Universality class of quantum chromodynamics and super-renormalizable nonpolynomial interactions

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The possibility that quantum chromodynamics (QCD) is merely a correct effective short-distance field theory for the strong interactions is investigated. In this context, the addition of super-renormalizable interactions to QCD is of interest. Conventional polynomial field theory does not include this possibility. Dimensional arguments suggest that certain nonpolynomial potentials can produce super-renormalizable interactions in four dimensions. Several classes of likely candidates are studied. It is shown that they actually produce no interaction at all. There is a universality principle at work that gives insight into the unique role of polynomial quantum field theories.

I. INTRODUCTION

This paper investigates the possibility that quantum chromodynamics (QCD) is merely a correct effective *short-distance* field theory for the strong interactions. There may exist many model theories that agree with QCD at short distance but differ substantially at large distance.

A general framework for the consideration of this problem is provided. One method for constructing such an alternative QCD′ is discussed in detail. Although the method appears promising at first, the studies presented here suggest that it is not likely to work. This is additional evidence for the uniqueness of QCD. More interestingly though, the investigation produces a general field-theory result that gives insight into the unique position of polynomial quantum field theories.

There are two observations that motivate this work. The first is that it would be useful to have a reasonable straw man against which to test the predictions of QCD. Second, the existence of theories that agree with QCD at short distance but differ at long distance would have evident bearing upon efforts to prove confinement in QCD.

Strong interactions are the topic in this paper, and for its duration the other forces will be neglected. In this realm, quantum chromodynamics is king. QCD is a quark model. It includes colored and flavored quarks interacting with gluons. For these general features of the model, there is substantial phenomenological support. Of course, QCD is much more than this. It also proposes a dynamics: a description of the interactions of the quarks with color gauge-field gluons. For this, the evidence is not yet compelling. The problem is a mismatch between theory and experiment. Current theoretical techniques can extract detailed dynamical predictions from QCD at short distance only. There are very few experiments that can be interpreted as clean probes of this region. Nevertheless, the situation is much better there than at long distance where there are no detailed dynamical calculations. In consideration of this, the (perhaps optimistic) position adopted here is that QCD gives a correct description of the strong interactions at short distances.

Before proceeding with the discussion of QCD, pause for a moment to recall the situation in an imaginary world of QED only. Here both theory and experiment are under control at long distances. The tests of QED are excellent. However, experiments do not probe arbitrarily short distances. And more importantly, theory cannot make predictions at arbitrarily short distances where the coupling becomes strong and theorists fail. In this situation, it can be said that QED is the experimentally correct effective *long-distance* field theory. It would be going too far to say that QED is the correct description of nature to arbitrarily short distances. Indeed the renormalization-group concept of universality[1] emphasizes the fact that there are whole classes of models that differ at short distance but give the same effective long-distance theory.

Comparison with QED shows that QCD presents the opposite situation. The theory is under control and seems to agree with experiment at short distances. The coupling becomes strong and theorists fail at long distance. We propose that QCD be viewed as merely a correct effective short-distance theory. Consideration will be given to the possibility that there are field theories with the short-distance behavior of QCD but with distinct long-distance behavior. This evidently bears upon the confinement problem.

To facilitate a more precise discussion, introduce an ultraviolet cutoff Λ to define the theory, and let M be a typical hadronic mass. For processes involving momenta q much greater than M and much less than Λ, QCD is assumed to be a correct effective theory. That is to say, the predictions of QCD are assumed to differ from experimental observations by at most small power-law corrections $[(M/q)^\gamma]$ that are asymptotically undetectable. The search is for a model QCD′ that is as good as QCD at short distance. In particular, it must preserve the asymptotic-freedom results of QCD. Thus QCD′ must have short-distance results differing from those of QCD by small power-law corrections. QCD′ may differ substantially from QCD at long distance. This is not necessarily unacceptable since the long-distance predictions of QCD have not yet been extracted and compared with experiment.

If the leading perturbative short-distance results of QCD are taken as a reference, then there are effects *within* QCD that are of order $(M/q)^\gamma$ at short distance. For example, there are contributions from quark masses, dynamical symmetry breaking, and instantons. These are not our interest. Rather the task is to find a model that reproduces up to power-law corrections the short-distance results of QCD and differs from QCD at long distance.

To study the problem, a conservative approach is adopted. We assume that QCD′ has the fields and interactions of QCD. In addition it has other interactions and perhaps other fields. Let λ represent the additional couplings. Let Γ' be a QCD′ n-point function with external quarks and gluons. Let Γ be the corresponding QCD object. The functional dependence is

$$\Gamma'(p,g,\lambda,m,\mu). \tag{1.1}$$

The renormalization-group result[2] is

$$\Gamma'(e^t p,g,\lambda,m,\mu)=e^{D(t)}\Gamma'(p,\bar{g}(t),\bar{\lambda}(t),e^{-t}\bar{m}(t),\mu). \tag{1.2}$$

Our scenario requires that at large t

$$\Gamma'(e^t p,g,\lambda,m,\mu)\to\Gamma(e^t p,g,m,\mu)+\Delta\Gamma \tag{1.3}$$

with

$$\Delta\Gamma/\Gamma\sim e^{-\gamma t}. \tag{1.4}$$

This will happen if λ is an effectively super-renormalizable interaction. Then

$$\bar{\lambda}(t)\sim e^{-\gamma t} \tag{1.5}$$

and

$$\Delta\Gamma\sim\bar{\lambda}(t). \tag{1.6}$$

The task is now refined to be that of adding couplings to QCD that are effectively super-renormalizable.

Consider this from the point of view of fixed points in the renormalization group.[1-3] In the space of all possible field-theory couplings, QCD gives a particular trajectory to the ultraviolet-stable fixed point at the origin as $t\to\infty$ (Fig. 1). What is the dimension of the critical surface that

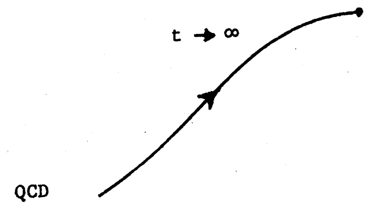

FIG. 1. The trajectory of QCD to the UV-stable fixed point at the origin.

contains this curve? If the dimension is greater than one, then there are QCD′ models of the type that we seek. A QCD′ trajectory would appear as in Fig. 2. (This counting ignores the obvious contributions from mass parameters and other parameters in QCD itself.) Where might these trajectories go in the infrared? Do they all lead to confinement?

The first observation is that there are no conventional, polynomial, super-renormalizable modifications of QCD. More exotic possibilities must be considered. The introduction of higher-derivative fields allows the introduction of polynomial super-renormalizable interactions. Unfortunately, there are difficult problems[4] associated with higher-derivative fields. Nonpolynomial interactions can contain couplings that by dimensional analysis seem to be super-renormalizable. This is the possibility that we have investigated.

The gauge-invariant nonpolynomial interactions that can be constructed with the fields of QCD all involve derivatives of the fields. This is surely a difficult situation, and we are not yet prepared to deal with it. A simpler possibility is to add to the QCD Lagrangian terms

$$\Delta\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+V(\phi)\bar{\psi}\psi. \tag{1.7}$$

V is a nonpolynomial function appropriately chosen so as to produce an effectively super-renormalizable interaction. We have chosen a closely related but even simpler case to investigate in detail. The model described by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+V(\phi) \tag{1.8}$$

will be studied in four Euclidean dimensions. V

FIG. 2. The trajectory of QCD′ must coincide with that that of QCD as $t\to\infty$.

is nonpolynomial. The purpose is to test the idea that nonpolynomial Lagrangians can lead to effectively super-renormalizable interactions in four dimensions. There is one trivial possibility, $V \sim \phi^3 + \phi^4$, that is not of interest and will be ruled out by imposing $\phi \to -\phi$ symmetry.

How should V be chosen? What forms can lead to super-renormalizable couplings? For guidance contrast ϕ^4 and ϕ^6. In a UV-cutoff-free theory, the vacuum expectation of ϕ^2 is of order Λ^2. The range of the fluctuations of ϕ is roughly $(-\Lambda, \Lambda)$, which diverges as the cutoff is removed. The field spends most of the "time" at large field values as $\Lambda \to \infty$. This suggests a connection between the large-cutoff (short-distance) behavior of a theory and the large-ϕ behavior of V. This is borne out by a comparison of ϕ^4 and ϕ^6. The short-distance effects of $(\partial\phi)^2$ and ϕ^4 are comparable and ϕ^4 is a renormalizable theory that can be treated perturbatively. At short distance ϕ^6 dominates ϕ^4 and $(\partial\phi)^2$. It cannot be considered a perturbation, and it gives a nonrenormalizable theory. Also ϕ^6 grows much faster than ϕ^4 at large ϕ. (At small ϕ, ϕ^6 is certainly a perturbation.) We conclude that potentials which grow less rapidly than ϕ^4 at large ϕ are potentially super-renormalizable. The conclusion is consistent with the situation in three dimensions where ϕ^6 is renormalizable and ϕ^4 is super-renormalizable.

Following this intuition, theories with potentials that approach constants at large ϕ are studied in Sec. II. With the further condition that the potential is cutoff independent, thoroughly nonperturbative techniques can be applied. It is shown that the large-cutoff limit is free-field theory. Section III allows cutoff dependence in V and a much larger class of functions. To eliminate nonrenormalizable interactions, we demand only that the short-distance behavior of the theory be no worse than ϕ^4. The price of this generality is that the techniques must be perturbative. (This is perturbation theory in powers of V. V is never expanded in powers of ϕ. Indeed, interesting possibilities such as $V \sim |\phi|^3$ have no such expansion.) The conclusion is that the potentials do not lead to super-renormalizable interactions. The potentials produce either no interactions at all or *renormalizable* interactions equivalent to ϕ^4.

This is an interesting quantum field theory result. It shows that even when the form of V is much more general than polynomial, still no super-renormalizable interactions can be obtained. Furthermore, there is a universality principle at work. Large classes of potentials are seen to be equivalent to either no potential at all or to ϕ^4. This elucidates the special role of polynomial theories. They certainly do not arise from any kind of small-ϕ expansion of more complicated Lagrangians. Rather they are the simplest representatives of large classes of equivalent theories.

This is an interesting general conclusion. However, within the context of the search for QCD′, it is not so interesting. Our results suggest that nonpolynomial modifications of QCD are not likely to produce a QCD′ with the required characteristics.

Remarks are in order concerning the relationship of this paper to earlier work on nonpolynomial potentials.[5] In that work, arbitrary features of the models were studied and to some extent controlled. Nevertheless the resulting models were still very singular at short distance and thus unacceptable to us. The earlier motivation was to control very singular nonpolynomial theories while ours is to find nonpolynomial theories that are inherently soft.

More closely related is the paper of Fried.[6] Our work goes somewhat beyond that paper in that more general potentials are considered and our method of handling the cutoff dependence is a significant improvement. Nevertheless, the idea of an equivalence between polynomial and nonpolynomial potentials is discussed by Fried.

Note added in proof. There are also papers by B. Schroer and R. Höegh-Krohn[7] that contain related results.

II. CUTOFF-INDEPENDENT POTENTIALS

Potentials $V(\phi)$ that are independent of the cutoff Λ will be considered in this section. It will be assumed that V is a locally integrable, even function of ϕ and that

$$V(\phi) \to V_\infty \quad \text{as } \phi \to \pm\infty. \tag{2.1}$$

Since the addition of a constant to V merely changes the irrelevant vacuum energy, it is no loss of generality to assume that V_∞ is zero.

The introductory discussion indicates that potentials of this form are superficially super-renormalizable. Recall that it is the large-ϕ behavior of V that is important. Then suppose, for instance, that at large ϕ

$$V(\phi) \sim g|\phi|^{-\alpha} \quad \text{with } \alpha > 0. \tag{2.2}$$

The immediate result for the dimension of g is

$$[g] = (\text{mass})^{4+\alpha}. \tag{2.3}$$

This suggests super-renormalizability. In fact, it will be shown that the field theory with potential V is noninteracting.

But first a short digression on normal ordering. If a potential of the type that is being considered has a Taylor expansion about the origin, then it can be normal ordered. However, this corresponds to the addition of an infinite number of Λ-dependent counterterms. We reject this procedure for three reasons: (1) With an infinite number of counterterms the original V and the normal-ordered V are only rather indirectly related. (2) It is our position that the primary purpose of counterterms is to manipulate results that are divergent when the cutoff is removed. An infinite number of divergent results will not be encountered. (3) Since the discussion concerns potentially super-renormalizable interactions, an infinite number of counterterms does not seem to be in the correct spirit. Further discussion of this point appears in Sec. III, where cutoff dependence in V is allowed.

The result that the field theory with potential V is free can be established in two ways. The first is nicer but requires that V be positive. The second seems cruder but does not require that restriction on V.

A. Positive potentials

In this subsection, it is assumed that V is even, that

$$V \geq 0, \tag{2.4}$$

and that

$$V \to 0 \quad \text{as } \phi \to \pm\infty. \tag{2.5}$$

In four-dimensional Euclidean space, introduce

$$S_0 = \int d^4x\, \tfrac{1}{2}\phi \hat{D} \phi \tag{2.6}$$

with

$$\hat{D} = -\partial^2 + m^2. \tag{2.7}$$

The interaction is

$$S_I = \int d^4x\, V(\phi), \tag{2.8}$$

so that

$$S = S_0 + S_I \tag{2.9}$$

and

$$S(j) = S - J \quad \text{with } J = \int d^4x\, \phi j. \tag{2.10}$$

The Euclidean generating functional is

$$Z(j) = \frac{\int D\phi\, e^{-S(j)}}{\int D\phi\, e^{-S(0)}}. \tag{2.11}$$

To get control of Z, a cutoff is introduced. Specifically

$$\hat{D}(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}(k^2+m^2) \tag{2.12}$$

is replaced by

$$\hat{D}(x,y,\Lambda) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} d(k,\Lambda) \tag{2.13}$$

with

$$d = \begin{cases} k^2+m^2, & k^2 < \Lambda^2 \\ \infty, & k^2 > \Lambda^2 . \end{cases} \tag{2.14}$$

This may appear a little strange, but it is just the statement that a sharp momentum cutoff has been introduced. After accounting for cancellation between numerator and denominator of (2.11), the products and integrals in momentum space extend up to Λ. For example,

$$\prod_k^{\Lambda} \int d\tilde{\phi}(k) \exp\left(-\int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{2} \tilde{\phi}(-k)(k^2+m^2)\tilde{\phi}(k)\right). \tag{2.15}$$

Equivalently, the free propagator is

$$D(x-y,\Lambda) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{\theta(\Lambda^2-k^2)}{k^2+m^2}. \tag{2.16}$$

Z can be written as a product of three factors

$$Z = Z_1 Z_2 Z_3 \tag{2.17}$$

with

$$Z_1 = \frac{\int D\phi\, e^{-S_0+J} e^{-S_I}}{\int D\phi\, e^{-S_0+J}}, \tag{2.18}$$

$$Z_2 = \frac{\int D\phi\, e^{-S_0+J}}{\int D\phi\, e^{-S_0}}, \tag{2.19}$$

$$Z_3^{-1} = \frac{\int D\phi\, e^{-S_0} e^{-S_I}}{\int D\phi\, e^{-S_0}} = Z_1\Big|_{j=0}. \tag{2.20}$$

It is clear that if $Z_0(j)$ is the free-field generating functional, then

$$Z_2 \to Z_0(j) \quad \text{as } \Lambda \to \infty. \tag{2.21}$$

Next it will be shown that

$$Z_1 \to 1 \quad \text{as } \Lambda \to \infty. \tag{2.22}$$

A similar analysis of Z_3^{-1} gives

$$Z_3^{-1} \to 1 \quad \text{as } \Lambda \to \infty. \tag{2.23}$$

To analyze Z_1, observe that

$$e^{-S_0+J} \geq 0, \tag{2.24}$$

and since

$$0 \leq S_I, \tag{2.25}$$

that

$$0 \leq e^{-S_I} \leq 1. \tag{2.26}$$

Thus

$$Z_1 \leqslant 1. \tag{2.27}$$

Jensen's inequality[8] gives

$$e^{-\bar{S}_I} \leqslant Z_1 \leqslant 1, \tag{2.28}$$

where

$$\bar{S}_I = \frac{\int D\phi\, e^{-S_0+J} S_I}{\int D\phi\, e^{-S_0+J}}. \tag{2.29}$$

This can be written

$$\bar{S}_I = S_1 S_2 \tag{2.30}$$

with

$$S_1 = \frac{\int D\phi\, e^{-S_0+J} S_I}{\int D\phi\, e^{-S_0}}, \tag{2.31}$$

$$S_2^{-1} = \frac{\int D\phi\, e^{-S_0+J}}{\int D\phi^{-S_0}}. \tag{2.32}$$

Clearly

$$S_2^{-1} \to Z_0(j) \quad \text{as } \Lambda \to \infty. \tag{2.33}$$

All of our effort will now go into showing that

$$S_1 \to 0 \quad \text{as } \Lambda \to \infty. \tag{2.34}$$

Introduce the Fourier transform of V,

$$V(\phi(z)) = \int \frac{d\omega}{2\pi} e^{-i\omega\phi(z)} \tilde{V}(\omega)$$

$$= \int \frac{d\omega}{2\pi} \tilde{V}(\omega) \exp\left(-i \int d^4x\, \bar{\omega}(z,x)\phi(x)\right) \tag{2.35}$$

with

$$\bar{\omega} = \omega \delta^4(z-x). \tag{2.36}$$

This gives

$$S_1 = \int d^4z \int \frac{d\omega}{2\pi} \tilde{V}(\omega) \frac{\int D\phi \exp(-S_0 + J - i\int \bar{\omega}\phi)}{\int D\phi\, e^{-S_0}} \tag{2.37}$$

$$= \int d^4z \int \frac{d\omega}{2\pi} \tilde{V}(\omega) Z_0(j - i\bar{\omega}) \tag{2.38}$$

and

$$Z_0(j - i\bar{\omega}) = \exp[\tfrac{1}{2}(j - i\bar{\omega})D(j - i\bar{\omega})]$$

$$= Z_0(j) \exp\left(-\tfrac{1}{2}\omega^2 D(0,\Lambda) - i\omega \int d^4x\, D(z-x)j(x)\right). \tag{2.39}$$

Inserting (2.39) in (2.38), undoing the Fourier transform, and doing the ω integration gives

$$S_1 = Z_0(j) \int d^4z \int dt\, \frac{e^{-(s-t)^2/2D}}{(2\pi D)^{1/2}} V(t) \tag{2.40}$$

with

$$D = D(0,\Lambda) \text{ and } s \equiv (Dj) \equiv \int d^4y\, D(z-y)j(y). \tag{2.41}$$

From (2.16), it is seen that

$$D(0,\Lambda) \sim \Lambda^2 \quad \text{as } \Lambda \to \infty. \tag{2.42}$$

Furthermore, since we are dealing with an ultraviolet problem, it is legitimate to handle any possible infrared divergences in the z integration by placing the system in a box of volume $\mathcal{V}$. There are no short-distance singularities in z. Then using (2.5), we see that the t integral in (2.40) will vanish when $\Lambda \to \infty$. This is evident since the integral is the solution to the heat equation for time D with initial data $V(t)$. Thus, (2.34) is obtained. Combining (2.34), (2.33), (2.30), (2.28), and (2.27), the result (2.22) is established.

The conclusion is that

$$\lim_{\Lambda \to \infty} Z(j,\Lambda) = Z_0(j). \tag{2.43}$$

Potentially super-renormalizable potentials of the class discussed in this subsection actually give no interaction at all.

In an aside, consider for a moment a potential $U = -V$. If (2.43) can be expanded in powers of V, then all terms beyond the first vanish as $\Lambda \to \infty$. Since these are the same (up to signs) as the terms that appear in the expansion of Z for the potential U, U will give no interaction order by order. Other methods, to which we now turn, must be used to establish the stronger nonperturbative result for U.

B. Bounded potentials

This subsection deals with a larger class of cutoff-independent potentials. It is convenient to introduce a coupling constant which, without loss of generality, can be assumed non-negative. The interaction is

$$S_I = \int d^4x\, gV(\phi(x)) \equiv g\bar{V}. \tag{2.44}$$

V is assumed to be a bounded

$$|V| \leq V_M \tag{2.45}$$

even function of ϕ satisfying

$$V \to 0 \text{ as } \phi \to \pm\infty . \tag{2.46}$$

The freedom to add a constant to V has been used again. The assumption that V is positive has been dropped.

Since the problems of interest are ultraviolet, we will work in a finite space-time volume $\mathcal{V}$ and take $\Lambda \to \infty$ before $\mathcal{V} \to \infty$.

Begin again from (2.6), (2.7), (2.44), and (2.9)–(2.20). The result (2.21) is again immediate. The goal is to establish (2.22) and (2.23) from which (2.43) follows. Thus consider

$$Z_1(g) = \frac{\int D\phi \, e^{-S_0+J} e^{-g\overline{V}}}{\int D\phi \, e^{-S_0+J}} . \tag{2.47}$$

Using the positivity of e^{-S_0+J}, $e^{-g\overline{V}}$, and the Λ-independent bound

$$|S_I| = g|\overline{V}| \leq gV_M \equiv v , \tag{2.48}$$

it follows that $Z_1(g)$ is continuous, that $\partial Z_1/\partial g$ exists and that it can be computed from

$$\frac{\partial Z_1}{\partial g} = -\frac{\int D\phi \, e^{-S_0+J} e^{-g\overline{V}}\overline{V}}{\int D\phi \, e^{-S_0+J}} . \tag{2.49}$$

Then

$$\left|\frac{\partial Z_1}{\partial g}\right| \leq e^{v} \frac{\int D\phi \, e^{-S_0+J}|\overline{V}|}{\int D\phi \, e^{-S_0+J}}$$

$$\leq e^{v} \frac{\int D\phi \, e^{-S_0+J}\int d^4z\, |V(\phi(z))|}{\int D\phi \, e^{-S_0+J}} . \tag{2.50}$$

Now observe that $|V|$ satisfies the conditions of the previous subsection and that (aside from the irrelevant multiplicative factor) (2.50) can be identified with (2.29). It follows that

$$\frac{\partial Z_1}{\partial g} \to 0 \text{ as } \Lambda \to \infty . \tag{2.51}$$

Thus

$$Z_1(g) \to Z_1(0) = 1 \text{ as } \Lambda \to \infty . \tag{2.52}$$

Similarly

$$Z_3^{-1} \to 1 \text{ as } \Lambda \to \infty . \tag{2.53}$$

The conclusion again is that

$$\lim_{\Lambda \to \infty} Z(g, \Lambda) = Z_0(j) . \tag{2.54}$$

This large class of superficially super-renormalizable, cutoff-independent potentials actually gives no interaction at all.

We interpret this result in the following way: As the cutoff is removed, the vacuum expectation value of the square of a free field diverges. The fluctuations of the field became infinitely large. The potentials considered are not strong enough to restrict the size of these fluctuations. The potential is significant in only a finite range. Thus the fraction of "time" that the field is in the potential approaches zero as $\Lambda \to \infty$, and the potential has no effect.

III. CUTOFF-DEPENDENT POTENTIALS

In Sec. II, the potential was cutoff independent. Potentials with cutoff dependence that can produce super-renormalizable or renormalizable, but not nonrenormalizable, interactions are investigated here. The comments at the close of Sec. II suggest that it may be possible to obtain nontrivial results if the strength of the potential is allowed to diverge appropriately as $\Lambda \to \infty$. This turns out to be true. More interestingly though, when the cutoff dependence of the potential is adjusted to obtain finite interaction, the resulting theory is always effectively renormalizable and never super-renormalizable. This contradicts the indications from the superficial dimensional analysis.

Consider for a moment the polynomial, renormalizable ϕ^4 theory. The cutoff-independent potential

$$V = \frac{\lambda}{4!} \phi^4 \tag{3.1}$$

gives nontrivial interaction in the first order that survives as $\Lambda \to \infty$. However, there are also divergent vacuum energy and mass shifts which require (3.1) to be modified by Λ-dependent counterterms. Higher-order calculations force more complicated Λ dependence.

The superficially super-renormalizable nonpolynomial potentials under consideration do not force cutoff dependence in this way. Indeed there is no interaction at all when $\Lambda \to \infty$. Nevertheless it is interesting to experiment with the introduction of cutoff dependence.

Some restrictions must be placed upon the cutoff dependence. Without restriction, the connection between the classical and the quantum theories will be tenuous. Also arbitrary cutoff dependence will generally give unacceptable nonrenormalizable interactions. In renormalizable theories, the introduction of the required Λ dependence does not disturb the functional form of the Lagrangian. The Λ dependence appears in a few field-independent parameters. In the search for super-renormalizable interactions, the required cutoff dependence should be no worse than that.

The restrictions upon the cutoff dependence of the potential prevent the appearance of nonrenormalizable interactions. In particular, the large-cutoff (short-distance) behavior should be no worse than that of ϕ^4. Two possibilities are considered. (1) As $\Lambda\to\infty$ the fluctuations of the free field diverge ,

$$\langle\phi^2\rangle \sim D \tag{3.2}$$

with

$$D \equiv D(0,\Lambda)\sim\Lambda^2. \tag{3.3}$$

As $\Lambda\to\infty$, the effect of the potential should be only a perturbation upon the controlling free-field action. The ϕ^4 potential is competitive with the free-field action at large Λ and therefore renormalizable. A natural condition is suggested: In the range of field values allowed by the free action, the Λ-dependent potential should have no more effect than the ϕ^4 potential as $\Lambda\to\infty$. From (3.2), the range of ϕ is roughly

$$-\sqrt{D}<\phi<\sqrt{D}. \tag{3.4}$$

A measure of the strength of ϕ^4 over this range is the average

$$\frac{1}{\sqrt{D}}\int_{-\sqrt{D}}^{\sqrt{D}} d\phi\,\phi^4 \sim D^2. \tag{3.5}$$

The restriction on the Λ-dependent potential is then

$$(1)\quad \frac{1}{\sqrt{D}}\int_{-\sqrt{D}}^{\sqrt{D}} d\phi\,|V(\phi,D)-V(0,D)|\lesssim D^2 \quad\text{as } \Lambda\to\infty. \tag{3.6}$$

It would seem that V must satisfy this in order to have any hope of being super-renormalizable.

(2) This will be a more precise condition upon the short-distance behavior of the interaction. Consider the second-order object

$$\begin{aligned}T_2&=\frac{1}{2}\,\frac{\int D\phi\, e^{-S_0+J}S_I{}^2}{\int D\phi\, e^{-S_0}}\\&=\tfrac{1}{2}\int d^4z_1\int d^4z_2\,\frac{\int D\phi\, e^{-S_0+J}V(\phi(z_1))V(\phi(z_2))}{\int D\phi\, e^{-S_0}}\\&=Z_0\tfrac{1}{2}\int d^4z_1\int d^4z_2\,Y(s_1,s_2,D)\end{aligned} \tag{3.7}$$

with

$$Y(s_1,s_2,D)=\int dt_1\int dt_2[(2\pi)^2\det\mathfrak{D}]^{-1/2}\exp[-\tfrac{1}{2}(s_i-t_i)\mathfrak{D}^{-1}{}_{ij}(s_j-t_j)]V(t_1,D)V(t_2,D) \tag{3.8}$$

and

$$\mathfrak{D}=\begin{pmatrix}D & D(z_1-z_2)\\ D(z_1-z_2) & D\end{pmatrix},\quad s_i=\int d^4x\,D(z_i-x)j(x). \tag{3.9}$$

We require that (3.8) have short-distance behavior no worse than the corresponding object in ϕ^4 theory. If V is square integrable against a Gaussian, then (3.8) can be evaluated at $z_1=z_2$. Sufficient information about the short-distance behavior is available by then looking at the $\Lambda\to\infty$ limit when $j=0$. (For potentials that do not satisfy the square-integrability condition, a slight modification of the method yields the same final results.) At $z_1=z_2=z$, (3.8) becomes

$$Y(s,D)=\int dt\,\frac{e^{-(s-t)^2/2D}}{(2\pi D)^{1/2}}\,V^2(t,D). \tag{3.10}$$

When this is evaluated at $s=0$, it gives the sum of the second-order vacuum-to-vacuum graphs. For the ϕ^4 potential, it is easy to show that

$$Y(0,D)\sim D^4. \tag{3.11}$$

Thus the second condition becomes

$$(2)\quad Y(0,D)\lesssim D^4 \quad\text{as } D\to\infty. \tag{3.12}$$

This is a condition that V must satisfy in order to have a chance of being super-renormalizable.

Condition (1) is more intuitive than (2); condition (2) is more precisely motivated than (1). It is not difficult to show that (except in certain contrived cases) conditions (1) and (2) give the same results. Since (2) rests on a firmer foundation, it will be favored in the following work. The discussion of (1) is given as an aid to intuitive understanding.

There is another more technical requirement that must also be imposed. The analysis to follow makes essential use of the Fourier transform of $V(\phi)$. Thus, we require that the Fourier transform of V exist at least in the distribution sense. This allows V to be any distribution and any of a very large class of functions. It excludes some functions such as e^{ϕ^2} that grow too fast. These require other techniques.

It must be emphasized that this section deals with a much broader class of potentials than did Sec. II. In particular, the requirement that V have a finite $\phi\to\infty$ limit is no longer imposed.

This section will be perturbative. Perturbation theory reveals useful information about polynomial renormalizable and super-renormalizable theories. Since the search is for nonpolynomial super-renormalizable interactions, perturbation theory is appropriate. Also the nonperturbative results of the last section show that the Λ-independent potentials considered had no effect. We can hope that as V is "turned up" by adjusting the Λ dependence, the initial effects will be small and revealed in perturbation theory.

The calculations will be done to first order in V, and certain equivalences will be established. The equivalences are valid to all orders. If for two potentials V_1 and V_2 all first-order n-point functions are equal, or equivalently if

$$\frac{\int D\phi\, e^{-S_0+J} S_{I_1}{}^m}{\int D\phi\, e^{-S_0}} = \frac{\int D\phi\, e^{-S_0+J} S_{I_2}{}^m}{\int D\phi\, e^{-S_0}} \tag{3.13}$$

for $m=1$ at fixed cutoff and for arbitrary source j, then it holds for all positive m. This is just the translation into the functional formalism of the statements that an operator is defined by its matrix elements and that polynomials are complete.

For each potential considered, an equivalent infinite-order, normal-ordered polynomial potential will be found. The coefficients in this polynomial will be strongly cutoff dependent. The results that have been claimed will follow from a study of this cutoff dependence.

Begin again with Z which can be written

$$Z(j)=\frac{T(j)}{T(0)} \tag{3.14}$$

with

$$T(j)=\frac{\int D\phi\, e^{-S_0} e^{J} e^{-S_I}}{\int D\phi\, e^{-S_0}}\,. \tag{3.15}$$

T has an expansion in powers of V. The first term is

$$T_0(j)=Z_0(j). \tag{3.16}$$

The second is

$$T_1(j)=-\frac{\int D\phi\, e^{-S_0} e^{J} S_I}{\int D\phi\, e^{-S_0}}\,. \tag{3.17}$$

T_2 was discussed above.

Now repeat the steps in the analysis of S_1 in Sec. II and find that

$$T_1=-Z_0\int d^4z\, X,$$
$$X(s,D)\equiv\int dt\,\frac{e^{-(s-t)^2/2D}}{(2\pi D)^{1/2}}\,V(t,D). \tag{3.18}$$

Again

$$D=D(0,\Lambda)\ \text{and}\ s=(Dj)=\int d^4y\, D(z-y)j(y). \tag{3.19}$$

$V(\phi,D)$ is the cutoff-dependent potential. The physics is contained in the coefficients of an expansion of X in powers of s. With the technical restriction of V, these exist.

Consider for a moment a slightly different problem. Find the first-order effect of a normal-ordered, infinite-order polynomial potential. Normal ordering corresponds to the particular choice of counterterms that removes contributions from graphs with loops that close upon themselves. Begin with a potential $v(\phi,D)$. The corresponding normal-ordered potential is a new function of ϕ, namely,

$$:v(\phi,D):=v(\phi,D)+\text{counterterms}. \tag{3.20}$$

The counterterms are arranged so that the first-order effect of $:v(\phi,D):$ corresponds to (3.18) is $v(s,D)$. Thus the potentials $V(\phi,D)$ and $:X(\phi,D):$ are equivalent.

This is a useful result. It shows that even for potentials that are not smooth enough to admit a small-ϕ expansion, an equivalent normal-ordered, infinite-order polynomial can be found. These are easier to understand.

Consider the expansion for X:

$$X(s,D)=\sum_{n=0}^{\infty}\frac{1}{n!}\,s^n X_n(D). \tag{3.21}$$

The potentials $V(\phi,D)$ and

$$:X(\phi,D):=\sum_{n=0}^{\infty}\frac{1}{n!}\,X_n(D):\phi^n: \tag{3.22}$$

produce the same effects. It is natural to think of the $X_n(D)$ as coupling constants. Of primary interest will be their behavior in the $D\to\infty$ limit that defines the field theory.

The expressions for the X_n follow:

$$X_n=D^{-n/2}\int dt\,\frac{e^{-t^2/2D}}{(2\pi D)^{1/2}}\,\mathrm{He}_n\!\left(\frac{t}{\sqrt{D}}\right)V(t,D). \tag{3.23}$$

The Hermite polynomials are

$$\mathrm{He}_0=1,\quad \mathrm{He}_2=x^2-1,\quad \mathrm{He}_4=x^4-6x^2+3,\ldots\,. \tag{3.24}$$

X_n is zero for odd n.

Condition (1) can be applied to (3.23) if it is assumed that the potential is well-enough behaved for $t\gtrsim\sqrt{D}$ to be dominated by the Gaussian. The behavior at large D can then be estimated. The Gaussian cuts off the integration for $t\gtrsim\sqrt{D}$ but is order one inside that range. Also the polynomial is at most of order one there. Since a shift of V can only effect X_0, $V(t,D)$ can be replaced by $V(t,D)-V(0,D)$ for $n>0$. Thus

$$|X_n| \lesssim D^{-n/2}\int_{-\sqrt{D}}^{\sqrt{D}} dt\,\frac{1}{\sqrt{D}}\,|V(t,D)-V(0,D)|,\quad n>0 \tag{3.25}$$

is a rough upper bound on X_n. Then (3.6) gives

$$|X_n(D)| \lesssim D^{2-n/2}. \tag{3.26}$$

The preferred analysis is to use condition (2). When $Y(0,D)$ is expressed in terms of X_n's, one gets

$$Y(0,D)=\sum_n \frac{1}{n!}\,X_n{}^2(D)D^n. \tag{3.27}$$

This can be seen from the graphs, but it also follows from (3.10) and (3.23). Since all terms in this sum are positive, each must satisfy (2):

$$X_n{}^2(D)D^n \lesssim D^4 \quad \text{as } D\to\infty. \tag{3.28}$$

Equation (3.26) is obtained again. This means that at most X_0, X_2, and X_4 do not vanish as $\Lambda\to\infty$. (X_0 and X_2 are easily made zero by the usual counterterms for the vacuum energy and mass shifts.)

The conclusion to this point is that the original potential $V(\phi,D)$ is equivalent to the potential $:X(\phi,D):$ of (3.22) with X_0 and X_2 zero and others satisfying (3.26). As $\Lambda\to\infty$, X_4 either vanishes or remains finite; all higher X's vanish. It is tempting to conclude from this that V theory is either trivial (if $X_4\to 0$ as $\Lambda\to\infty$) or that it is equivalent to ϕ^4 theory (if X_4 remains finite as $\Lambda\to\infty$). This conclusion turns out to be justified, but a final step in the argument must be made.

The problem is that the $\Lambda\to\infty$ limit must be saved until last. At fixed Λ, calculate the graphs for a process to a given order in $:X:$. The result will have Λ dependence from the X_n and from the cutoff of the momentum integrations. Now take $\Lambda\to\infty$. Since the integrations in the graphs with six- and higher-line vertices are more divergent, there is the possibility that this could balance the fall of the X_n's as $\Lambda\to\infty$. A power-counting argument shows that this can happen only in contributions to the usual three counterterms of ϕ^4 theory.

Consider a graph with N external lines, with any number of ϕ^4 vertices, with n_6 ϕ^6 vertices, etc. The overall degree of divergence is

$$d=4-N+2n_6+4n_8+\cdots\,. \tag{3.29}$$

If $d<0$, the integrals converge to finite values as $\Lambda\to\infty$. However, the integrals are multiplied by

$$X_6{}^{n_6}X_8{}^{n_8}\cdots\,. \tag{3.30}$$

Thus if any of the $\{n_6,n_8,\ldots\}$ are positive, there will be no contribution as $\Lambda\to\infty$. (The argument is being run as if there were no divergences from subintegrations. It will be easy to see at the end of the argument that these are of the usual three kinds. Consequently all subintegrations can be made finite by the standard procedure before the behavior of the whole graph is considered.) If $d\geq 0$ and $N>4$, the large-Λ behavior is

$$X_6{}^{n_6}X_8{}^{n_8}\cdots\Lambda^d=(X_6\Lambda^2)^{n_6}(X_8\Lambda^4)^{n_8}\cdots\Lambda^{4-N}. \tag{3.31}$$

Equation (3.26) shows that this vanishes as $\Lambda\to\infty$. For $N=4$, the once-subtracted graphs behave as if $N=6$; the preceding argument applies. But (3.31) shows that the *counterterm* can get contributions from graphs with higher vertices. For $N=2$, similar arguments show that twice-subtracted graphs vanish if there are higher vertices in them. The wave-function counterterm gets contributions from graphs with higher vertices. The mass counterterm is quadratically divergent and the graphs with higher vertices contribute to the finite coefficient of the divergence. Thus as $\Lambda\to\infty$, graphs with the higher vertices can contribute only to the unobservable counterterms. The theory is equivalent to $(g/4!)\phi^4$ if $X_4\to g$ as $\Lambda\to\infty$. This completes the power-counting argument.

The conclusion is that the potentials under consideration lead to renormalized theories that are either free or equivalent to ordinary ϕ^4 theory. Again, super-renormalizable interactions do not emerge from superficially super-renormalizable potentials.

This suggests a kind of universality. There is a very large class of potentials that is equivalent to the ϕ^4 potential. It also suggests a reason for the preeminent role of polynomial field theories. They do not arise from some sort of small field expansion (indeed $\langle\phi^2\rangle\to\infty$). Rather they are the simplest representatives of large classes of equivalent potentials.

IV. CONCLUSION

Field theories in four Euclidean dimensions of the form

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+V(\phi)$$

have been studied. If V is independent of Λ and has a finite $\phi\to\infty$ limit, then the theory is free. (Section II gives the precise discussion.) Perturbation theory is not used. When Λ dependence is allowed in V, perturbation theory in V is employed (Sec. III). Potentials can be divided into those whose short-distance behavior is (1) softer than ϕ^4, (2) the same as ϕ^4, or (3) worse than ϕ^4. The first give free-field theory. The second are equivalent to ϕ^4. The third are nonrenormalizable. There is no super-renormalizable class.

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