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Dyson-Schwinger equations approach to the large-N limit: Model systems and string representation of Yang-Mills theory

Spenta R. Wadia
*The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637**
and Tata Institute of Fundamental Research, Bombay, India-400005
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Simple model systems like the O(N) σ model, the Gross-Neveu model, and the random matrix model are solved at $N\to\infty$ using Dyson-Schwinger equations and the fact that the Hartree-Fock approximation is exact at $N\to\infty$. The complete string equations of the $U(\infty)$ lattice gauge theory are presented. These must include both string rearrangement and splitting. Comparison is made with the "continuum" equations of Makeenko and Migdal which are structurally different. The difference is ascribed to inequivalent regularization procedures in the treatment of string splitting or rearrangement at intersections.

I. INTRODUCTION

In quantum chromodynamics (QCD) the gauge-invariant Wilson loop operator creates a string of chromoelectric flux from the vacuum. With hindsight from the string model, the proposal of Nambu, Gervais and Neveu, and Polyakov[1] has been to study directly the local variations of the Wilson loop. The precise study of the local-variations problem is tantamount to establishing the Dyson-Schwinger (DS) equations for the loop operator which dictates the definition of the derivative operator in loop space.[2,3] This definition depends on the gauge group. The gauge group of interest here is U(N).

In the case of a general intersecting loop the DS equation is not closed as an equation for real functions defined on the set of all four-dimensional loops. The crucial observation in this direction due to Migdal and Makeenko[3] is that as $N\to\infty$ the Hartree-Fock approximation becomes exact. Using this fact it becomes possible to have a representation of QCD entirely in terms of real functions on loops: a string theory. The functional integral representation of these equations in loop space would represent a dual model. In fact Migdal has proposed such a dual model involving fermionic strings.[4] The problem of finding the correct dual model and working out its consequences seems to be central in getting at a working theory of the strong interactions based on QCD. Such a theme has motivated this work.

An alternative formulation of the $N=\infty$ limit of the gauge theory is due to Sakita.[5] It involves a change of variables from gauge fields to gauge-invariant loop operators at a given time slice in the Hamiltonian formulation. The basic difference with the DS equations formulation is that here at a *given time slice* the space of states of the transfer matrix can involve gauge-noninvariant string operators.[6] Hence the type of physical questions that can be answered in the two formulations are different.

The topics treated in this paper are the following.

(1) The DS equations for far simpler systems such as the O(N) σ model, the Gross-Neveu model, and the random matrix model are derived and solved at $N=\infty$. Well-known results are reproduced in this approach.

(2) The DS equations in a U(N) gauge theory are reexamined. They are first discussed in the context of lattice gauge theories which offer a sound platform to discuss these matters. We briefly review the equation for a simple loop. This mainly serves to establish the notation of the derivative operator in loop space corresponding to the gauge group U(N). The DS equations for simple loops are insufficient to determine them. This is explicitly illustrated in the exactly soluble case of two dimensions.

Next we study intersecting loops—where a given link is traversed more than once in the same or opposite direction. To have closed equations all such loops must be included because even starting from simple loops such configurations can be obtained by successive applications of the derivative operator. The general structure of the string equations involves the action of the second-derivative operator producing string rearrangement or splitting at intersections depending upon whether the common link is traversed in the same or opposite direction. The string equations on a lattice presented by Makeenko and Migdal[3] are only valid for the class of intersecting loops in which repeated traverses of a given link are in the *same* direction. For these, only string rearrangement occurs. Such has been the case where these equations have correctly reproduced the known results, e.g., of the one-plaquette world.[7] In general, this subclass of equations cannot be closed because the derivative operator can lead to loops

where string splitting occurs.

After this we briefly discuss the continuum string equations which are structurally different from the lattice equations. Roughly speaking the basic difference lies in smearing the "δ functions" at string intersections which lead to string splitting and rearrangement. In the lattice theory the δ functions always come smeared because strings interact when links overlap. In the continuum derivation string splitting is allowed to occur at a point. The singular δ-function interaction is smeared only after evaluating quantum expectation values. It is not surprising that the continuum equations cannot be derived by taking the continuum limit of the lattice equations.

The basic steps involved in the DS equations approach are the following.

(i) To write down the exact DS equations for the relevant operators. The method used here is a version previously used in the study of toy non-Abelian models.[8]

(ii) Use the fact that at $N=\infty$ the Hartree-Fock approximation is exact[3,9] to get a closed equation for the operators. We note that this approach completely bypasses the problem of extracting the entropy per invariant configuration, so crucial in formulating the $N=\infty$ limit in the path integral. Further it is easy to formulate the problem of higher-order corrections.

II. MODEL SYSTEMS

A. The O(N) nonlinear σ model

The degrees of freedom are normalized N-dimensional spins at each lattice site:

$$\vec{S}_m=(S_m^1,S_m^2,\ldots,S_m^N),\quad \vec{S}_m{}^2=1.$$

The correlation function of interest is

$$\langle\vec{S}_m\cdot\vec{S}_n\rangle=\frac{1}{Z}\int d\mu\,\vec{S}_m\cdot\vec{S}_n\exp(A/T),\tag{1}$$

where

$$A=\sum_{\langle m,n\rangle}\vec{S}_n\cdot\vec{S}_m\tag{2}$$

is the action with nearest-neighbor couplings and

$$d\mu=\prod_n d\vec{S}_n\,\delta(\vec{S}_n{}^2-1)\tag{3}$$

is the O(N)-invariant measure. Z is the partition function.

DS equations

Case 1. When $n\neq m$ the equation is given by

$$\sum_\alpha\frac{\partial}{\partial\Lambda_\alpha}\left\langle\vec{S}_m^\Lambda\cdot L^{(\alpha)}\vec{S}_n\exp\left(\frac{1}{T}\sum_\mu(\vec{S}_m^\Lambda-\vec{S}_m)\cdot\vec{S}_{m+\mu}\right)\right\rangle_{\Lambda=0}=0,\tag{4}$$

$L^{(\alpha)}$ are the generators of O(N):

$$\sum_\alpha L_{ab}^{(\alpha)}L_{cd}^{(\alpha)}=\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}\tag{5}$$

and

$$\vec{S}_m^\Lambda=\vec{S}_m+\sum_\alpha\Lambda_\alpha L^{(\alpha)}\vec{S}_m.\tag{6}$$

Working out the details in (4) we arrive at

$$\frac{1}{T(N-1)}\left(\sum_\mu\langle\vec{S}_n\cdot\vec{S}_{m+\mu}\rangle-\sum_\mu\langle\vec{S}_m\cdot\vec{S}_{m+\mu}\vec{S}_n\cdot\vec{S}_m\rangle\right)=\langle\vec{S}_n\cdot\vec{S}_m\rangle.\tag{7}$$

This is the DS equation when $n\neq m$. It has previously been used to study correlation inequalities.[10] Note that (7) is not valid at $n=m$ and further (7) is not sufficient to determine $\langle\vec{S}_n\cdot\vec{S}_m\rangle$ at $N=\infty$. This is a simple but important point and we shall come back to it in the gauge-theory context.

Case 2. When $n=m$,

$$\langle\vec{S}_n\cdot\vec{S}_m\rangle=1.\tag{8}$$

Combining (7) and (8) we get the complete DS equation

$$\frac{1}{T(N-1)}\left(\sum_\mu\langle\vec{S}_n\cdot\vec{S}_{m+\mu}\rangle-\sum_\mu\langle\vec{S}_m\cdot\vec{S}_{m+\mu}\vec{S}_n\cdot\vec{S}_m\rangle\right)=\langle\vec{S}_n\cdot\vec{S}_m\rangle-\delta_{mn}.\tag{9}$$

This done we use the fact that at $N=\infty$ the Hartree-Fock approximation is exact:

$$\langle\vec{S}_m\cdot\vec{S}_{m+\mu}\vec{S}_n\cdot\vec{S}_m\rangle=\langle\vec{S}_m\cdot\vec{S}_{m+\mu}\rangle\langle\vec{S}_n\cdot\vec{S}_m\rangle.\tag{10}$$

This is the key to the solubility of this model and one obtains a closed equation for the correlation:

$$\sigma_{n-m}=\langle\vec{S}_n\cdot\vec{S}_m\rangle=\int_{-\pi}^{\pi}\frac{d\vec{k}}{(2\pi)^d}\sigma_k\exp[ik\cdot(n-m)],$$

viz.,

$$\frac{1}{TN}\left[\sum_\mu\sigma_{n-m-\mu}-\left(\sum_\mu\sigma_\mu\right)\sigma_{n-m}\right]=\sigma_{n-m}-\delta_{nm},\tag{11}$$

or equivalently in Fourier space

$$\frac{1}{TN}\left[\sum_\mu 2\cos k_\mu\sigma_k-\int_{-\pi}^{\pi}\frac{d\vec{l}}{(2\pi)^d}\sum_\mu 2\cos l_\mu\sigma_l\sigma_k\right]=\sigma_k-1,\tag{12}$$

which leads to

$$\sigma_k=\frac{TN}{(4+H)-\sum_\mu 2\cos k_\mu},\tag{13}$$

$$4+H=TN\left(1+\int\frac{d\vec{l}}{(2\pi)^d}\sum_\mu 2\cos l_\mu\sigma_l\right).$$

The constant $(4+H)$ is determined by

$$\langle \vec{S}_n \cdot \vec{S}_n \rangle = \int \frac{d\vec{k}}{(2\pi)^d} \frac{NT}{(4+H) - \sum_\mu 2\cos k_\mu} = 1 . \tag{14}$$

This is the result of Berlin and Kac and of Stanley.[11] Finally we remark that (9) can be used as a starting point for a systematic $1/N$ expansion.

B. The Gross-Neveu model[12]

The degress of freedom are N-component fermion fields at each space-time point. The correlation function of interest is

$$\sigma(x-y) = \sum_a \langle \bar{\psi}_a(x)\psi_a(y)\rangle = \frac{1}{Z}\int \prod_{x,a} d\bar{\psi}_a(x) d\psi_a(x) \sum_a \bar{\psi}_a(x)\psi_a(y)\exp(-A) . \tag{15}$$

The action is

$$A = \int d^2x \left[\sum_a \bar{\psi}^a\left(i\gamma^\mu \frac{\partial}{\partial x_\mu}\psi^a\right) + \frac{g}{2N}\left(\sum_a \bar{\psi}_a\psi_a\right)^2\right] . \tag{16}$$

The DS equation is

$$\int \prod_x d\bar{\psi} d\psi \sum_a \frac{\delta}{\delta\bar{\psi}_a(y)}[\bar{\psi}_a(x)\exp(-A)] = 0 . \tag{17}$$

(17) implies

$$\left\langle \sum_a \frac{\delta\bar{\psi}_a(x)}{\delta\bar{\psi}_a(y)}\right\rangle = \left\langle \sum_a \bar{\psi}_a(x)\frac{\delta A}{\delta\bar{\psi}_a(y)}\right\rangle .$$

A simple calculation gives

$$N\delta(x-y) = i\gamma^\mu \frac{\partial}{\partial y_\mu}\left\langle \sum_a \bar{\psi}_a(x)\psi_a(y)\right\rangle + \frac{g}{N}\left\langle \sum_a \bar{\psi}_a(x)\psi_a(y)\sum_b \bar{\psi}_b(y)\psi_b(y)\right\rangle . \tag{18}$$

For large N the Hartree-Fock approximation is exact:

$$\left\langle \sum_a \bar{\psi}_a(x)\psi_a(y)\sum_b \bar{\psi}_b(y)\psi_b(y)\right\rangle = \left\langle \sum_a \bar{\psi}_a(x)\psi_a(y)\right\rangle \left\langle \sum_b \bar{\psi}_b(y)\psi_b(y)\right\rangle ,$$

and (18) becomes a closed equation for the correlation $\sigma(x-y)$:

$$N\delta(x-y) = i\not{\partial}_y \sigma(x-y) + \frac{g}{N}\sigma(x-y)\sigma(0) . \tag{19}$$

Taking Fourier transforms we have

$$N = -\not{k}\sigma_k + \frac{g}{N}\sigma_k\left(\int_{-\Lambda}^{\Lambda} \frac{d^2k}{(2\pi)^2}\sigma_k\right) . \tag{20}$$

Λ is an ultraviolet cutoff. Calling

$$\frac{g}{N}\int_{-\Lambda}^{\Lambda} \frac{d^2k}{(2\pi)^2}\sigma_k = m , \tag{21}$$

(20) and (21) imply

$$\int_{-\Lambda}^{\Lambda} \frac{d^2k}{(2\pi)^2}\frac{2}{k^2+m^2} = \frac{1}{g} , \tag{22}$$

with solution

$$m = \Lambda\exp(-\pi/g) , \tag{23}$$

which is the well-known result for the ground state.

C. Random Hermitian matrix model

The degrees of freedom consist of a $N\times N$ Hermitian matrix M_{ij}. The correlation function of interest is

$$u_k = \frac{1}{Z}\int dM \frac{1}{N}\operatorname{tr}(e^{ikM})\exp(-A/T) , \qquad k = 0, \pm1, \pm2, \ldots . \tag{24}$$

The action is

$$A = \tfrac{1}{2}\operatorname{tr}M^2 + \frac{g}{4}\operatorname{tr}M^4 . \tag{25}$$

This is the model previously considered by Brézin *et al.*[13] The DS equation is

$$\sum_{ij}\int dM \frac{\partial}{\partial M_{ij}}[(e^{ikM})_{ij}\exp(-A/T)] = 0 . \tag{26}$$

This implies

$$\left\langle \sum_{ij}\frac{\partial}{\partial M_{ij}}(e^{ikM})_{ij}\right\rangle = \left\langle \sum_{ij}(e^{ikM})_{ij}\frac{\partial A}{\partial M_{ij}}\right\rangle . \tag{27}$$

Using the formula

$$\frac{\partial}{\partial M_{ml}}(e^{ikM})_{ij} = \int_0^1 dt\,(ik)(e^{ikMt})_{im}(e^{ik(1-t)M})_{lj} ,$$

(27) becomes

$$\frac{1}{T}\left\langle \frac{1}{N}\operatorname{tr}[e^{ikM}(M+gM^3)]\right\rangle = \frac{ik}{N}\int_0^1 dt\langle \operatorname{tr}e^{iktM}\operatorname{tr}e^{ik(1-t)M}\rangle . \tag{28}$$

Note that the right-hand side of (28) is the analog of the δ function that appears in the DS equation for vector degrees of freedom. For large N, the Hartree-Fock approximation is exact, hence

$$\langle \operatorname{tr}e^{iktM}\operatorname{tr}e^{ik(1-t)M}\rangle = \langle \operatorname{tr}e^{iktM}\rangle\langle \operatorname{tr}e^{ik(1-t)M}\rangle ,$$

and (28) can be written as a closed equation for u_k:

$$-g\frac{d^3}{dk^3}u_k+\frac{d}{dk}u_k+k\int_0^1 dt\,u_{kt}u_{k(1-t)}=0\,. \tag{29}$$

We have put $TN=1$ for convenience. Taking the Fourier transform

$$u_k=\int_{-2a}^{2a}\frac{d\lambda}{2\pi}u(\lambda)e^{ik\lambda}\,, \tag{30}$$

we get from (29), after some algebra, either $u(\lambda)=0$, or

$$\tfrac{1}{2}(\lambda+g\lambda^3)=P\int_{-2a}^{2a}\frac{d\lambda'}{2\pi}\frac{u(\lambda')}{\lambda-\lambda'}\,,$$
$$\int_{-2a}^{2a}\frac{d\lambda}{2\pi}u(\lambda)=u_{k=0}=1\,. \tag{31}$$

This is a singular integral equation for the density of eigenvalues of the matrix M. It was obtained in Ref. 13 using different methods. The solution proceeds by the Riemann-Hilbert method. We quote the result

$$u(\lambda)=(1+2ga^2+g\lambda^2)(4a^2-\lambda^2)^{1/2}\,,\quad |\lambda|\le 2a \tag{32}$$

with a determined by

$$3ga^4+a^2-1=0\,.$$

III. YANG-MILLS THEORY ON A LATTICE

Gauge degrees of freedom are $N\times N$ unitary matrices defined on the links of a space-time lattice. The correlation functions of interest are Wilson loops. Let C be an arbitrary closed loop and consider the group element

$$U(C)=\prod_{l\in\Gamma}U(l)\,, \tag{33}$$

which is a directed product of group elements along the loop. A given link can occur more than once along the contour. The Wilson loop is given by

$$W(C)=\left\langle\frac{\mathrm{tr}}{N}U(C)\right\rangle=\frac{1}{Z}\int\prod_l dU_l\frac{\mathrm{tr}}{N}U(C)\exp\left(\frac{1}{g^2}A\right)\,. \tag{34}$$

The action is

$$A=\sum_P[\mathrm{tr}U(p)-N]\,,$$

where $U(p)$ is a product of group elements at the four oriented links of the plaquette P. $dU(1)$ denotes the left and right invariant measures at the link 1.

The DS equations

We first discuss the well-known case of simple loops and indicate the derivative operator. Then we present three simple examples to make our point about intersecting loops. After that the general result is evident.

Case 1. We will call a loop simple if each link occurs only once along the contour. Examples are given in Fig. 1. Denote the link at which the equation is required by U. Its coordinates are $(n,n+\nu)$. Let us isolate it in the action

$$A=A'+\sum_{\mu\neq\nu}\mathrm{tr}(UV_\mu^\dagger+U^\dagger V_\mu)\,. \tag{35}$$

A' does not contain U. V_μ is the product of the three links which together with U forms a plaquette in the μ direction. Denote the part of the loop without U by $U(C')$ so that

$$U(C)=U(C')U\,.$$

The DS equation is then given by

$$\sum_\alpha\frac{\partial}{\partial\Lambda_\alpha}\left\langle\mathrm{tr}[U(C')t^{(\alpha)}U_\Lambda]\exp\left[\frac{1}{g^2}\sum_{\mu\neq\nu}\mathrm{tr}(U_\Lambda V_\mu^\dagger+U_\Lambda^\dagger V_\mu)-\mathrm{tr}(UV_\mu^\dagger+U^\dagger V_\mu)\right]\right\rangle_{\Lambda=0}=0\,. \tag{36}$$

$t^{(\alpha)}$ is a U(N) generator:

$$\sum_{\alpha=1}^{N^2}t_{ab}^{(\alpha)}t_{cd}^{(\alpha)}=-\delta_{ad}\delta_{bc} \tag{37}$$

and

$$U_\Lambda=\left(1+\sum_\alpha\Lambda_\alpha t^{(\alpha)}\right)U\,,$$
$$U_\Lambda^\dagger=U^\dagger\left(1-\sum_\alpha\Lambda_\alpha t^{(\alpha)}\right)\,.$$

Working out the details one gets

$$\frac{1}{g^2N}\sum_{\mu\neq\nu}d_\mu(n,n+\nu)W(C)=W(C)\,, \tag{38}$$

$d_\mu(n,n+\nu)$ is the derivative operator in the $\mu\neq\nu$ direction at the link $(n,n+\nu)$ given by (Fig. 2)

$$d_\mu(n,n+\nu)W(C)=\langle\mathrm{tr}U(C)[U(\gamma_\mu)-U^\dagger(\gamma_\mu)]\rangle$$
$$=W(C+\gamma_\mu)-W(C+\gamma_\mu^{-1})\,. \tag{39}$$

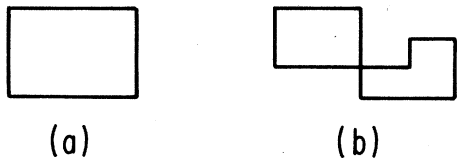

FIG. 1. Simple loops where each link occurs only once along the contour.

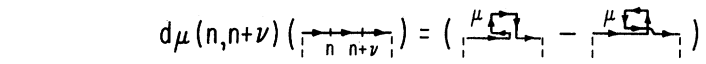

FIG. 2. Lattice derivative operator for the unitary group.

$U(\gamma_\mu)=U^\dagger V_\mu$ is the group element corresponding to the little loop γ_μ at n in the μ direction. γ_μ^{-1} is the oppositely traversed little loop. Also note that $[U(\gamma_\mu)-U^\dagger(\gamma_\mu)]_{ij}=-\delta A/\delta a_{\nu ij}$ is the equation of motion term with $\delta a_{\nu ij}=(U^\dagger dU)_{ij}$.

The equation for the simple loop (37) is the analog of (7) for spin systems. However, the factorization property which essentially solves the vector model is not valid here (see Fig. 3):

$$\langle \mathrm{tr}[U(C)U(\gamma_\mu)^\dagger]\rangle \neq \langle \mathrm{tr}U(C)\rangle\langle \mathrm{tr}U(\gamma_\mu)\rangle\,. \tag{40}$$

Hence the equation for $W(C)$ is not closed. It gets related to nearby loops in a complicated way. The point of similarity is that like (10) Eq. (37) is insufficient to determine $W(C)$. This is presumably because both these equations are homogeneous and do not contain information about coalescing degrees of freedom. In the Appendix we demonstrate this explicitly in the context of the soluble two-dimensional lattice gauge theory. There we also establish (40).

Case 2. We provide three examples of intersecting loops where links do occur more than once along the curve.

(i) The link U with coordinates $(n,n+\nu)$ is traversed twice in *opposite* directions [Fig. 4(a)]. We have denoted the two closed contours by C_1 and C_2 so that the whole contour C is given by

$$C=C_1(n+\nu,n)C_2(n,n+\nu) \tag{41}$$

and

$$U(C)=U(C_1)UU(C_2)U^\dagger\,. \tag{42}$$

The DS equation at all links $\neq(n,n+\nu)$ is the same as (38). At $(n,n+\nu)$ we have another term. The equation is

$$\sum_\alpha \frac{\partial}{\partial\Lambda_\alpha}\left\langle \mathrm{tr}[U(C_1)t^{(\alpha)}U_\Lambda U(C_2)U_\Lambda^\dagger]\exp\left(\frac{1}{g^2}\sum_{\mu\neq\nu}\mathrm{tr}(U_\Lambda V_\mu^\dagger+U_\Lambda^\dagger V_\mu)-\mathrm{tr}(UV_\mu^\dagger+U^\dagger V_\mu)\right)\right\rangle_{\Lambda=0}=0\,. \tag{43}$$

The steps are the same as before. Noting that

$$\sum_\alpha[\langle \mathrm{tr}U(C_1)t^{(\alpha)}t^{(\alpha)}UU(C_2)U^\dagger\rangle-\langle \mathrm{tr}U(C_1)t^{(\alpha)}UU(C_2)U^\dagger t^{(\alpha)}\rangle]=-N\langle \mathrm{tr}U(C_1)UU(C_2)U^\dagger\rangle+\langle \mathrm{tr}U(C_1)\,\mathrm{tr}U(C_2)\rangle\,,$$

we arrive at

$$\frac{1}{g^2N}\sum_{\mu\neq\nu}d_\mu(n,n+\nu)W(C)=W(C)-\left\langle\frac{\mathrm{tr}}{N}U(C_1)\frac{\mathrm{tr}}{N}U(C_2)\right\rangle. \tag{44}$$

Now using the factorization property as $N\to\infty$,

$$\left\langle\frac{\mathrm{tr}}{N}U(C_1)\frac{\mathrm{tr}}{N}U(C_2)\right\rangle=\left\langle\frac{\mathrm{tr}}{N}U(C_1)\right\rangle\left\langle\frac{\mathrm{tr}}{N}U(C_2)\right\rangle, \tag{45}$$

we have the closed equation

$$\frac{1}{g^2N}\sum_{\mu\neq\nu}d_\mu(n,n+\nu)W(C)=W(C)-W(C_1)W(C_2)\,. \tag{46a}$$

Also,

$$\frac{1}{g^2N}\sum_{\mu\neq\nu}d(n+\nu,n)W(C)=W(C)-W(C_1)W(C_2)\,, \tag{46b}$$

and if $m\neq n$ or $n+\nu$ as we mentioned,

$$\frac{1}{g^2N}\sum_{\mu\neq\nu}d(m,m+\nu)W(C)=W(C)\,. \tag{46c}$$

We note that at the line $(n,n+\nu)$ the string has split [Fig. 4(b)].

(ii) The link U with coordinates $(n,n+\nu)$ is traversed twice in the *same* direction. See Fig. 5(a). C_1 and C_2 are the two *closed* curves that comprise the curve C. Let C_1' and C_2' be the segments which when completed by the link $(n,n+\nu)$ give the closed

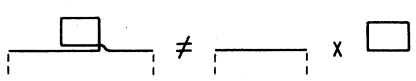

FIG. 3. Small loop does not decouple from big loop.

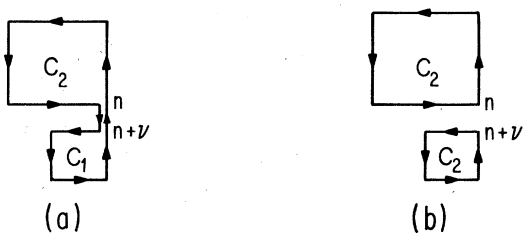

FIG. 4. String splitting.

curves C_1 and C_2:

$$C = C_1'(n, n+\nu) C_2'(n, n+\nu) = C_1 C_2$$

and

$$U(C) = U(C_1') U U(C_2') U .$$

The DS equation at all links $\neq (n, n+\nu)$ is the same as (37). At $(n, n+\nu)$ we have another term. The DS equation is given in the usual notation by

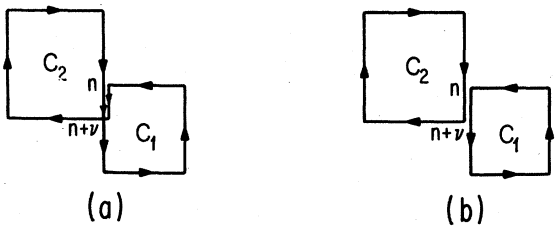

FIG. 5. String rearrangement.

$$\sum_\alpha \frac{\partial}{\partial \Lambda_\alpha} \left\langle \mathrm{tr}[U(C_1') t^{(\alpha)} U_\Lambda U(C_2') U_\Lambda] \exp\left(\frac{1}{g^2} \sum_{\mu \neq \nu} \mathrm{tr}(U_\Lambda V_\mu^\dagger + U_\Lambda^\dagger V_\mu) - \mathrm{tr}(U V_\mu^\dagger + U^\dagger V_\mu) \right) \right\rangle_{\Lambda=0} = 0 . \tag{47}$$

Working out the details and noting that

$$\sum_\alpha [\langle \mathrm{tr} U(C_1') t^{(\alpha)} t^{(\alpha)} U U(C_2') U \rangle + \langle \mathrm{tr} U(C_1') t^{(\alpha)} U U(C_2') t^{(\alpha)} U \rangle] = -[N \langle \mathrm{tr} U(C) \rangle + \langle \mathrm{tr} U(C_1) \, \mathrm{tr} U(C_2) \rangle] ,$$

we arrive at

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (n, n+\nu) W(C) = W(C) + W(C_1) W(C_2) . \tag{48a}$$

(We have used the Hartree-Fock approximation as $N \to \infty$.) The same equation is valid on both traverses. To complete the equations for $m \neq n$ or $n+\nu$ we have

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (m, m+\nu) W(C) = W(C) . \tag{48b}$$

We note that at the link $(n, n+\nu)$ the string has rearranged itself [Fig. 5(b)].

There is a crucial sign difference in the right sides of (46) and (48). These equations can all be compactly written in the notation of Makeenko and Migdal[3]

$$\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (n, n+\nu) W(C) = \sum_{m \in C} [\delta(n, n+\nu \,|\, m, m+\nu) - \delta(n, n+\nu \,|\, m, m-\nu)] W(C_{nm}) W(C_{mn}) . \tag{49}$$

The first δ function always has at least one contribution when $C_{nm} = C$. It also contributes when a link in C is traversed more than once and string rearrangement occurs in C at the link $(n, n+\nu)$ leading to C_{nm} and C_{mn}. The second δ function contributes when a link in C is traversed more than once and string splitting occurs at the link $(n, n+\nu)$ leading to C_{nn} and C_{mn}.

By working our various examples of increasing complexity in which one has several traverses we can convince ourselves that (49) is applicable in all cases and in fact it is the general string equation that describes QCD at $N = \infty$. The equation of Makeenko and Migdal corresponds only to the possibility of string rearrangement. As already mentioned, such a subclass of equations cannot be closed because by successive applications of the derivative operator we can indeed reach cases where string splitting occurs (see Fig. 6).

Finally we illustrate the general equation in a more complicated situation. The loop C is given in Fig. 7. Its three constituent directed loops are C_1, C_2, and C_3. We suppose that they all start and end at the point n. We denote the three string bits that pass through $(n, n+\nu)$ by 1, 2, and 3. The DS equations are derived in the same manner as before. They turn out to be

$$\begin{aligned}
&\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (m, m+\nu) W(C) = W(C) , \\
&\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (n, n+\nu)_1 W(C) = W(C) - W(C_2) W(C_1 C_3) - W(C_1) W(C_3 C_2) , \\
&\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (n, n+\nu)_2 W(C) = W(C) - W(C_2) W(C_1 C_3) + W(C_3) W(C_1 C_2) , \\
&\frac{1}{g^2 N} \sum_{\mu \neq \nu} d_\mu (n, n+\nu)_3 W(C) = W(C) - W(C_1) W(C_2 C_3) + W(C_2 C_1) W(C_3) .
\end{aligned} \tag{50}$$

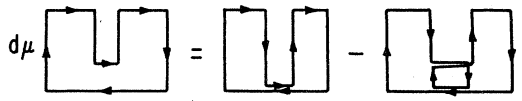

FIG. 6. Derivative operator connects the two different classes of strings.

The string splitting and rearrangement terms carry opposite signs—negative for strong splitting and positive for string rearrangement. These equations can once more be written compactly as Eq. (49).

We believe that Eq. (49) is complete since it contains all gauge-invariant operators which are all independent as $N\to\infty$. Operators like $\det U(C)$ can be considered as dependent because they are linear combinations of the operator $\mathrm{tr}[U(C)^k]$ which create strings of flux which wind k times around the loop C. Further backtracking paths must be excluded from the definitions of $W(C)$. The string equation (49) is a diffusion equation and it is natural to require a boundary condition. One proposal is to fix the value of a particular Wilson loop, e.g., an elementary plaquette. This number may be determined by other methods, e.g., for small g^2N one can use perturbation theory.

IV. CONTINUUM THEORY

Up until now we have worked with the lattice cutoff. This lattice procedure was *also* used to define the notion of a differential or derivative in loop space [Eq. (39)]. Further in deriving the DS equations all expectation values were evaluated without sending the cutoff to infinity.

Now consider the continuum theory defined by the Wilson-loop expectation values:

$$W(C)=\frac{1}{Z}\int\prod_x dA_\mu(x)\frac{1}{N}\,\mathrm{tr}U(C)\exp\left(-\frac{1}{g^2}S\right), \tag{51}$$

where $S=\int d^4x\,\mathrm{tr}F_{\mu\nu}{}^2$ is the Yang-Mills action and

$$U(C)=P\prod_{n\in C}\exp[iA_\mu(n)t_\mu(n)b]$$

is a path-ordered product along C which has been made discrete in segments of size b. $A_\mu(n)$ is the gauge field at the point n on C and $t_\mu(n)$ is the unit tangent to C at n. The continuum theory is regulated by a cutoff $\Lambda=1/a$. The DS equation for $W(C)$ is given by

$$\left\langle\sum_{ij}\frac{\delta}{\delta A_\mu(x)_{ij}}U(C)_{ij}\right\rangle=\frac{1}{g^2}\left\langle\sum_{ij}U(C)_{ij}\frac{\delta}{\delta A_\mu(x)_{ij}}S\right\rangle. \tag{52}$$

The right-hand side of (52) is the string derivative term: $\partial_\nu(\delta/\delta\sigma_{\nu\mu})W(C)$.[3] It can be shown to be the continuum limit of the lattice derivative:

$$\lim_{a\to 0}\frac{1}{a^3}\sum_{\nu\neq\mu}d_\nu(n,n+\mu)=2\partial_\nu\frac{\delta}{\delta\sigma_{\nu\mu}}.$$

The main emphasis of this discussion is on the evaluation of the left side of (52). The procedure of Mâkeenko and Migdal is to consider $b\ll a$ and take the limit $b\to 0$ within the expectation values.

To illustrate the point consider the loop of the type given in Fig. 8(a). In the lattice theory there is neither string splitting nor rearrangement for this loop at the point of intersection. However, if the limit $b\to 0$ is taken *within* the expectation value one has

$$\left\langle\frac{\delta U_{ij}(C)}{\delta A_\mu(x)_{ij}}\right\rangle=\left\langle\lim_{b\to 0}\frac{\delta}{\delta A_\mu(x)_{ij}}\left(\prod_n\exp[iA_\mu(n)t_\mu(n)b]\right)\right\rangle$$
$$=\oint d\tau t_\mu(\tau)\delta^{(4)}(x-y(\tau))\langle\mathrm{tr}U(C_{xy})\mathrm{tr}U(C_{yx})\rangle. \tag{53}$$

The δ function in (53) is a result of taking the limit $b\to 0$. This leads to x and y being coincident points. $\mathrm{tr}U(C_{xy})$ and $\mathrm{tr}U(C_{yx})$ are then gauge in-

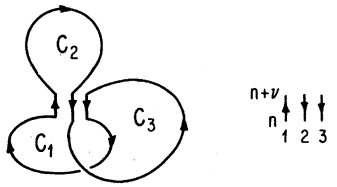

FIG. 7. Loop where a given link is traversed 3 times.

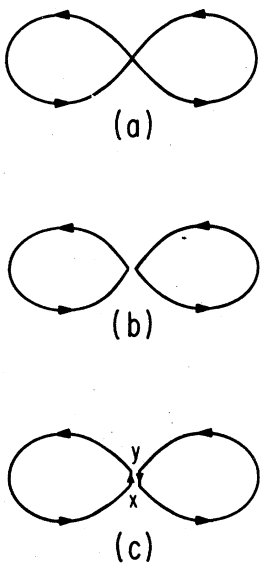

FIG. 8. Continuum string splitting and regularization.

variant and one has as $N\to\infty$ the continuum result of Ref. 3:

$$\left\langle \frac{\delta U(C)_{ij}}{\delta A_\mu(x)_{ij}} \right\rangle = \oint d\tau\, t_\mu(\tau)\delta^{(4)}(x-y(\tau))W(C_{xy})W(C_{yx}) . \tag{54}$$

Clearly the string has split at $x=y$ [Fig. 8(b)]. But (54) is very singular and the proposal of Makeenko and Migdal is to smear out the δ function over length scales of the order of a. Now since $x\neq y$, $\mathrm{tr}U(C_{xy})$ and $\mathrm{tr}U(C_{yx})$ are not gauge invariant unless one introduces by hand small string bits from $x\to y$ and $y\to x$ [Fig. 8(c)]. This is the meaning of (54). In the lattice theory the δ function is smeared out *before* taking the expectation values (Figs. 4 and 5). In the continuum derivation sketched above this is done *after* taking the expectation values. This is the basic point of difference. This also indicates why taking the lattice spacing $a\to 0$ in the general equation (49) cannot give us (54). In fact we do not know how to take the continuum limit of (49). How does the continuum theory remember the different signs which occur at string splitting and rearrangement in the lattice theory? Finally we wish to be explicit: our analysis of the lattice and continuum $U(\infty)$ string theory must not be understood as passing judgment on the two formulations but only as a comparative study.

CONCLUSION

We have seen that in vector models the $N=\infty$ DS equations for correlation functions are truly closed. This is the key to the solubility of these models. In a gauge theory (and even in chiral models) the $N=\infty$ DS equations relate nearby loops in a complicated way. The complete string equation for the $U(\infty)$ gauge theory has been presented on the lattice. It contains amplitudes for both string splitting and string rearrangements with opposite signs—the basic difference between the lattice and continuum derivations as ascribed to inequivalent regularization procedures in the treatment of string splitting or rearrangement at intersections. The question of the continuum limit of the lattice string equations remains. It may be interesting to look for a *new* field-theoretic representation of the string equation very much like the description of polymers in terms of scalar field theory.[15]

Note Added. The sign difference that occurs for doubly traversed links going in the same or opposite direction has also been noted by D. Foerster [Nucl. Phys. B170, 107 (1980)]. We were unaware of this at the time of writing this paper.

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APPENDIX

Here we show in the context of two-dimensional lattice gauge theories[15] that the DS equation (39) for the simple loop is insufficient to determine it. In (39) μ is either of $\pm\hat{x}$ or $\pm\hat{y}$. The equation is depicted in Fig. 9. Since we are in two dimensions we can choose the generalized axial gauge: all links in the y direction and all links in the x direction at a particular y are set equal to the identity. Then it is easy to show that for a simple Wilson loop

$$W(C)=\omega_1^A ,$$

A is the area of the loop and

$$\omega_K=\frac{1}{z}\int dU\,\frac{1}{N}\,\mathrm{tr}U^K \exp\left(\frac{1}{g^2}\,\mathrm{tr}(U+U^\dagger)\right)$$

are the one-plaquette correlations. Further, (39) can be shown to be

$$\frac{1}{g^2N}(\omega_1^{A+1}+\omega_1^{A-1}-\omega_1^{A+1}-\omega_1^{A-1}\omega_2)=\omega_1^A ,$$

which implies

$$\frac{1}{g^2N}(1-\omega_2)=\omega_1 .$$

This is a relation between W_1 and W_2 which cannot determine W_1 and hence $W(C)$. We also note that (40) is true:

$$\langle \mathrm{tr}U(C')UV_{\pm\hat{x}}U\rangle=\omega_1^A\omega_2\neq\langle \mathrm{tr}U(C)\rangle\langle \mathrm{tr}UV_{\pm\hat{x}}U\rangle .$$

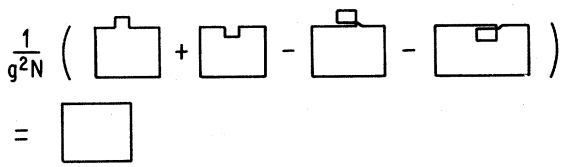

FIG. 9. DS equation in two space-time dimensions.

*Permanent address.
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