lambda_R$. The behavior of the solution of this equation is a little easier to see if we introduce another scaling parameter[19] t defined by

$$\kappa = e^t .$$

Then $\kappa d/d\kappa = d/dt$, and $\kappa = 1$ corresponds to $t = 0$, and $\kappa \to 0$ corresponds to $t \to \infty$. In this variable

$$\frac{d\lambda}{dt} = \beta(\lambda) \tag{3.1}$$

and $\lambda(t=0) = \lambda_R$. We have seen that β is given by the expression

$$\beta(\lambda) = \left[1 - \lambda \frac{d}{d\lambda}\right] a_1(\lambda) , \tag{2.24}$$

where a_1 is the coefficient of the $(n-4)^{-1}$ singularity in Z_λ. a_1 is computed in perturbation theory[4,6,7] to be $-3\lambda^2/16\pi^2$. Thus, to second order in λ,

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} . \tag{3.2}$$

As is expected, there are no λ^0 or λ^1 terms in β. Let us now assume that λ_R lies between zero and the first positive zero of the β function, if one exists. Then the first (i.e., quadratic) term in the β function determines the $\kappa \to 0$ behavior of λ. Thus consider integrating equation (2.22a) from $\kappa = 1$ (or $t = 0$) to $\kappa = 0$ (or $t = -\infty$). By assumption $\beta(\lambda) > 0$ in this range so that

$$d\lambda = \beta(\lambda) dt < 0 .$$

Thus, as $t \to -\infty$, λ moves from λ_R down to the IR fixed point $\lambda = 0$. Thus, as $\kappa \to 0$, the Green's functions look like those of the free field theory.

Equation (2.22a) can be written

$$\frac{d\lambda}{dt} = \beta = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) . \tag{3.3}$$

This equation is easily integrated to give

$$\lambda(t) = \frac{16\pi^2 \lambda_R}{1 - 3\lambda_R t} \tag{3.4}$$

or

$$\lambda(\kappa) = \frac{16\pi^2 \lambda_R}{1 - 3\lambda_R \ln\kappa} . \tag{3.4$'$}$$

This gives the explicit κ dependence of λ for small λ.

The renormalization-group equation for the effective mass is

$$\kappa \frac{\partial m(\kappa)}{\partial \kappa} = -[1 + \gamma_m] m(\kappa) \tag{2.22b}$$

with $m(\kappa = 1) = m_R$. Since small λ corresponds to small κ, it is the small-κ limit of $m(\kappa)$ that we can hope to extract with confidence from the first few terms in the perturbation expansion of γ_m. We have found that

$$\gamma_m = \frac{\lambda_R}{m_R} \frac{\partial}{\partial \lambda} b_1 , \tag{2.25}$$

where now b_1 is the coefficient of the $(n-4)^{-1}$ singularity in the mass renormalization constant Z_m. Again b_1 has been computed to second order in λ to be[4,6,7]

$$b_1 = -\frac{\lambda}{16\pi^2} + \frac{5\lambda^2}{12(16\pi^2)^2} + O(\lambda^3) , \tag{3.5}$$

hence,

$$\gamma_m = -\frac{\lambda}{16\pi^2} + \frac{5\lambda^2}{6(16\pi^2)^2} + O(\lambda^3) . \tag{3.6}$$

The equation for m is then

$$\begin{aligned} \kappa \frac{d}{d\kappa} m(\kappa) &= -[1 + \gamma_m] m(\kappa) \\ &= -\left[1 - \frac{\lambda}{16\pi^2} + \frac{5\lambda^2}{6(16\pi^2)^2} + O(\lambda^3)\right] m . \end{aligned} \tag{3.7}$$

We can use the renormalization-group equation for λ to write

$$\kappa \frac{d}{d\kappa} = \beta(\lambda) \frac{d}{d\lambda} \tag{3.8}$$

or

$$\frac{dm(\lambda)}{d\lambda} = \frac{-1}{\beta(\lambda)} \left[1 - \frac{\lambda}{(16\pi^2)} + \frac{5\lambda^2}{6(16\pi^2)^2} + O(\lambda^3)\right] m(\lambda) . \tag{3.9}$$

The β function is itself a power series in λ, beginning with $3\lambda^2/16\pi^2$. Thus

$$\frac{dm}{d\lambda} = \left[-\frac{16\pi^2}{3\lambda^2} - \frac{A}{3\lambda} + \frac{B}{18(16\pi^2)} + O(\lambda)\right] m \tag{3.10}$$

(where A and B are constants, the values of which depend on the coefficients of the λ^3 and λ^4 terms of the B function) with $m(\lambda_R) = m_R$. Now clearly if $m_R = 0$, then $m(\kappa) = 0$ for all κ. If, however, $m_R \neq 0$, the equation for $m(\kappa)$ is easily integrated to give

$$\ln\left[\frac{m}{m_R}\right] = \frac{16\pi^2}{3}\left[\frac{1}{\lambda} - \frac{1}{\lambda_R}\right] + \frac{A}{3} \ln\left[\frac{\lambda}{\lambda_R}\right] + O(\lambda) \tag{3.11}$$

As $\kappa \to 0$, $\lambda \to 0$ and $16\pi^2/3\lambda$ dominates the expression for $\ln(m/m_R)$. In the limit $\kappa \to 0$, $m \to \infty$. Thus, in the long-distance limit, not only does the effective coupling constant vanish, the effective mass becomes infinite. It is the 1 term in the RG equation (2.22b) for m that dominates the $\kappa \to 0$ behavior of m. This term comes from the naive scaling equation (2.19) for Γ and is present even in a free field theory on a flat spacetime. The meaning of the $m \to \infty$ result is then clear: As the scale of the momentum (κ) is taken to zero, the mass scales like κ^{-1} so that the two point function

$$G(p^2) = \frac{1}{p^2 - m^2}$$

stays on the mass shell. In the case of the $\lambda\phi^4$ self-interacting field, we find that the anomalous scaling of m (given by γ_m) is not sufficiently strong to overcome the naive scaling of m as $\kappa \to 0$. The same result is true in QED.

Finally, we come to the effective coupling to the curvature, $\xi(\kappa)$. Again it is the small-κ limit that is accessible to us using perturbation theory. The renormalization-group equation for $\xi(\kappa)$,

$$\kappa \frac{d\xi(\kappa)}{d\kappa} = -\gamma_\xi \xi , \tag{2.22c}$$

is more interesting than the ones for λ and m since now γ_m depends on ξ. As with β and γ_m, γ_ξ can be written in terms of the coefficient d_1 of the $(n-4)^{-1}$ singularity in Z_ξ. d_1 can be computed using perturbation theory[4,6,7] and the first two terms are

$$d_1 = -\frac{(\xi_R - \frac{1}{6})\lambda_R \xi_R}{16\pi^2} + \frac{5(\xi_R - \frac{1}{6})\lambda_R{}^2}{6(256\pi^4)} - \frac{\lambda_R{}^2 \xi_R}{36(256\pi^4)} .$$

In terms of d_1, γ_m is

$$\gamma_m = \frac{\lambda}{\xi} \frac{d(d_1)}{d\lambda} . \tag{2.30}$$

Thus γ_m is

$$\gamma_m = -\frac{(\xi - \frac{1}{6})\lambda}{16\pi^2} + \frac{5(\xi - \frac{7}{30})\lambda^2}{6(16\pi^2)^2} + O(\lambda^3) . \tag{3.12}$$

(Although the first two terms are linear in ξ, it is not known what ξ dependence the higher-order terms have.) In terms of λ, the renormalization-group equation for ξ is

$$\frac{d\xi(\lambda)}{d\lambda} = -\frac{\xi}{\beta(\lambda)} \gamma_\xi(\xi,\lambda) . \tag{3.13}$$

Both γ_ξ and β are power series in λ, γ_ξ beginning with λ and β with λ^2. Thus we can expand $1/\beta$ and combine terms in like powers of λ to get

$$\frac{d\xi}{d\lambda} = \frac{\xi}{3\lambda}[(\xi - \tfrac{1}{6}) + A_1(\xi)\lambda + A_2(\xi)\lambda^2 + \dots] . \tag{3.14}$$

A_1 will depend on the coefficient of the λ^3 terms in β, but will nonetheless have the structure

$$A_1(\xi) = A_{10} + A_{11}\xi , \tag{3.15}$$

where A_{10} and A_{11} are constants. The other A terms will likewise be power series in ξ. It is clear from the $(\xi/3\lambda)(\xi - \frac{1}{6})$ terms that as $\lambda \to 0$ ξ must go to either 0 or $\frac{1}{6}$ in order to be well defined at $\lambda = 0$. In fact, this first term is easily integrated to give

$$\xi(\lambda) = \sim \frac{1}{6}\left[1 - \left[\frac{\xi_R - \frac{1}{6}}{\xi_R}\right]\left[\frac{\lambda}{\lambda_R}\right]^{1/18}\right]^{-1}$$
$$= \frac{1}{6}\left[1 + \left[\frac{\xi_R - \frac{1}{6}}{\xi_R}\right]\left[\frac{\lambda}{\lambda_R}\right]^{1/18} + O(\lambda^{1/9}) + \dots\right] . \tag{3.16}$$

Thus to this order, ξ does indeed go to $\frac{1}{6}$ as κ (and hence λ) goes to zero. Further, ξ can be written as a power series in $\lambda^{1/18}$. Because of the structure of the A_1 term in (3.14), we can again integrate the differential equation for ξ with this additional $A_1\lambda$ term. Again, we find that regardless of λ_R, λ has the form of $\frac{1}{6}$ plus a power series in $\lambda^{1/18}$.

We point out that the factor $\frac{1}{6}$ in the above expression for ξ comes from the $\frac{1}{6}$ that appears in the $(\xi - \frac{1}{6})\lambda/16\pi^2$ term of γ_m. As noted earlier, γ_m is calculated from the d_1 coefficient for Z_ξ and arose from an expression for the scalar propagator in Riemann normal coordinates.[7] As such, the factor $\frac{1}{6}$ is the same, regardless of the physical spacetime dimension. [Thus, although the $\kappa = 0$ limits of λ and ξ might suggest a conformally invariant theory (for $m = 0$) in the IR limit, the fact that we get $\xi(\kappa = 0) = \frac{1}{6}$ for any spacetime dimension n instead of the general conformally invariant coupling constant $\xi = \frac{1}{4}(n-2)/(n-1)$ means this is true only for $n = 4$.]

IV. CONCLUSION

We have noted that for the $\lambda\phi^4$ theory in curved spacetimes, the (infinite) renormalization constants Z_λ and Z_m are the same (to second order, at least) as in the flat-spacetime case. Only the finite part of the Green's functions are affected by the coupling to the nonflat metric. Since the renormalization-group equations for λ and m are determined by the $(n-4)^{-1}$ coefficients of Z_λ and Z_m, these equations are the same as in flat spacetime. The new feature that arises in the renormalization of the $\lambda\phi^4$ theory in curved spacetimes is the necessity of introducing counterterms proportional to $R\phi^2$, even if no such terms were present in the original Lagrangian. Thus, we were led to the investigation of the renormalization-group equation for the new coupling constant ξ. The key result is that at low energies (large distances), $\xi=\frac{1}{6}$. This curious fact does not come about from any assumptions about conformal invariance but rather from an asymptotic expression for the Feynman propagator.[7] This result has been already noted by Gass[12] though our expression for the low-energy behavior of ξ differs from his.

In the case of electrodynamics on a curved spacetime, again the renormalization constants are the same, at least for one-loop diagrams, as in flat spacetime.[8] General arguments[8] suggest that this will be true to all orders. Thus, the renormalization-group equations for e and m_e will not be affected by the presence of a curved background metric. In the case of electrodynamics the further requirement of gauge invariance leads to the result that, unlike in the $\lambda\phi^4$ theory, there are no new divergences in the curved-spacetime case. Thus, there are no new coupling constants to study.

We do not know if these results are true also for non-Abelian gauge theories either without or with symmetry breaking. At present we can only caution that each theory needs to be investigated carefully to ascertain, or rule out, the effects of curvature on the renormalization-group equations for the usual parameters and to know whether or not new interaction terms are required by the curved-space renormalization scheme. Recent results of Gass and Dresden[20] show that for a field theory with a cubic self-interaction in six dimensions, the presence of a curved background results in a nonrenormalizable theory.

Note added in proof. It is interesting to note that if the coupling constant λ has a negative sign, then our model theory (although unreasonable in some respects) is asymptotically free, i.e., $\lambda(\kappa)\to 0$ as $\kappa\to\infty$. Further, in the large-κ limit, $m(\kappa)\to 0$, as one would hope; but, as in the positive-λ theory, the coupling to the curvature $\xi(\kappa)$ again goes to its conformally invariant value $\frac{1}{6}$. Thus in the presence of (classical) gravity, an asymptotically free model becomes asymptotically conformally invariant rather than asymptotically minimally coupled as one might naively expect.

ACKNOWLEDGMENTS

We have benefitted from discussions with Michael Allen, Carleton DeTar, Karel Kuchař, Richard H. Price, and Lee Smolin. We would like to thank Richard Gass for sending us copies of his work prior to publication. This research was supported by NSF Grant No. PHY81-06909 to the University of Utah.

[1] I. T. Drummond, Nucl. Phys. B95, 115 (1975).
[2] I. T. Drummond and G. M. Shore, Ann. Phys. (N.Y.) 117, 89 (1979).
[3] N. D. Birrell and J. G. Taylor, J. Math. Phys. 21, 1740 (1980).
[4] N. D. Birrell, J. Phys. A 13, 569 (1980).
[5] T. S. Bunch, P. Panangaden, and L. Parker, J. Phys. A 13, 901 (1980).
[6] T. S. Bunch and P. Panangaden, J. Phys. A 13, 919 (1980).
[7] T. S. Bunch and L. Parker, Phys. Rev. D 20, 2499 (1979).
[8] P. Panangaden, Phys. Rev. D 23, 1735 (1981).
[9] T. S. Bunch, Liverpool University report (unpublished).
[10] T. S. Bunch, Liverpool University report (unpublished).
[11] L. S. Brown and J. C. Collins, Ann. Phys. (N.Y.) 130, 215 (1980).
[12] R. Gass, SUNY, Stony Brook Report No. ITP-SB-80-28, 1980 (unpublished).
[13] G. 't Hooft, Nucl. Phys. B61, 455 (1973).
[14] S. Weinberg, Phys. Rev. D 8, 3497 (1973).
[15] J. C. Collins and A. J. Macfarlane, Phys. Rev. D 10, 1201 (1974).
[16] K. Fujikawa, Phys. Rev. D 23, 2262 (1981).
[17] I. T. Drummond and G. M. Shore, Phys. Rev. D 19, 1134 (1979); N. D. Birrell and P. C. W. Davies, *ibid.* 22, 322 (1980).
[18] M. Yoshimura, Phys. Rev. Lett. 41, 281 (1978); 42, 746 (E) (1979); S. Dimopoulos and L. Susskind, Phys.

Rev. D 18, 4500 (1978); D. Toussaint, S. B. Treiman, F. Wilczek, and A. Zee, *ibid.* 19, 1036 (1979); S. Weinberg, Phys. Rev. Lett. 42, 850 (1979); N. J. Papastamatiou and L. Parker, Phys. Rev. D 19, 2283 (1979).

[19]D. J. Gross, in *Methods in Field Theory,* edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[20]R. Gass and M. Dresden, SUNY, Stony Brook Report No. ITP-SB-81-35, 1981 (unpublished).