Asymptotic analysis of the monopole structure

Thomas W. Kirkman and Cosmas K. Zachos
Physics Department, University of Wisconsin, Madison, Wisconsin 53706
(Received 23 April 1981)

We analyze the behavior of the magnetic-monopole equations as a function of the ratio of Higgs-scalar-to-vector-meson mass M_H/M_W. For very small M_H/M_W, we find that the mass of the monopole deviates from the Prasad-Sommerfield limit M_W/α by $M_H/2\alpha$, independently of the particular details of the symmetry-breaking potential. In the opposite limit $M_H/M_W \to \infty$, the scalar field contributes negligibly to the mass, which is found to be $(M_W/\alpha)(1.787 - 2.228\, M_W/M_H + \cdots)$.

Gauge theories with scalar fields admit classical solutions whose energy is finite and localized in a small region. Magnetic monopoles are such configurations of vector and scalar fields,[1] which are stably intertwined as a result of their nontrivial topology.[2] There exist numerous general analyses of the symmetry-breaking patterns, the representations of and *Ansätze* for the fields, the conditions imposed on the Higgs potentials, and the topology on which monopole configurations are predicated, notably within the context of grand unified theories.[3]

The size, the mass, and most of the features of monopoles are usually abstracted from the complicated nonlinear equations that describe the prototype system of this kind, namely an SO(3) gauge theory with the Higgs scalar in the adjoint representation.[1] An analytic solution is available in a special limit,[4] as well as useful numerical work for various values of the arbitrary ratio of Higgs-scalar-to-vector-meson mass.[5]

Nonetheless, the essential features of these solutions are obtainable immediately through asymptotic analysis,[6] which is simple, powerful, and provides physical intuition on the structure of the configuration considered. Below, we illustrate the use of asymptotic techniques in the two extreme limits of the free parameter $\epsilon \equiv M_H/M_W$. For $\epsilon \to 0$, we find that the monopole mass exceeds the lower bound M_W/α of the Prasad-Sommerfield solution by $(M_W/\alpha)\epsilon/2 = M_H/2\alpha$. To $O(\epsilon)$, the correction is due only to the scalar field, but it does not depend on the particular form of the Higgs potential. As ϵ increases, the spatial extent of the energy concentration of the monopole decreases and its mass increases, while their scale is always set by M_W. For $\epsilon \to \infty$, we show that the contribution of the scalar field to the monopole mass vanishes as the upper bound $1.787\, M_W/\alpha$ is reached, in agreement with an estimate of Bogomol'nyi and Marinov[5]; the first correction to this limit, found to be $2.228\, M_W^2/M_H\alpha$, is also due to the Higgs field only. We comment briefly on the physical interpretation of these results, and suggest applications of these methods to more general monopole systems.

We restrict our discussion to the SO(3) Georgi-Glashow model with Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \tfrac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a - \frac{\lambda}{8}\left(\phi^a\phi^a - \frac{m^2}{\lambda}\right)^2, \tag{1}$$

where the Higgs field is in the adjoint representation. Since the scalar has a vacuum expectation value $\langle\phi^2\rangle = m^2/\lambda$, the symmetry is broken down to O(2), and two components of the vector field develop a mass

$$M_W = \frac{em}{\sqrt{\lambda}} \tag{2}$$

while the mass of the Higgs field is

$$M_H = m\,. \tag{3}$$

The monopole we discuss is a static, spherically symmetric solution specified by the "hedgehog" *Ansätz*[1]

$$\phi^a = \hat{r}^a\frac{H(r)}{er}\,,\quad A_0^a = 0\,,\quad A_i^a = \epsilon^{aij}\hat{r}_j\frac{1-K(r)}{er}\,. \tag{4}$$

$H(r)$ and $K(r)$ are dimensionless radial functions which extremize the action, or, equivalently, since the system is presumed to be static, minimize the energy (mass) of the monopole

$$E = -\int d^3x\,\mathcal{L} = \frac{4\pi}{e^2}\int_0^\infty dr\left[K'^2 + \frac{(K^2-1)^2}{2r^2} + \frac{H^2K^2}{r^2} + \frac{(rH'-H)^2}{2r^2} + \frac{\lambda r^2}{8e^2}\left(\frac{H^2}{r^2} - \frac{m^2e^2}{\lambda}\right)^2\right]. \tag{5}$$

It is convenient to define the dimensionless ratio

$$\epsilon \equiv \frac{M_H}{M_W} = \frac{\sqrt{\lambda}}{e} \tag{6}$$

and to rescale the radial coordinate by M_W, so that distance is expressed in units of $1/M_W$. Then the mass of the monopole is given by the minimum of

$$E(\epsilon)=\frac{M_W}{e^2/4\pi}\int_0^\infty dr\left[K'^2+\frac{(K^2-1)^2}{2r^2}+\frac{H^2K^2}{r^2}+\frac{(rH'-H)^2}{2r^2}+\frac{\epsilon^2 r^2}{8}\left(\frac{H^2}{r^2}-1\right)^2\right]$$

$$\equiv\frac{M_W}{\alpha}C(\epsilon)\,. \tag{7}$$

The boundary conditions imposed by the finite-mass requirement and topological stability[1,2] are

$$\frac{H(r)}{r}\to\begin{cases}1 \text{ as } r\to\infty\\ 0 \text{ as } r\to 0,\end{cases}\qquad K(r)\to\begin{cases}0 \text{ as } r\to\infty\\ 1 \text{ as } r\to 0.\end{cases} \tag{8}$$

The equations of motion are

$$K''=\frac{K(K^2-1)}{r^2}+K\left(\frac{H}{r}\right)^2, \tag{9a}$$

$$H''=2\frac{HK^2}{r^2}+\frac{\epsilon^2}{2}H\left(\frac{H^2}{r^2}-1\right). \tag{9b}$$

For a general ϵ, the two-scale problem (8), (9) is not easy to solve. However, when the mass of the Higgs field is switched off ($\epsilon=0$), an analytic solution may be found,[4] which we proceed to review. Equation (7) may be recast as follows[7]:

$$C(\epsilon)=\int_0^\infty dr\left\{\left(K'+\frac{KH}{r}\right)^2+\frac{1}{2}\left[H'-\frac{H}{r}-\frac{(1-K^2)}{r}\right]^2+\partial_r\left[\frac{H(1-K^2)}{r}\right]+\frac{\epsilon^2r^2}{8}\left(\frac{H^2}{r^2}-1\right)^2\right\}. \tag{10}$$

By the boundary conditions (8), the surface term is equal to 1. In the limit $\epsilon\to 0$, it is possible to nullify the first two positive-definite terms of the integral by satisfying the two first-order equations

$$K'=-\frac{KH}{r},\quad H'=\frac{H+1-K^2}{r} \tag{11}$$

[which, of course, reproduce (9) with $\epsilon=0$]. The following solution of Prasad and Sommerfield[4] satisfies *all* boundary conditions (8):

$$K=\frac{r}{\sinh r},\quad \frac{H}{r}=\coth r-\frac{1}{r}\,. \tag{12}$$

From the form of (10), it is apparent that this limit bounds the monopole mass from below:

$$C(0)=1\leq C(\epsilon)\,. \tag{13}$$

The functions K, H/r, and the energy density $\mathcal{E}$—the integrand of (7)—for $\epsilon=0$ are plotted in Figs. 1 and 2. Although the energy density displays a long-range ($1/r^2$) component as a result of the massless scalar and the em fields, it is dominated by a region of size 1 (characteristic of $1/M_W$), and the mass of the monopole has its scale set by M_W. Even as $\epsilon\to\infty$, we see below that this fact is not modified: the mass and the size of the monopole are determined by the mass of the vector boson, light or heavy.

Asymptotic analysis easily provides the first correction to the Prasad-Sommerfield mass, for

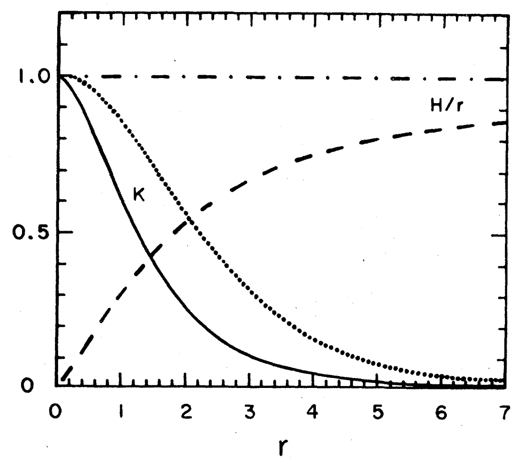

FIG. 1. The radial variation of the vector and scalar fields. For $\epsilon=0$, K is represented by a dotted line and H/r by a dashed line. For $\epsilon\to\infty$, K is represented by a solid line and H/r by a dash-dot line depicting a step function.

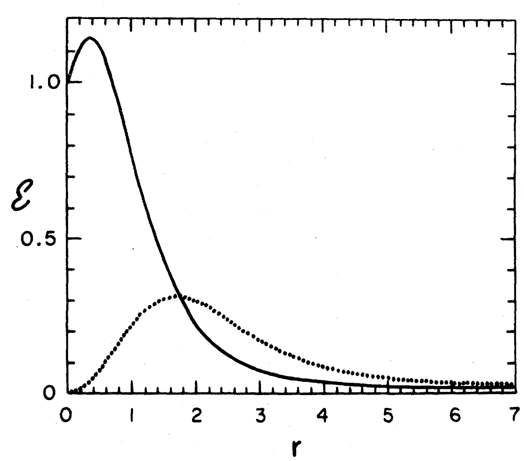

FIG. 2. The radial variation of the energy density $\mathcal{E}$ of the monopole in the $\epsilon=0$ limit (dotted line) and the $\epsilon\to\infty$ limit (solid line). Both cases involve a long-range ($1/r^2$) component. The integrals under the entire curves are 1 and 1.787, respectively.

sufficiently small ϵ. Taking the boundary conditions (8) into account and studying the dominant balance[6] of each term in Eqs. (9), we readily determine the regions where ϵ contributes to the solutions.

For large r, the scalar field is best split into its vacuum expectation value and the (small) deviation from it

$$\frac{H}{r} \equiv 1 - \frac{h}{r}. \tag{14}$$

Its equation of motion (9b) then becomes

$$h'' = -2\left(1-\frac{h}{r}\right)\frac{K^2}{r} + \frac{\epsilon^2}{2}(r-h)\left[1-\left(1-\frac{h}{r}\right)^2\right]$$

$$= -2\left(1-\frac{h}{r}\right)\frac{K^2}{r} + \epsilon^2 h\left(1-\frac{h}{r}\right)\left(1-\frac{h}{2r}\right). \tag{15}$$

For $r \gg 1$, the second term on the right-hand side (rhs) may not be regarded as a "small perturbation," since it, in fact, dominates:

$$h'' \sim \epsilon^2 h. \tag{16}$$

Using the boundary condition, it follows that

$$h \sim c_1 e^{-\epsilon r}, \quad \frac{H}{r} \sim 1 - c_1 \frac{e^{-\epsilon r}}{r}, \tag{17}$$

consistent with dropping the subdominant terms in (15).

As expected, the scalar field exhibits the Yukawa long-distance behavior controlled by M_H (ϵr represents distance in units of $1/M_H$), in contrast to (12). Note that if one had attempted to solve (15) by perturbation theory, each term in the perturbation expansion would be secular, i.e., it would grow uncontrollably with r in the asymptotic region $\gg 1/\epsilon$, as evoked by (17). Given (17), in the equation for the vector field (9a) the dominant balance yields

$$K'' \sim K, \quad K \sim c_2 e^{-r} \tag{18}$$

which decays exponentially on the shorter scale of $1/M_W$ and does not contribute to (15).

For ϵr sufficiently small, we may determine the constant c_1 by asymptotic matching[6]; in the region $1 \ll R \ll 1/\epsilon$, both (17) and the Prasad-Sommerfield solution (12) are simultaneously valid:

$$\frac{H}{R} \sim \coth R - \frac{1}{R} \sim 1 - \frac{c_1 e^{-\epsilon R}}{R}, \quad \text{hence } c_1 = 1,$$
$$K \sim 0. \tag{19}$$

We are then able to integrate the contribution of these asymptotic forms to (10), past $1/\epsilon$ and up to infinity, thereby obtaining $C(\epsilon)$ to $O(\epsilon)$ (Ref. 8):

$$C(\epsilon) = 1 + \int_R^\infty dr\, \frac{1}{2}\left[\left(\epsilon e^{-\epsilon r} + \frac{e^{-\epsilon r}}{r} - \frac{1}{r}\right)^2 + \epsilon^2 e^{-2\epsilon r}\left(1 - \frac{e^{-\epsilon r}}{r} + \frac{e^{-2\epsilon r}}{4r^2}\right)\right] \quad \left(1 \ll R \ll \frac{1}{\epsilon}\right),$$

$$= 1 + \frac{\epsilon}{2}\left(e^{-2\epsilon R} - \frac{(1-e^{-\epsilon R})^2}{\epsilon R}\right) + O(\epsilon^2)$$

$$= 1 + \frac{\epsilon}{2} + O(\epsilon^2). \tag{20}$$

Hence, to lowest order in ϵ, the correction to the monopole mass is a long-distance effect due to the Higgs field. The lower limit of integration R may be taken to be large, since at shorter distances the perturbation term in (15) is not dominant and regular perturbation theory yields a correction of $O(\epsilon^2)$; note, for instance, that the dependence on R in (20) only enters in $O(\epsilon^2)$.

We conclude that the mass of the monopole is

$$\frac{4\pi}{e^2} M_W C(\epsilon) = \frac{M_W}{\alpha}\left(1 + \frac{M_H}{2M_W} + O(\epsilon^2)\right) \tag{21}$$

in the limit $\epsilon \to 0$. In order for the matching procedure (19) to be valid to ~1%, $R \gtrsim 2.5$. Even the smallest $\epsilon = (0.1)^{1/2}$ considered in the numerical analyses[5] corresponds to $e^{-2.5(0.1)^{1/2}} \sim e^{-0.8} \sim 0.45$, which is too different from 1 to furnish a reasonably accurate estimate. For a 1% error in matching, ϵ must be $\lesssim \ln 1.01/\text{arc}\coth 1.01 \sim 4\times 10^{-3}$.

Observe that the above correction is a broader result, since it does not depend crucially on the particular form of the symmetry-breaking Higgs potential. Any potential normalized to 0 at its absolute minimum which involves a bilinear in the deviation from the vacuum expectation value for the scalar

$$V(H) = \frac{\epsilon^2}{2}\frac{h^2}{r^2} + O(h^3) \tag{22}$$

will be subject to the above analysis. For instance, a logarithmic potential of the Coleman-Weinberg type

$$V(H) = \frac{\epsilon^2 r^2}{8}\left[\frac{H^4}{r^4}\left(\ln\frac{H^2}{r^2} - \frac{1}{2}\right) + \frac{1}{2}\right] \tag{23}$$

supports a monopole solution with the above mass for sufficiently small ϵ.[9]

As ϵ increases, the mass of the monopole increases monotonically, since

$$\frac{d}{d\epsilon}C(\epsilon)=\frac{\partial C(\epsilon)}{\partial\epsilon}+\int_0^\infty dr\left(\frac{\partial H}{\partial\epsilon}\frac{\delta C}{\delta H}+\frac{\partial H'}{\partial\epsilon}\frac{\delta C}{\delta H'}+\frac{\partial K}{\partial\epsilon}\frac{\delta C}{\delta K}+\frac{\partial K'}{\partial\epsilon}\frac{\delta C}{\delta K'}\right)$$
$$=\int_0^\infty dr\frac{\epsilon r^2}{4}\left(\frac{H^2}{r^2}-1\right)^2, \tag{24}$$

where the equations of motion have been used, as well as the fact that, by (8), the functions H and K cannot vary with ϵ on the boundary. Since the rhs of (24) is positive definite, $C(\epsilon)$ is a nondecreasing function of ϵ. One may reproduce the correction (20) by evaluating (24) to zero order in ϵ, which leaves only the long-distance contribution to the integral. Around $\epsilon=0$,

$$\frac{d}{d\epsilon}C(\epsilon)=\int_R^\infty dr\,\epsilon e^{-2\epsilon r}\left(1-\frac{e^{-2\epsilon r}}{2r}\right)^2+O(\epsilon)$$
$$=\frac{1}{2}\int_{2\epsilon R}^\infty dr\,e^{-r}+O(\epsilon)=\tfrac{1}{2}+O(\epsilon). \tag{25}$$

For higher values of ϵ, numerical solutions[5] indicate that the scalar field rises with r to a level close to its vacuum value within a region of size $\sim 1/\epsilon$, while the vector field varies significantly less on this scale. Moreover, $C(\epsilon)$ varies very slowly with ϵ. We thus proceed to the asymptotics in the limit $\epsilon\to\infty$ (i.e., $M_H \gg M_W$) which bring out these features strikingly, and provide the upper bound for $C(\epsilon)$. For sufficiently large r ($\epsilon r\gg 1$ automatically)

$$K''\sim K,\ \text{hence}\ K\sim c_3e^{-r} \tag{26}$$

as before. However, the dominant balance analysis of Eq. (15) is now different, because $h\sim e^{-\epsilon r}$ makes the second term on the rhs of Eq. (15) subdominant, while e^{-2r}/r does the same to the first term. The only self-consistent analysis results in $h''\sim 0$, whence

$$h\sim\frac{2c_3{}^2e^{-2r}}{\epsilon^2 r},\quad\text{i.e.,}\quad \frac{H}{r}\sim 1-\frac{2c_3{}^2e^{-2r}}{\epsilon^2r^2}. \tag{27}$$

Asymptotically, both scalar and vector fields are controlled by the smaller mass M_W.

Remarkably, by the same reasoning, this behavior ($h''\sim 0$, $h\sim 2K^2/\epsilon^2 r$) is valid for small r as well, down to $r\sim O(1/\epsilon)$. Because of this smallness of the overall coefficient of h, the contributions of the Higgs field to the monopole mass of the last two terms in (7) are negligible in the region $r\gg 1/\epsilon$: they are suppressed by $1/\epsilon^4$ and $1/\epsilon^2$, respectively. The contribution of the third term is likewise $K^2+O(\epsilon^{-2})$.

Even at distances of order $1/\epsilon$ and below, very little energy resides in the scalar field. Specifically, H/r must rise from 0 at $r=0$ to 1 within a distance of order $1/\epsilon$—the precise behavior is computed later. As a consequence, the last three terms in the integral (7) are at most $O(1/\epsilon)$ and may be ignored in the limit $\epsilon\to\infty$. Consequently, to $O(\epsilon^0)$, the Higgs field is pegged to its vacuum value 1 at the minimum of the potential and decouples from the problem—the vector field does not "see" the Higgs spike at the origin. The mass of the monopole is then given by the minimum of the one-dependent-variable, one-scale problem

$$C(\infty)=\int_0^\infty dr\left[K'^2+\frac{(K^2-1)^2}{2r^2}+K^2\right]. \tag{28}$$

The equations of motion for K,

$$K''=K\left(1+\frac{K^2-1}{r^2}\right);\quad K(0)=1,\quad K(\infty)=0, \tag{29}$$

may be solved numerically. The solution K and the energy density $\mathscr{E}$ are plotted in Figs. 1 and 2. [At $r\to 0$, $K\sim 1+r^2(\frac{1}{3}\ln r+0.342+\cdots)$; and at $r\to\infty$, $K\sim e^{-r}2.275(1-1/2r+\cdots)$.] The integral in (28) is then evaluated to yield the upper bound to the monopole mass

$$C(\infty)=1.787 \tag{30}$$

to an accuracy of 10^{-3}, in good agreement with the number quoted in the first of Refs. 5.[10]

As in the small-ϵ limit, the first correction to this value of order $1/\epsilon$ is given by the Higgs field through Eq. (24):

$$C(\epsilon)=C(\infty)+\frac{1}{\epsilon}\frac{d}{d(1/\epsilon)}C(\epsilon)\bigg|_{\epsilon=\infty}+O(\epsilon^{-2})$$
$$=C(\infty)-\frac{1}{\epsilon}\int_0^\infty dr\frac{r^2\epsilon^3}{4}\left(\frac{H^2}{r^2}-1\right)^2+O(\epsilon^{-2})$$
$$=C(\infty)-\frac{1}{\epsilon}\int_0^\infty d\rho\frac{\rho^2}{4}\left(\frac{\overline{H}^2}{\rho^2}-1\right)^2+O(\epsilon^{-2}). \tag{31}$$

Since the natural scale of the Higgs field is $\rho\equiv\epsilon r$, we have rescaled the radial coordinate, which is now measured in units of $1/M_H$. To preserve the vacuum value 1, we also rescale $\overline{H}\equiv\epsilon H$. The Higgs equation of motion now reads

$$\frac{d^2}{d\rho^2}\overline{H}(\rho)=\frac{2\overline{H}}{\rho^2}K^2\left(\frac{\rho}{\epsilon}\right)+\frac{\overline{H}}{2}\left(\frac{\overline{H}^2}{\rho^2}-1\right);$$
$$\frac{\overline{H}}{\rho}\to 0,\qquad\text{for }\rho=0, \tag{32}$$
$$\frac{\overline{H}}{\rho}\to 1-\frac{2K^2}{\rho^2},\quad\text{for }\rho=\infty.$$

The "potential" K^2 in Eq. (32) may be set equal to 1, since K does not vary appreciably over the scale under consideration, for sufficiently large

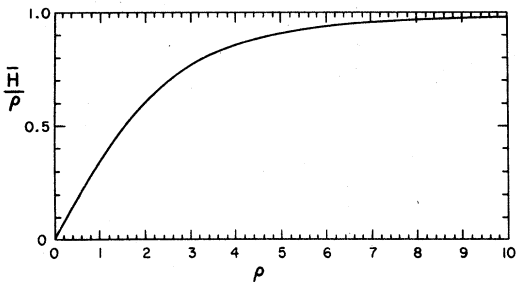

FIG. 3. The radial variation of the Higgs field $\bar{H}/\rho$ for very large ϵ plotted on the scale characteristic of the Higgs mass: $\rho = \epsilon r$. In the $\epsilon \to \infty$ limit this curve is identified with the dash-dot line of Fig. 1.

ϵ. Equation (32) is solved numerically for $\bar{H}(\rho)$, which is plotted in Fig. 3 (for $\rho \to 0$, $\bar{H}/\rho \to 0.358\rho$; for $\rho \to \infty$, $\bar{H}/\rho \sim 1 - 2/r^2 - 6/r^4 + \cdots$).

One may observe that, upon setting $K=1$, (32) reduces to the description of an isolated static Higgs system in three space-time dimensions with a "hedgehog" *Ansätz* (4). Its equation of motion then results from the following integral, which may also be obtained by subtracting (28) from (7), after suitable rescalings, and setting $K=1$:

$$\bar{C} = \int_0^\infty d\rho \left[\frac{\rho^2}{8}\left(\frac{\bar{H}^2}{\rho^2} - 1\right)^2 + \frac{1}{2}\left(\bar{H}' - \frac{\bar{H}}{\rho}\right)^2 + \left(\frac{\bar{H}^2}{\rho^2} - 1\right)\right]. \tag{33}$$

Note that the constant term -1 in the integrand is indispensable in order to preserve the finiteness of $\bar{C}$, given the boundary conditions. It is also necessary in order to escape a naive interpretation of Derrick's scaling argument,[11] which does *not* exclude the solution obtained, since the rightmost term in parentheses in the integrand is negative. In fact, since the solution $\bar{H}(\rho)$ extremizes $\bar{C}$, if $\bar{H}$ is varied by a scale transformation of its argument $\rho \to \rho/a$, the derivative of $\bar{C}$ with respect to a must vanish at $a=1$. This leads to the following expression, which was also checked numerically to an accuracy of $O(10^{-3})$,

$$\int_0^\infty d\rho \left[\frac{3\rho^2}{8}\left(\frac{\bar{H}^2}{\rho^2} - 1\right)^2 + \frac{1}{2}\left(\bar{H}' - \frac{\bar{H}}{\rho}\right)^2 + \left(\frac{\bar{H}^2}{\rho^2} - 1\right)\right] = 0. \tag{34}$$

Through this expression, $\bar{C}$ is readily identifiable with the coefficient of $1/\epsilon$ in (31).

This correction is evaluated directly to yield

$$C(\epsilon) = 1.787 - 2.228\frac{1}{\epsilon} + O(\epsilon^{-2}) \tag{35}$$

for large ϵ, in analogy to the small-ϵ result (21). In contrast to (21), however, this correction receives substantial contributions from the small-ρ region, where the scalar field differs appreciably from its vacuum value. Thus, in general, the correction is not independent of the form of the Higgs potential.

The purpose of this paper is to indicate how to use asymptotic analysis for a rapid determination of the limiting behavior of the fields and the mass of the simplest monopole system. In particular, in the $\epsilon \to \infty$ limit the two fields decouple allowing a relatively straightforward solution to the problem. We observe that the field configurations for various values of ϵ generated numerically[5] are enveloped by the limiting curves in Fig. 1. It is striking that, even though the scalar field provides stability and magnetic charge through its boundary condition at the origin,[1,2] the Higgs mass scale affects the results very little: the overall variation of the mass is by less than a factor of 2 between the extreme cases discussed.

In conclusion, we point out that a similar analysis could be implementable in determining the approximate nature of the monopole solutions sustained by the larger groups and the multiple mass scales of grand unified theories.[3]

We have profited from conversations with L. Durand, K. Miller, D. Wilkinson, and especially C. Goebel. This research was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Department of Energy under Contract No. DE-AC02 76ER00881-207.

[1] G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[2] J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. 16, 433 (1975).

[3] P. Goddard and D. Olive, Rep. Prog. Phys. 41, 1357 (1978) and references therein. A partial list includes E. Corrigan, D. Olive, D. Fairlie, and J. Nuyts, Nucl. Phys. B106, 475 (1976); E. Corrigan and D. Olive, *ibid.* B110, 237 (1976); A. Goldhaber and D. Wilkinson, *ibid.* B114, 317 (1976); Z. Horvath and L. Palla, *ibid.* B116, 500 (1976); C. P. Dokos and T. N. Tomaras, Phys. Rev. D 21, 2940 (1980).

[4] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[5] E. B. Bogomol'nyi and M. S. Marinov, Yad. Fiz. 23, 676 (1976) [Sov. J. Nucl. Phys. 23, 355 (1976)]; F. A. Bais and J. Primack, Phys. Rev. D 13, 819 (1976).

[6] For a very useful review of asymptotic analysis techniques, see C. Bender and S. Orszag, *Advanced Mathe-*

matical Methods for Scientists and Engineers, (Mc-Graw-Hill, New York, 1978).

[7]E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].

[8]The Hamiltonian $\frac{1}{2}\int_R^\infty dr[(H' - H/r - 1/r)^2 + (\epsilon^2 r^2/4)(H^2/r^2 - 1)^2]$ generates both the corrections to the mass and the long-range equations of motion for H—an analogous situation is observed in the large-ϵ limit.

[9]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973). We are merely using this potential, which arises from radiative corrections, as an illustration of a potential of the general type discussed. We are not considering radiative corrections to the kinetic terms in the effective action, nor are we making a statement of physical significance.

[10]The authors of that reference do not explain how they have obtained this value. Equation (29) and its solution are given by P. Vinciarelli, in *Understanding the Fundamental Constituents of Matter*, edited by A. Zichichi (Plenum, New York, 1978) in a somewhat different context. Equation (29) of that reference contains a sign error.

[11]G. H. Derrick, J. Math. Phys. 5, 1252 (1964). Also, see S. Coleman in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1977).